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A Note on the Projective Representations of Finite Groups.

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ABSTRACT - In this paper, we describe exactly the greatest common divisor of the degrees of projective representations of finite groups.

1. Introduction.

All representations and characters, studied in this paper, are taken over the complex numbers, and all considered groups are finite. For basic definitions concerning projective representations, see [1]. If G is a group and α is a cocycle of G, we denote by $\mathrm{Proj}(G, \alpha) = \{\tau_1, \tau_2, \ldots, \tau_t\}$ the set of irreducible projective characters of G with cocycle α, where (of course) t is the number of α-regular conjugacy classes of G, $\tau_i(1)$ being called the *degree* of τ_i. Also as normal, $M(G)$ will denote the Schur multiplier of G, $[\alpha]$ the cohomology class of α, and $[1, G]$ the cohomology class of the trivial cocycle of G.

The main result of this paper exactly describes the greatest common divisor of the degrees of $\mathrm{Proj}(G, \alpha)$.

2. Main result.

Our main result is the following

THEOREM. *Let $p_1, p_2, \ldots, p_n$ be the prime divisors of $|G|$, with $P_1, P_2, \ldots, P_n$ corresponding Sylow p_i-subgroups of G. Let M_i be a sub-*

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group of P_i of minimal index such that $[a_{Mi}] = [1, G]$. Then, the greatest common divisor of the degrees of Proj(G, a) *is equal to* $\Pi_{i=1}^{n}[P_i: M_i]$. ■

We start by defining

$$s(G, a) = \min\{\tau_i(1): 1 \leqslant i \leqslant t\}$$

and

$$c(G, a) = g.c.d.\ \{\tau_i(1): 1 \leqslant i \leqslant t\}.$$

It is obvious that if $[a] = [1, G]$, then $c(G, a) = s(G, a) = 1$. Thus, we are only really interested in non-trivial cocycles of G.

Now, we quote the following well-known result, which, to my known, belongs to the *folklore* of this topic (e.g., for the part (a), see [1, 6.2.6]).

LEMMA 1. *Let a be a cocycle of G with $o([a]) = e$ in $M(G)$. Then,*

(a) $e\,|\,c(G, a)$;

(b) *if p is a prime number such that $p\,|\,c(G, a)$, then $p\,|\,e$.* ■

We note here that it is not true, in general, that $c(G, a) = e$, or indeed that, if some integer m divides $c(G, a)$, then $m\,|\,e$. For, according to [2], there exists a cocycle a of $G = 2^4$ with $o([a]) = 2$, but $c(G, a) = 4$. In other words, the corresponding central extension of the elementary abelian group G, an extraspecial group of order 32, has an unique ordinary irreducible nonlinear character, which is of a degree 4.

Now, we show that to analyse $c(G, a)$ we should consider the prime divisors of $o([a])$ and $s(P, a_P)$ for the corresponding Sylow subgroups P of G.

PROPOSITION 1. *Let $c = c(G, a)$. Then, the p-th part of c, c_p, is equal to $s(P, a_P)$ for P a Sylow p-subgroup of G.*

PROOF. Let $P \in \mathrm{Syl}_p(G)$ and $\mathrm{Proj}(P, \alpha_P) = \{\gamma_1, \gamma_2, \ldots, \gamma_r\}$. Now, let $\tau \in \mathrm{Proj}(G, \alpha)$ such that $(\tau(1))_p = c_p$. Then, $\tau_P = \sum_{j=1}^{r} b_j \gamma_j$, where the b_j's are non-negative integers so that

$$c_P = s(P, \alpha_P)\left(\sum_{j=1}^{r} b_j (\gamma_j(1)/s(P, \alpha_P))\right)_p .$$

Hence, $s(P, \alpha_P) \mid c_p$.

On the other hand, let $\gamma \in \mathrm{Proj}(P, \alpha_P)$ be such that $\gamma(1) = s(P, \alpha_P)$. Then, $\gamma^G = \sum_{i=1}^{t} a_i \tau_i$ for some non-negative integers a_i, and so, comparing the p-th parts of the degrees, we obtain

$$s(P, \alpha_P) = c_p\left(\sum_{i=1}^{t} a_i (\tau_i(1)/c)\right)_p .$$

Hence, $c_p \mid s(P, \alpha_P)$. ■

Thus, we are left with the task of describing $s(P, \alpha_P) = c(P, \alpha_P)$ for $P \in \mathrm{Syl}_p(G)$. However, we shall actually consider a more general situation than this. Recall that $\tau \in \mathrm{Proj}(G, \alpha)$ is called *monomial*, if it is induced from a projective character of degree 1 of a subgroup, and G is said to be a *PM-group* if all its irreducible projective characters are monomial.

PROPOSITION 2. *Let M be a subgroup of G of minimal index such that $[\alpha_M] = [1, G]$. Then, we have:*

(a) $s(G, \alpha) \leqslant [G: M]$ *and* $c(G, \alpha) \mid [G: M]$;

(b) *if* $c(G, \alpha) = [G: M]$, *then* $c(G, \alpha) = s(G, \alpha)$;

(c) $s(G, \alpha) = [G: M]$ *if and only if there exists a monomial character* $\tau \in \mathrm{Proj}(G, \alpha)$ *with* $\tau(1) = s(G, \alpha)$.

PROOF. Let $\tau' \in \mathrm{Proj}(G, \alpha)$ such that $\tau'(1) = s(G, \alpha)$, and $\lambda \in \mathrm{Proj}(M, \alpha_M)$ with $\lambda(1) = 1$. Then, $\lambda^G = \sum_{i=1}^{t} a_i \tau_i$, for some non-negative integers a_i, and so

$$(1) \qquad \lambda^G(1) = [G: M] = c(G, \alpha)\left(\sum_{i=1}^{t} a_i (\tau_i(1)/c(G, \alpha))\right) \geqslant \tau'(1),$$

proving (a). Since $c(G, \alpha) \mid s(G, \alpha)$, we have that (b) is immediate from (a).

Now, suppose that equality holds in (1). Then, we must have that λ^G is irreducible. Conversely, if $\tau \in \mathrm{Proj}(G, \alpha)$ is monomial and $\tau(1) = s(G, \alpha)$, then, by definition, there exists a subgroup L of G and $\mu \in \mathrm{Proj}(L, \alpha_L)$ with $\mu(1) = 1$ such that $\mu^G = \tau$. Obviously, then $[\alpha_L] = [1, G]$, from Lemma 1(a). Also, $[G: L] = s(G, \alpha) \leqslant [G: M]$, by (a), and, hence, by hypothesis, $[G: L] = [G: M]$. ■

Of course, equality in Proposition 2(c) does occur when G is a *PM*-group and, in particular, when G is supersolvable (see [1, 6.5.11]). However, if $G = A_4$, $o([\alpha]) = 2$, then $s(G, \alpha) = c(G, \alpha) = 2$, but A_4 has no subgroup of index 2, so that equality does not always hold.

The proof of the main theorem is now yielded by the above remarks in conjunction with Propositions 1 and 2.

We mention three applications of the above results.

COROLLARY 1. *Let L be a cyclic subgroup of G. Then, $s(G, \alpha) \leqslant [G: L]$ and $c(G, \alpha) \mid [G: L]$ for all cocycles α of G.*

PROOF. Since L is cyclic, $M(L)$ is trivial, and, hence, $[\alpha_L] = [1, G]$ for all cocycles α of G. Thus, the result is immediate, from Proposition 2(a). ■

Now, we show that a slightly weaker version of Proposition 2(a) gives an alternative proof for the final assertion of [1, 4.1.9].

COROLLARY 2. *Let e denote the exponent of $M(G)$, α be a cocycle of G with $o([\alpha]) = e$, and L be a subgroup of G such that $[\alpha_L] = [1, G]$. Then, $e \mid [G: L]$. In particular, e divides the index of each cyclic subgroup of G.*

PROOF. By Lemma 1(a) and Proposition 2(a), we have $e \mid c(G, \alpha) \mid [G: L]$. ■

Finally, the following type of result is useful in constructing the projective representations of a given group with specified Sylow structure.

COROLLARY 3. *Let α be a cocycle of G with $2 \mid o([\alpha])$, and suppose that G has a dihedral Sylow 2-subgroup. Then,* $(c(G, \alpha))_2 = 2$.

PROOF. Let $P \in \mathrm{Syl}_2(G)$. The restriction mapping from $\mathrm{Syl}_2(M(G))$ into $M(P)$ is a monomorphism. Hence, since P has a cyclic subgroup of index 2, we have, by Proposition 1 and Corollary 1, that $(c(G, \alpha))_2 = s(P, \alpha_P) = 2$. ■

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