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Prime Rings with Hypercommuting Derivations on a Lie Ideal.

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ABSTRACT - Let R be a prime ring with no non-zero nil right ideals, d a non-zero derivation of R, L a non-central Lie ideal of R. If d satisfies $[d(u^m), u^m]_k = 0$ for all $u \in L$, $m = m(u) \geq 1$, $k = k(u) \geq 1$, then R is an order in a simple algebra of dimension at most 4 over its center.

1. Introduction.

The classical hypercenter theorem, proved by I. N. Herstein [10], asserts that for a ring R not containing non-zero nil two-sided ideals, an inner derivation d_a, induced by $a \in R$, satisfying $d_a(x^m) = 0$, $m = m(x) \geq 1$, for all $x \in R$, must vanish identically on the whole ring R, i.e. $a \in Z(R)$.

Moreover in [4, theorem 4] Chuang and Lin proved that for a ring R not containing non-zero nil right ideals, the inner derivation d_a, induced by $a \in R$, satisfying $[d_a(x^m), x^m]_k = 0$, $m = m(x) \geq 1$, $k = k(x) \geq 1$, for all $x \in R$, must vanish identically on the whole ring R.

Later Chuang generalized the results above to arbitrary derivation d, defined in a prime ring R not containing non-zero nil right ideals. He proved in [3, corollary 2] that if $[d(x^m), x^m]_k = 0$, $m = m(x) \geq 1$,

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$k = k(x) \geq 1$, for all $x \in R$, then either d vanishes identically on the whole ring R or R is commutative.

The starting point of this paper is the following result, obtained in [5]:

Let R be a prime ring, with no non-zero nil right ideals, L a non-central Lie ideal of R, d_a an inner derivation induced by $a \in R$. If $[d_a(u^m), u^m]_k = 0$, for all $u \in L$, $m = m(u) \geq 1$, $k = k(u) \geq 1$, then either d_a vanishes identically on the whole ring R, that is $a \in Z(R)$, or R satisfies $S_4(x_1, x_2, x_3, x_4)$, the standard identity of degree 4.

The last conclusion is well known to be equivalent, by Posner's theorem, to saying that R is an order in a simple algebra at most 4-dimensional over its center.

Moreover we remark that if $R = M_2(C)$, the ring of all 2×2 matrices over a commutative ring C, then for any $u \in [R, R]$ one has $u^2 \in Z(R)$ (see example (3) page 12 in [12]), and so our condition $[d(u^m), u^m]_k = 0$ holds for $m = 2$ and $k = 1$.

The purpose of this note is to generalize the result obtained in [5] to arbitrary derivation d. We will prove the following:

THEOREM. *Let R be a prime ring with no non-zero nil right ideal, L a non-central Lie ideal of R, d a non-zero derivation of R satisfying $[d(u^m), u^m]_k = 0$, for all $u \in L$, $m = m(u) \geq 1$, $k = k(u) \geq 1$. Then R satisfies $S_4(x_1, x_2, x_3, x_4)$, the standard identity of degree* 4.

2. Preliminares.

The proof of our theorem is based upon two results obtained respectively in [3] and [5].

The first one is a result of Chuang [3, proposition 2] concerning a careful analysis of derivations satisfying a particulary property on semiprime rings.

The related objects we need to mention are the left Utumi quotient ring U, and also the two-sided Utumi quotient ring Q of a ring R (sometimes, as in [1], U and Q are called the maximal left ring of quotients and the symmetric ring of quotients respectively).

The definitions, the axiomatic formulations and the properties of these quotient rings U, Q can be found in [15], [8], [1].

For istance U, the left Utumi quotient ring of R, exists if and only if R is right faithful, that is for any $a \in R$, $Ra = 0$ implies $a = 0$.

In the same way we can define Q if R is both right and left faithful.

In any case, when R is a prime ring, all that we need here about these objects is that

1) $R \subseteq Q \subseteq U$.

2) U and Q are prime rings [8, page 74].

3) For all $q \in Q$ there exists a dense left ideal M of R such that $Mq \subseteq R$, moreover if $Mq = 0$, for some dense left ideal M of R, then $q = 0$.

4) The center of U, denoted by C, coincides with the center of Q.

C is a field which is called the extended centroid of R [1, pages 68-70].

Moreover if R is a prime P.I. ring then, by Posner's theorem [9, theorem 1.4.3 page 40], C is the quotient field of $Z(R)$ and

$$RZ^{-1} = \{rz^{-1}: r \in R,\ z \in Z(R) - \{0\}\} = RC$$

is a simple algebra finite dimensional over its center.

In this case it is easy to see that $RC = Q = U$.

Finally we recall that a map $d: R \to R$ is a derivation if, for any $x, y \in R$, $d(x+y) = d(x) + d(y)$ and $d(xy) = d(x)\,y + xd(y)$. Each derivation of a prime ring R can be uniquely extended to a derivation of its Utumi quotient ring U and thus all derivations of R will be implicitly assumed to be defined on the whole U (see [15, page 101] or [16, lemma 2]).

Now we are ready to state the result of Chuang ([3, proposition 1, page 46]) for prime rings. In this case as we said above U and Q are prime too and so any central idempotent is trivial.

Hence, for any $a \in U - \{0\}$, the norm $\|a\|$ of a, defined in [3, page 39, (7)], is always 1 and of course $\|0\| = 0$.

PROPOSITION 2.1. *Let R be a prime ring with extended centroid C. Let d a derivation of U, the left Utumi quotient of R, satisfying $d(a)(1+a)^{-1} \in C$, for any $a \in R$, with $a^2 = 0$. Let Q be the two-sided Utumi quotient ring of R. Then*

either the ring Q is reduced, that is Q does not have any non-zero nilpotent element

or $Q = U = M_2(C)$, the ring of all two by two matrices over C

or the derivation d is the inner derivation induced by a square-zero *element* $c \in U$, *satisfying the property that, for any* $x, y \in Q$ *with* $xy = 0$, *we also have* $xc = cy = 0$.

The second result concerns the generalized hypercentralizer of a non-central Lie ideal of R.

More precisely the generalized hypercentralizer of an arbitrary subset S of R is the following subring of R:

$$H_R(S) = \{a \in R\colon \text{ for all } s \in S \text{ there exist } n = n(a, s) \geq 1,$$
$$k = k(a, s) \geq 1 \text{ such that } [a, s^n]_k = 0\}.$$

In [5] we proved that if L is a non-central Lie ideal of a prime ring R with no non-zero nil right ideal, then either $H_R(L) = Z(R)$ or R satisfies $S_4(x_1, x_2, x_3, x_4)$.

Finally we remark that an important tool in our proof will be the theory of differential identities initiated by Kharchenko [13].

3. Proof of the theorem.

Throught this paper we will use the following notation:

R will always be an associative prime ring, with no non-zero nil right ideal, L will be a non-central Lie ideal of R and d will be a derivation of R satisfying $[d(u^m), u^m]_k = 0$, for any $u \in L$, $m = m(u) \geq 1$, $k = k(u) \geq 1$. U will be the left Utumi quotient ring of R, and Q will be the two-sided Utumi quotient ring of R.

We start with an easy remark:

REMARK 1. *If R has characteristic $p \neq 0$ then for all $u \in L$ there exists $n = n(u) \geq 1$ such that $d(u^n) = 0$.*

PROOF. Let $u \in L$ be arbitrarily given. There exist $m = m(u) \geq 1$, $k = k(u) \geq 1$ such that $[d(u^m), u^m]_k = 0$. Pick an integer $t \geq 1$ such that $p^t \geq k$. Then

$$0 = [[d(u^m), u^m]_k, u^m]_{p^t - k} = [d(u^m), u^m]_{p^t}.$$

Since R is of characteristic $p > 0$, we have

$$0 = [d(u^m), u^m]_{p^t} = [d(u^m), u^{mp^t}]$$

and this implies immediately $[d(u^{mp^t}), u^{mp^t}] = 0$. Let $a = u^{mp^t}$. We obtain $[d(a), a] = 0$.

Using the commutativity we have $d(a^p) = pa^{p-1} d(a) = 0$, that is

$$d(u^{mp^{t+1}}) = 0$$

and so we have shown that for all $u \in L$ there exists $l = l(u) \geq 1$ such that $d(u^l) = 0$. ∎

Notice that in this case, if $p \neq 2$, then our result follows immediately by theorem 2 of [7].

Now we make some other reductions.

It is well known that if L is a non-central Lie ideal of a prime ring R then either R satisfies $S_4(x_1, x_2, x_3, x_4)$ or there exists a non-zero two-sided ideal I of R such that $[I, R] \subseteq L$ and $[I, R] \not\subseteq Z(R)$.

Therefore we will assume, in all that follows, that $L = [I, R]$ for some non-zero two-sided ideal I of R (see for istance Lemma 2 and Proposition 1 in [6]).

In this case L is invariant under any inner automorphism induced by an invertible (or quasi-invertible) element of R.

Moreover, if $Z(R) \neq 0$, then we can consider

$$\overline{R} = \{rz^{-1}: r \in R, z \in Z(R) - \{0\}\}$$

$$\overline{L} = \{uz^{-1}: u \in L, z \in Z(R) - \{0\}\}$$

which are the localizations at $Z(R)$ of R and L respectively. Since $L = [I, R]$ is a $Z(R)$-submodule of R, we have

REMARK 2.

1) $\overline{L}$ is a non-central Lie ideal of the prime ring $\overline{R}$

2) the derivation d extends uniquely to a derivation on $\overline{R}$ as follows

$$d(rz^{-1}) = (d(r)\,z - rd(z))z^{-2}$$

3) the derivation d, defined on $\overline{R}$, satisfies our assumptions on $\overline{L}$, that is for any $\overline{u} \in \overline{L}$, there exist $n = n(\overline{u}) \geq 1$, $k = k(\overline{u}) \geq 1$ such that $[d(\overline{u}^n), \overline{u}^n]_k = 0$.

LEMMA 3.1. *Let $a \in R$. If a is invertible then $d(a)\,a^{-1} \in H_R(L)$, if a is quasi-invertible then $d(a)(1+a)^{-1} \in H_R(L)$.*

PROOF. First we assume that a is invertible, then, as we said above, $aLa^{-1} = L$, hence for any $u \in L$ there exist $m, k, n, h \geq 1$ such that

$$0 = \left[d((a^{-1}ua)^m), (a^{-1}ua)^m\right]_k = 0$$

and

$$[d(u^n), u^n]_h = 0 .$$

Hence for $s = nm$ and $t = \max\{h, k\}$ we also have:

$$\left[d((a^{-1}ua)^s), (a^{-1}ua)^s\right]_t = 0 = [d(u^s), u^s]_t.$$

It follows that:

$$0 = [d(a^{-1}u^s a), a^{-1}u^s a]_t =$$

$$= [d(a^{-1})\, u^s a + a^{-1} d(u^s)\, a + a^{-1} u^s d(a), a^{-1} u^s a]_t =$$

$$= [-a^{-1}d(a)\, a^{-1}u^s a + a^{-1}u^s d(a), a^{-1}u^s a]_t + [a^{-1}d(u^s)\, a, a^{-1}u^s a]_t =$$

$$= -a^{-1}[d(a)\, a^{-1}u^s - u^s d(a)\, a^{-1}, u^s]_t a = -a^{-1}[d(a)\, a^{-1}, u^s]_{t+1} a .$$

Hence $[d(a)\, a^{-1}, u^s]_{t+1} = 0$ that is $d(a)\, a^{-1} \in H_R(L)$.

A similar proof holds if a is a quasi-invertible element of R. ∎

We remark that any square-zero element a of R is quasi-invertible with quasi-inverse $-a$.

Therefore, by [5], either R satisfies $S_4(x_1, x_2, x_3, x_4)$ and we are done or $d(a)\, a^{-1} \in Z(R)$, that is the derivation d satisfies the hypothesis of Chuang's result. In this case one of the three conclusions of the Proposition 2.1 must hold.

Now we treat each case separately.

Of course if $U = M_2(C)$, the ring of all 2×2 matrices over C, then it satisfies the standard identity $S_4(x_1, x_2, x_3, x_4)$ and we are done again, since $R \subseteq U$.

In the second case we have:

PROPOSITION 3.1. *If the derivation d is the inner derivation defined by a square-zero element c in U, satisfying $xc = cy = 0$ for any $x, y \in Q$, with $xy = 0$, then d vanishes identically on R.*

PROOF. Since c is an element of the left Utumi quotient ring of R, there exists a left dense ideal M of R such that $Mc \subseteq R$ (see proposition 2.1.7 in [1]).

Moreover, since R is a prime ring, IM is again a left dense ideal of R and, of course, $IMc \subseteq IR \subseteq I$.

In other words we can assume that there exists M left dense ideal of R such that $Mc \subseteq I$ and so $[Mc, Mc] \subseteq L$. Therefore for any $x, y \in M$ there exist $m = m(c, x, y) \geq 1$, $k = k(c, x, y) \geq 1$ such that

$$[d([xc, yc]^m), [xc, yc]^m]_k = 0 .$$

Moreover $d([xc, yc]^m) = [c, [xc, yc]^m] = c[xc, yc]^m$. Therefore

$$0 = [c[xc, yc]^m, [xc, yc]^m]_k =$$

$$= \sum_{h=0}^{k} \binom{k}{h} (-1)^h [xc, yc]^{mh} (c[xc, yc]^m)[xc, yc]^{m(k-h)} = c[xc, yc]^{m(k+1)}.$$

Thus $[xc, yc][xc, yc]^{m(k+1)} = 0$, that is $[xc, yc]$ is a nilpotent quasi-invertible element. By lemma 3.1 and the main theorem in [5], either R satisfies $S_4(x_1, x_2, x_3, x_4)$ or $d([xc, yc]) = \alpha(1 + [xc, yc])$, where $\alpha \in Z(R)$.

In the first case R is a prime PI ring and so, by Posner's theorem, $RC = S = Q = U$ is a central simple algebra finite dimensional over its center C.

Since U satisfies $S_4(x_1, x_2, x_3, x_4)$, if c is a non-zero square-zero element, then we have $U = M_2(C)$, the ring of 2×2 matrices over C.

Since $xc = cy = 0$, for any $x, y \in Q = U$ such that $xy = 0$, then $e_{11} c = e_{21} c = e_{22} c = e_{12} c = 0$, that is $M_2(C)c = 0$ and so $c = 0$, a contradiction.

In the second case we know that $d([xc, yc]) = [c, [xc, yc]] = c[xc, yc]$.

So $\alpha^2(1 + [xc, yc])^2 = (\alpha(1 + [xc, yc]))^2 = (d([xc, yc]))^2 = (c[xc, yc])^2 = 0$.

Since $[xc, yc]$ is quasi-invertible, then $\alpha = 0$. Thus $0 = d([xc, yc]) = c[xc, yc]$. Hence, for any x, y in M, $c[xc, yc] = 0$.

Let x, y, z be in R, t in M. Since M is a left dense ideal of R, we have that xt, yt, zt fall in M and so $ztc[xtc, ytc] = 0$, that is R is GPI [2].

In this case, by Martindale's result the central closure $S = RC$ is a primitive ring, containing a minimal right ideal eS, such that eSe is a division algebra finite dimensional over C, for any minimal idempotent e of S [9, theorem 1.3.2].

If $e=1$ then S is a finite dimensional division algebra over C. Therefore S is PI and so R is PI too. As we said above in this case $S=Q=U$ and so $c\in S$ which is a division ring. Hence $c=0$ and consequentely $d=d_c=0$.

Now we may suppose $e\neq 1$. We known that $xc=cy=0$, for any x, y in Q, with $xy=0$.

Let $x^2=0$. Since $xc=cx=0$ then $d(x)=[c,x]=0$, that is c commutes with every square-zero element x in Q.

Let A be the subring generated by the elements of square zero. A is invariant under all automorphisms of Q. By our assumption there are non-trivial idempotent in the prime ring Q and so A contains a non-zero ideal J of Q by [11].

Now, since $0=d(A)\supseteq d(J)\supseteq d(JQ)=Jd(Q)$, by the primeness of Q we obtain $d(Q)=0$, that is $d=0$ in Q and so in R too. ∎

REMARK. The last case is the one in which Q is a reduced ring. Since Q is also a prime ring then it must be a domain. In fact, let $x, y\in Q$ be such that $xy=0$ and $y\neq 0$. Then, for any $z\in Q$, we have $(yzx)^2=$ $=yzxyzx=0$ and so $yzx=0$, that is $yQx=0$ and $x=0$ because Q is prime.

DEFINITION. For $a\in R$ let

$$H(a)=\{r\in R:[r,a]_m=0 \text{ for some integer } m=m(r)\geq 1\}.$$

Of course $H(a)$ is a subring of R.

We also have:

LEMMA 3.2. *Let R be a domain of characteristic zero and let d be the derivation satisfying our assumption. If a is an element of I such that*

$$[d(a),a]_l=0 \textit{ for some } l=l(a)\geq 1$$

then $H(a)$ is invariant under the derivation d and moreover $d(a)$ is in the center of $H(a)$.

PROOF. By localizing at non-zero integers we may assume that R is an algebra over the field of the rational numbers.

By [3, assertion 2] it follows that $H(a)$ is invariant under d. Now, we put $\delta=d_a$, the inner derivation induced by a.

Of course the derivation δ restricted to $H(a)$, which we also denote δ, is nil and hence for any integer λ, the derivation $\lambda\delta$ is also nil on $H(a)$.

Since $H(a)$ is an algebra over the field of the rational numbers, the map $\exp(\lambda)$ is an automorphism of $H(a)$ (see [3, proposition 2]), hence the map $d_\lambda = \exp(\lambda\delta)\, d \exp(-\lambda\delta)$ is a derivation of $H(a)$.

Obviously $I \cap H(a)$ is a two-sided ideal of $H(a)$ which is invariant under the action of $\exp(\lambda\delta)$.

Hence $L_1 = [I \cap H(a), H(a)] \subseteq [I, R] = L$ is a Lie ideal of $H(a)$, moreover, for any $u \in L_1$, there exist some integers $n = n(u) \geq 1$, $k = k(u) \geq 1$ such that $[d_\lambda(u^n), u^n]_k = 0$.

Now, given $u \in L_1$, there exist integers $n = n(u) \geq 1$, $m = m(u) \geq 1$, $k = k(u) \geq 1$, $h = h(u) \geq 1$ such that

$$[d(u^m), u^m]_h = 0 = [d_\lambda(u^n), u^n]_k$$

hence, as in the proof of lemma 3.1, for $s = nm$ and $t = \max\{h, k\}$ we also have

$$[d(u^s), u^s]_t = [d_\lambda(u^s), u^s]_t$$

that is $[(d_\lambda - d)(u^s), u^s]_t = 0$.

By [3, proposition 2, (3)] the derivation $d_\lambda - d$ is the inner derivation induced by the element $b_\lambda = \sum_{n \geq 1} ([\delta(\lambda a), \lambda a]_{n-1})/n!$, and so b_λ is in the generalized hypercentralizer of L_1 in $H(a)$.

If $I \cap H(a)$ is the zero ideal of $H(a)$, then $a = 0$ since it is in $I \cap H(a)$ and of course $d(a) = 0 \in Z(H(a))$.

If $I \cap H(a)$ is non-zero then, by [5, proposition 4.1], either $H_{H(a)}(L_1) = \; = Z(H(a))$ or $H(a)$ satisfies $S_4(x_1, x_2, x_3, x_4)$.

In the first case we may conclude, by a Vandermonde determinant argument, $d(a) \in Z(H(a))$.

In the other case, by localizing at the center of $H(a)$, we may assume that $H(a)$ is a division algebra of dimension at most 4 over its center $Z(H(a))$. It follows that there exists $m \geq 1$ such that $\delta^m(r) = 0$, for any $r \in H(a)$, that is δ is a nil of bounded index on $H(a)$.

By [9, lemma 1.1.9] there exists $z \in Z(H(a))$ such that $a - z$ is nilpotent and so $a - z = 0$, because $H(a)$ is a division ring. Hence $a \in Z(H(a))$. Therefore, for any $r \in H(a)$, $0 = d([r, a]) = d(ra - ar) = [r, d(a)]$, that is $d(a) \in Z(H(a))$. ■

LEMMA 3.3. Let R be a domain. For any $x, y \in I$ there exists $m = m(x, y) \geq 1$ such that $C_R([x, y]^m) = \{r \in R : [[x, y]^m, r] = 0\}$ is invariant under derivation d, that is $d(C_R([x, y]^m)) \subseteq C_R([x, y]^m)$.

PROOF. If char. $R > 0$ then, as we said in Remark 1, our assumption about the derivation d implies that for any $x, y \in R$ there exists $m = m(x, y) \geq 1$ such that $d([x, y]^m) = 0$. For any $r \in C_R([x, y]^m)$ we have

$$0 = d([[x, y]^m, r]) = [[x, y]^m, d(r)]$$

that is $d(r) \in C_R([x, y]^m)$.

Now let char. $R = 0$. For any $r \in C_R([x, y]^m)$ one has

$$0 = d([[x, y]^m, r]) = [d([x, y]^m), r] + [[x, y]^m, d(r)].$$

Since by previous lemma $d([x, y]^m) \in Z(H([x, y]^m))$ then $[[x, y]^m, d(r)] = 0$, that is $d(r) \in C_R([x, y]^m)$. ∎

The last step in our proof is the following:

PROPOSITION 3.2. *Let Q be a domain, then R satisfies $S_4(x_1, x_2, x_3, x_4)$.*

PROOF. First we show that for all $x, y \in I$ one has:

$$[[[x, y], d([x, y])]^2, [x, y]] = 0.$$

In fact given $x, y \in I$, by previous lemma there exists an integer $m = m(x, y) \geq 1$ such that $d(C_R([x, y]^m)) \subseteq C_R([x, y]^m)$, and of course we can assume $[x, y] \neq 0$. We denote $A = C_R([x, y]^m)$, therefore $[x, y]^m$ is a non-zero element of $Z(A)$ and $I \cap A$ is a non-zero two-sided ideal of A. By localizing A at $Z(A)$ we obtain a domain D whose center is a field containing $[x, y]^m$, moreover $D = \{rz^{-1} : r \in A, z \in Z(A) - \{0\}\}$. As we said in Remark 2 d extends uniquely to a derivation on D, which we will also denote d and moreover d satisfies our assumption on D with respect to the Lie ideal $\overline{L}$ which is the localization of $[I \cap A, A] \subseteq [I, R] = L$.

Of course $[x, y]$ is invertible in D, therefore by lemma 3.1 and main result in [5], either $d([x, y]) = \alpha[x, y]$, for some $\alpha \in Z(A)$ or D satisfies $S_4(x_1, x_2, x_3, x_4)$.

In the first case $[[x, y], d([x, y])] = 0$ and a fortiori

$$[[[x, y], d([x, y])]^2, [x, y]] = 0.$$

In the second case D is a division algebra of dimension at most 4 over its center. Moreover we know that in this case, for any $a, b \in D$, $[a, b]^2 \in Z(D)$.

This implies $[[x, y], d([x, y])]^2 \in Z(A)$, because $[x, y] \in A \subseteq D$ and $d([x, y]) \in A \subseteq D$.

In particular the following holds

$$\left[[[x, y], d([x, y])]^2, [x, y]\right] = 0 .$$

Therefore, in any case, we have

$$\left[[[x, y], [d(x), y] + [x, d(y)]]^2, [x, y]\right] = 0$$

for all $x, y \in I$.

In other words

$$\phi(x_1, x_2, d(x_1), d(x_2)) = \left[[[x_1, x_2], [d(x_1), x_2] + [x_1, d(x_2)]]^2, [x_1, x_2]\right]$$

is a differential identity for I.

Because any non-zero two-sided ideal of a prime ring R is also a dense (or rational, see [8] page 50) R-submodule of U, then, by [16, theorem 2], $\phi(x_1, x_2, d(x_1), d(x_2))$ is a differential identity for U.

By theorem 1 of [16] (or theorem 2 in [14]) it follows that either d is an inner derivation of U or U satisfies the polynomial identity

$$\phi(z_1, z_2, z_3, z_4) = \left[[[z_1, z_2], [z_3, z_2] + [z_1, z_4]]^2, [z_1, z_2]\right].$$

If d is an inner derivation induced by some $q \in U$ then

$$\left[[[x, y], [q, [x, y]]]^2, [x, y]\right] = 0$$

for all $x, y \in U$.

In particular this one holds in R and so R is a GPI-ring [2], its central closure $S = RC$ is a primitive ring having minimal right ideal, moreover, for any minimal idempotent $e = e^2 \neq 0$, eSe is a division algebra finite dimensional over its center $eCe \cong C$ [9, theorem 1.3.2].

Because $S = RC \subseteq Q$ and Q is a domain then S is a domain and so any idempotent element e of S is trivial.

This implies that S is a division algebra finite dimensional over C, that is R is a PI-ring and C is the quotient field of $Z(R)$.

It follows that $RC = S = Q = U$.

Moreover $RC = S = \overline{R} = \{rz^{-1}: r \in R, z \in Z(R) - \{0\}\}$ by Posner's theorem and so, for any $u \in \overline{L} = \{uz^{-1}: u \in L, z \in Z(R) - \{0\}\}$, there exist integers m, k such that $[d(u^m), u^m]_k = 0$ (see Remark 2).

Because d is the inner derivation in U induced by $q \in U$, we obtain that $q \in H_U(\overline{L})$, that is either $q \in Z(U)$ or U satisfies $S_4(x_1, x_2, x_3, x_4)$. In this last case we are done because $R \subseteq U$. If $q \in Z(U)$ then $d = 0$ in U, and this is a contradiction. Now we have to analize the only case in which $\phi(z_1, z_2, z_3, z_4)$ is a polynomial identity of U. In this case $R \subseteq U$ satisfies the blended component $[[[z_1, z_2], [z_3, z_2]]^2, [z_1, z_2]]$ of the polynomial identity $\phi(z_1, z_2, z_3, z_4)$.

Since R is prime there exists a field F such that R and $M_k(F)$, the ring of all $k \times k$ matrices over F, satisfy the same polynomial identities (see [12]).

Suppose $k \geq 3$. Let e_{ij} the matrix unit with 1 in (i, j) entry and 0 elsewhere.

Let $z_1 = e_{13} + e_{22}$, $z_2 = e_{21} + e_{33}$, $z_3 = e_{32} + e_{31}$. By calculation we obtain

$$[z_3, z_2] = -e_{32}$$

$$[z_1, z_2] = e_{21} - e_{23} + e_{13}$$

$$[[z_1, z_2], [z_3, z_2]] = e_{22} - e_{12} + e_{31} - e_{33}$$

$$([[z_1, z_2], [z_3, z_2]])^2 = e_{22} - e_{12} - e_{32} - e_{31} + e_{33}$$

$$[([[z_1, z_2], [z_3, z_2]])^2, [z_1, z_2]] = e_{12} - e_{31} \neq 0$$

and this is a contradiction. So $k \leq 2$ and R satisfies $S_4(x_1, x_2, x_3, x_4)$. ∎

At this point the proof of our theorem is complete and we state it here again for sake of clearness:

THEOREM 3.1. *Let R be a prime ring with no non-zero nil right ideals, d a non-zero derivation of R, L a non-central Lie ideal of R. If d satisfies $[d(u^m), u^m]_k = 0$ for all $u \in L$, $m = m(u) \geq 1$, $k = k(u) \geq 1$, then R satisfies $S_4(x_1, x_2, x_3, x_4)$.*

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