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Rendiconti del Seminario Matematico della Università di Padova, tome 102 (1999), p. 305-317

<http://www.numdam.org/item?id=RSMUP_1999__102__305_0>
Prime Rings with Hypercommuting Derivations on a Lie Ideal.

V. DE FILIPPIS(*) - O. M. DI VINCENZO(**)(***)

ABSTRACT - Let $R$ be a prime ring with no non-zero nil right ideals, $d$ a non-zero derivation of $R$, $L$ a non-central Lie ideal of $R$. If $d$ satisfies $[d(u^m), u^m]_k = 0$ for all $u \in L$, $m = m(u) \geq 1$, $k = k(u) \geq 1$, then $R$ is an order in a simple algebra of dimension at most 4 over its center.

1. Introduction.

The classical hypercenter theorem, proved by I. N. Herstein [10], asserts that for a ring $R$ not containing non-zero nil two-sided ideals, an inner derivation $d_a$, induced by $a \in R$, satisfying $d_a(x^m) = 0$, $m = m(x) \geq 1$, for all $x \in R$, must vanish identically on the whole ring $R$, i.e. $a \in Z(R)$.

Moreover in [4, theorem 4] Chuang and Lin proved that for a ring $R$ not containing non-zero nil right ideals, the inner derivation $d_a$, induced by $a \in R$, satisfying $[d_a(x^m), x^m]_k = 0$, $m = m(x) \geq 1$, $k = k(x) \geq 1$, for all $x \in R$, must vanish identically on the whole ring $R$.

Later Chuang generalized the results above to arbitrary derivation $d$, defined in a prime ring $R$ not containing non-zero nil right ideals. He proved in [3, corollary 2] that if $[d(x^m), x^m]_k = 0$, $m = m(x) \geq 1$,

(**) Dipartimento di Matematica, Università della Basilicata, Via N. Sauro 85, 85100 Potenza.
(***) Research supported by a grant from M.U.R.S.T.
\( k = k(x) \geq 1 \), for all \( x \in R \), then either \( d \) vanishes identically on the whole ring \( R \) or \( R \) is commutative.

The starting point of this paper is the following result, obtained in [5]:

Let \( R \) be a prime ring, with no non-zero nil right ideals, \( L \) a non-central Lie ideal of \( R \), \( d_a \) an inner derivation induced by \( a \in R \). If \([d_a(u^m), u^m]_k = 0\), for all \( u \in L \), \( m = m(u) \geq 1 \), \( k = k(u) \geq 1 \), then either \( d_a \) vanishes identically on the whole ring \( R \), that is \( a \in Z(R) \), or \( R \) satisfies \( S_4(x_1, x_2, x_3, x_4) \), the standard identity of degree 4.

The last conclusion is well known to be equivalent, by Posner's theorem, to saying that \( R \) is an order in a simple algebra at most 4-dimensional over its center.

Moreover we remark that if \( R = M_2(C) \), the ring of all \( 2 \times 2 \) matrices over a commutative ring \( C \), then for any \( u \in [R, R] \) one has \( u^2 \in Z(R) \) (see example (3) page 12 in [12]), and so our condition \([d(u^m), u^m]_k = 0\) holds for \( m = 2 \) and \( k = 1 \).

The purpose of this note is to generalize the result obtained in [5] to arbitrary derivation \( d \). We will prove the following:

\textbf{Theorem.} Let \( R \) be a prime ring with no non-zero nil right ideal, \( L \) a non-central Lie ideal of \( R \), \( d \) a non-zero derivation of \( R \) satisfying \([d(u^m), u^m]_k = 0\), for all \( u \in L \), \( m = m(u) \geq 1 \), \( k = k(u) \geq 1 \). Then \( R \) satisfies \( S_4(x_1, x_2, x_3, x_4) \), the standard identity of degree 4.

\section{Preliminary.

The proof of our theorem is based upon two results obtained respectively in [3] and [5].

The first one is a result of Chuang [3, proposition 2] concerning a careful analysis of derivations satisfying a particular property on semiprime rings.

The related objects we need to mention are the left Utumi quotient ring \( U \), and also the two-sided Utumi quotient ring \( Q \) of a ring \( R \) (sometimes, as in [1], \( U \) and \( Q \) are called the maximal left ring of quotients and the symmetric ring of quotients respectively).

The definitions, the axiomatic formulations and the properties of these quotient rings \( U, Q \) can be found in [15], [8], [1].

For instance \( U \), the left Utumi quotient ring of \( R \), exists if and only if \( R \) is right faithful, that is for any \( a \in R \), \( Ra = 0 \) implies \( a = 0 \).
In the same way we can define $Q$ if $R$ is both right and left 
faithful.
In any case, when $R$ is a prime ring, all that we need here about these 
objects is that 

1) $R \subseteq Q \subseteq U$. 

2) $U$ and $Q$ are prime rings [8, page 74].

3) For all $q \in Q$ there exists a dense left ideal $M$ of $R$ such that 
$Mq \subseteq R$, moreover if $Mq = 0$, for some dense left ideal $M$ of $R$, then 
$q = 0$. 

4) The center of $U$, denoted by $C$, coincides with the center of $Q$. 

$C$ is a field which is called the extended centroid of $R$ [1, pages 
68-70].
Moreover if $R$ is a prime P.I. ring then, by Posner's theorem [9, theo-
rem 1.4.3 page 40], $C$ is the quotient field of $Z(R)$ and 

$$RZ^{-1} = \{ rz^{-1} : r \in R, z \in Z(R) - \{ 0 \} \} = RC$$

is a simple algebra finite dimensional over its center. 

In this case it is easy to see that $RC = Q = U$. 

Finally we recall that a map $d : R \rightarrow R$ is a derivation if, for any $x, y \in 
R$, $d(x + y) = d(x) + d(y)$ and $d(xy) = d(x)y + xd(y)$. Each derivation of 
a prime ring $R$ can be uniquely extended to a derivation of its Utumi quo-
tient ring $U$ and thus all derivations of $R$ will be implicitly assumed to be 
defined on the whole $U$ (see [15, page 101] or [16, lemma 2]).

Now we are ready to state the result of Chuang ([3, proposition 1, 
page 46]) for prime rings. In this case as we said above $U$ and $Q$ are 
prime too and so any central idempotent is trivial. 

Hence, for any $a \in U - \{ 0 \}$, the norm $\|a\|$ of $a$, defined in [3, page 39, 
(7)], is always 1 and of course $\|0\| = 0$.

**Proposition 2.1.** Let $R$ be a prime ring with extended centroid $C$. 

Let $d$ a derivation of $U$, the left Utumi quotient of $R$, satisfying 
$d(a)(1 + a)^{-1} \in C$, for any $a \in R$, with $a^2 = 0$. Let $Q$ be the two-sided Utu-
mi quotient ring of $R$. Then 

either the ring $Q$ is reduced, that is $Q$ does not have any non-zero 
ilpotent element 
or $Q = U = M_2(C)$, the ring of all two by two matrices over $C$ 
or the derivation $d$ is the inner derivation induced by a square-zero
element \( c \in U \), satisfying the property that, for any \( x, y \in Q \) with \( xy = 0 \), we also have \( xc = cy = 0 \).

The second result concerns the generalized hypercentralizer of a non-central Lie ideal of \( R \).

More precisely the generalized hypercentralizer of an arbitrary subset \( S \) of \( R \) is the following subring of \( R \):

\[
H_R(S) = \{ a \in R : \text{ for all } s \in S \text{ there exist } n = n(a, s) \geq 1, \ k = k(a, s) \geq 1 \text{ such that } [a, s^n]_k = 0 \}.
\]

In [5] we proved that if \( L \) is a non-central Lie ideal of a prime ring \( R \) with no non-zero nil right ideal, then either \( H_R(L) = Z(R) \) or \( R \) satisfies \( S_4(x_1, x_2, x_3, x_4) \).

Finally we remark that an important tool in our proof will be the theory of differential identities initiated by Kharchenko [13].

3. Proof of the theorem.

Throught this paper we will use the following notation:

\( R \) will always be an associative prime ring, with no non-zero nil right ideal, \( L \) will be a non-central Lie ideal of \( R \) and \( d \) will be a derivation of \( R \) satisfying \([d(u^m), u^m]_k = 0\), for any \( u \in L \), \( m = m(u) \geq 1 \), \( k = k(u) \geq 1 \).

\( U \) will be the left Utumi quotient ring of \( R \), and \( Q \) will be the two-sided Utumi quotient ring of \( R \).

We start with an easy remark:

\textbf{Remark 1.} If \( R \) has characteristic \( p \neq 0 \) then for all \( u \in L \) there exists \( n = n(u) \geq 1 \) such that \( d(u^n) = 0 \).

\textbf{Proof.} Let \( u \in L \) be arbitrarily given. There exist \( m = m(u) \geq 1 \), \( k = k(u) \geq 1 \) such that \([d(u^m), u^m]_k = 0\). Pick an integer \( t \geq 1 \) such that \( p^t \geq k \). Then

\[ 0 = [[d(u^m), u^m]_k, u^m]_{p^t-k} = [d(u^m), u^m]_{p^t}. \]

Since \( R \) is of characteristic \( p > 0 \), we have

\[ 0 = [d(u^m), u^m]_{p^t} = [d(u^m), u^{mp^t}] \]
and this implies immediately $[d(u^{mp^l}), u^{mp^l}] = 0$. Let $a = u^{mp^l}$. We obtain $[d(a), a] = 0$.

Using the commutativity we have $d(a^p) = pa^{p-1}d(a) = 0$, that is

$$d(u^{mp^l+1}) = 0$$

and so we have shown that for all $u \in L$ there exists $l = l(u) \geq 1$ such that $d(u^l) = 0$. ■

Notice that in this case, if $p \neq 2$, then our result follows immediately by theorem 2 of [7].

Now we make some other reductions.

It is well known that if $L$ is a non-central Lie ideal of a prime ring $R$ then either $R$ satisfies $S_4(x_1, x_2, x_3, x_4)$ or there exists a non-zero two-sided ideal $I$ of $R$ such that $[I, R] \subseteq L$ and $[I, R] \notin Z(R)$.

Therefore we will assume, in all that follows, that $L = [I, R]$ for some non-zero two-sided ideal $I$ of $R$ (see for instance Lemma 2 and Proposition 1 in [6]).

In this case $L$ is invariant under any inner automorphism induced by an invertible (or quasi-invertible) element of $R$.

Moreover, if $Z(R) \neq 0$, then we can consider

$$\overline{R} = \{rz^{-1} : r \in R, z \in Z(R) - \{0\}\}$$

$$\overline{L} = \{uz^{-1} : u \in L, z \in Z(R) - \{0\}\}$$

which are the localizations at $Z(R)$ of $R$ and $L$ respectively. Since $L = [I, R]$ is a $Z(R)$-submodule of $R$, we have

**Remark 2.**

1) $\overline{L}$ is a non-central Lie ideal of the prime ring $\overline{R}$

2) the derivation $d$ extends uniquely to a derivation on $\overline{R}$ as follows

$$d(rz^{-1}) = (d(r)z - rd(z))z^{-2}$$

3) the derivation $d$, defined on $\overline{R}$, satisfies our assumptions on $\overline{L}$, that is for any $\overline{u} \in \overline{L}$, there exist $n = n(\overline{u}) \geq 1$, $k = k(\overline{u}) \geq 1$ such that $[d(\overline{u}^n), \overline{u}^n]_k = 0$.

**Lemma 3.1.** Let $a \in R$. If $a$ is invertible then $d(a)a^{-1} \in H_R(L)$, if $a$ is quasi-invertible then $d(a)(1 + a)^{-1} \in H_R(L)$. 
PROOF. First we assume that \( a \) is invertible, then, as we said above, 
\[ aLa^{-1} = L, \] hence for any \( u \in L \) there exist \( m, k, n, h \geq 1 \) such that 
\[ 0 = \left[ d((a^{-1}ua)^m), (a^{-1}ua)^n \right]_k = 0 \]
and 
\[ [d(u^n), u^n]_h = 0. \]
Hence for \( s = nm \) and \( t = \max\{h, k\} \) we also have:
\[ \left[ d((a^{-1}ua)^s), (a^{-1}ua)^s \right]_t = 0 = [d(u^s), u^s]_t. \]
It follows that:
\[ 0 = [d(a^{-1}u^s a), a^{-1}u^s a]_t = \]
\[ = [d(a^{-1}) u^s a + a^{-1} d(u^s) a + a^{-1} u^s d(a), a^{-1} u^s a]_t = \]
\[ = [-a^{-1} d(a) a^{-1} u^s a + a^{-1} u^s d(a), a^{-1} u^s a]_t + [a^{-1} d(u^s) a, a^{-1} u^s a]_t = \]
\[ = -a^{-1} [d(a) a^{-1} u^s - u^s d(a) a^{-1} u^s] a = -a^{-1} [d(a) a^{-1} u^s]_t a. \]
Hence \([d(a) a^{-1}, u^s]_t + 1 = 0\) that is \( d(a) a^{-1} \in H_R(L) \).

A similar proof holds if \( a \) is a quasi-invertible element of \( R \).

We remark that any square-zero element \( a \) of \( R \) is quasi-invertible with quasi-inverse \(-a\).

Therefore, by [5], either \( R \) satisfies \( S_4(x_1, x_2, x_3, x_4) \) and we are done or 
\( d(a) a^{-1} \in Z(R) \), that is the derivation \( d \) satisfies the hypothesis of Chuang's result. In this case one of the three conclusions of the Proposition 2.1 must hold.

Now we treat each case separately.

Of course if \( U = M_2(C) \), the ring of all \( 2 \times 2 \) matrices over \( C \), then it satisfies the standard identity \( S_4(x_1, x_2, x_3, x_4) \) and we are done again, since \( R \subseteq U \).

In the second case we have:

**Proposition 3.1.** If the derivation \( d \) is the inner derivation defined by a square-zero element \( c \) in \( U \), satisfying \( xc = cy = 0 \) for any \( x, y \in Q \), with \( xy = 0 \), then \( d \) vanishes identically on \( R \).
PROOF. Since $c$ is an element of the left Utumi quotient ring of $R$, there exists a left dense ideal $M$ of $R$ such that $Mc \subseteq R$ (see proposition 2.1.7 in [1]).

Moreover, since $R$ is a prime ring, $IM$ is again a left dense ideal of $R$ and, of course, $IMc \subseteq IR \subseteq I$.

In other words we can assume that there exists $M$ left dense ideal of $R$ such that $Mc \subseteq I$ and so $[Mc, Mc] \subseteq L$. Therefore for any $x, y \in M$ there exist $m = m(c, x, y) \geq 1$, $k = k(c, x, y) \geq 1$ such that

$$[d([xc, yc]^m), [xc, yc]^m]_k = 0.$$ 

Moreover $d([xc, yc]^m) = [c, [xc, yc]^m] = c[xc, yc]^m$. Therefore

$$0 = [c[xc, yc]^m, [xc, yc]^m]_k =$$

$$= \sum_{k = 0}^{k} \binom{k}{h} (-1)^h [xc, yc]^m(h)c[xc, yc]^m)([xc, yc]^m(k-h)) = c[xc, yc]^m(k+1).$$

Thus $[xc, yc][xc, yc]^m(k+1) = 0$, that is $[xc, yc]$ is a nilpotent quasi-invertible element. By lemma 3.1 and the main theorem in [5], either $R$ satisfies $S_4(x_1, x_2, x_3, x_4)$ or $d([xc, yc]) = \alpha(1 + [xc, yc])$, where $\alpha \in e \in Z(R)$.

In the first case $R$ is a prime PI ring and so, by Posner's theorem, $RC = S = Q = U$ is a central simple algebra finite dimensional over its center $C$.

Since $U$ satisfies $S_4(x_1, x_2, x_3, x_4)$, if $c$ is a non-zero square-zero element, then we have $U = M_2(C)$, the ring of $2 \times 2$ matrices over $C$.

Since $xc = cy = 0$, for any $x, y \in Q = U$ such that $xy = 0$, then $e_{11}c = e_{21}c = e_{22}c = e_{12}c = 0$, that is $M_2(C)c = 0$ and so $c = 0$, a contradiction.

In the second case we know that $d([xc, yc]) = [c, [xc, yc]] = c[xc, yc]$.

So $\alpha^2(1 + [xc, yc])^2 = (\alpha(1 + [xc, yc]))^2 = (d([xc, yc]))^2 = (c[xc, yc])^2 = 0$.

Since $[xc, yc]$ is quasi-invertible, then $\alpha = 0$. Thus $0 = d([xc, yc]) = c[xc, yc]$. Hence, for any $x, y$ in $M$, $c[xc, yc] = 0$.

Let $x, y, z$ be in $R$, $t$ in $M$. Since $M$ is a left dense ideal of $R$, we have that $xt, yt, zt$ fall in $M$ and so $ztc[xtc, ytc] = 0$, that is $R$ is GPI [2].

In this case, by Martindale’s result the central closure $S = RC$ is a primitive ring, containing a minimal right ideal $eS$, such that $eSe$ is a division algebra finite dimensional over $C$, for any minimal idempotent $e$ of $S$ [9, theorem 1.3.2].
If \( e = 1 \) then \( S \) is a finite dimensional division algebra over \( C \). Therefore \( S \) is PI and so \( R \) is PI too. As we said above in this case \( S = Q = U \) and so \( c \in S \) which is a division ring. Hence \( c = 0 \) and consequently \( d = d_c = 0 \).

Now we may suppose \( e \neq 1 \). We known that \( xc = cy = 0 \), for any \( x, y \) in \( Q \), with \( xy = 0 \).

Let \( x^2 = 0 \). Since \( xc = cx = 0 \) then \( d(x) = [c, x] = 0 \), that is \( c \) commutes with every square-zero element \( x \) in \( Q \).

Let \( A \) be the subring generated by the elements of square zero. \( A \) is invariant under all automorphisms of \( Q \). By our assumption there are non-trivial idempotent in the prime ring \( Q \) and so \( A \) contains a non-zero ideal \( J \) of \( Q \) by [11].

Now, since \( 0 = d(A) \supset d(J) \supset d(JSQ) = JD(Q) \), by the primeness of \( Q \) we obtain \( d(Q) = 0 \), that is \( d = 0 \) in \( Q \) and so in \( R \) too. ■

REMARK. The last case is the one in which \( Q \) is a reduced ring. Since \( Q \) is also a prime ring then it must be a domain. In fact, let \( x, y \in Q \) be such that \( xy = 0 \) and \( y \neq 0 \). Then, for any \( z \in Q \), we have \( (yzx)^2 = yzxyzx = 0 \) and so \( yzx = 0 \), that is \( yQx = 0 \) and \( x = 0 \) because \( Q \) is prime.

DEFINITION. For \( a \in R \) let

\[ H(a) = \{ r \in R : [r, a]_m = 0 \text{ for some integer } m = m(r) \geq 1 \}. \]

Of course \( H(a) \) is a subring of \( R \).

We also have:

LEMMA 3.2. Let \( R \) be a domain of characteristic zero and let \( d \) be the derivation satisfying our assumption. If \( a \) is an element of \( I \) such that

\[ [d(a), a]_l = 0 \text{ for some } l = l(a) \geq 1 \]

then \( H(a) \) is invariant under the derivation \( d \) and moreover \( d(a) \) is in the center of \( H(a) \).

PROOF. By localizing at non-zero integers we may assume that \( R \) is an algebra over the field of the rational numbers.

By [3, assertion 2] it follows that \( H(a) \) is invariant under \( d \). Now, we put \( \delta = d_a \), the inner derivation induced by \( a \).

Of course the derivation \( \delta \) restricted to \( H(a) \), which we also denote \( \delta \),
is nil and hence for any integer $\lambda$, the derivation $\lambda \delta$ is also nil on $H(a)$.

Since $H(a)$ is an algebra over the field of the rational numbers, the map $\exp(\lambda)$ is an automorphism of $H(a)$ (see [3, proposition 2]), hence the map $d_\lambda = \exp(\lambda \delta) \cdot d \cdot \exp(-\lambda \delta)$ is a derivation of $H(a)$.

Obviously $I \cap H(a)$ is a two-sided ideal of $H(a)$ which is invariant under the action of $\exp(\lambda \delta)$.

Hence $L_1 = [I \cap H(a), H(a)] \subseteq [I, R] = L$ is a Lie ideal of $H(a)$, moreover, for any $u \in L_1$, there exist some integers $n = n(u) \geq 1$, $k = k(u) \geq 1$ such that $[d_\lambda(u^n), u^n]_k = 0$.

Now, given $u \in L_1$, there exist integers $n = n(u) \geq 1$, $m = m(u) \geq 1$, $k = k(u) \geq 1$, $h = h(u) \geq 1$ such that

$$[d(u^m), u^m]_h = 0 = [d_\lambda(u^n), u^n]_k$$

hence, as in the proof of lemma 3.1, for $s = nm$ and $t = \max \{h, k\}$ we also have

$$[d(u^s), u^s]_t = [d_\lambda(u^s), u^s]_t$$

that is $[(d_\lambda - d)(u^s), u^s]_t = 0$.

By [3, proposition 2, (3)] the derivation $d_\lambda - d$ is the inner derivation induced by the element $b_\lambda = \sum_{n \geq 1} (\delta(\lambda a), \lambda a)_n^{-1}/n!$, and so $b_\lambda$ is in the generalized hypercentralizer of $L_1$ in $H(a)$.

If $I \cap H(a)$ is the zero ideal of $H(a)$, then $a = 0$ since it is in $I \cap H(a)$ and of course $d(a) = 0 \in Z(H(a))$.

If $I \cap H(a)$ is non-zero then, by [5, proposition 4.1], either $H_{H(a)}(L_1) = Z(H(a))$ or $H(a)$ satisfies $S_4(x_1, x_2, x_3, x_4)$.

In the first case we may conclude, by a Vandermonde determinant argument, $d(a) \in Z(H(a))$.

In the other case, by localizing at the center of $H(a)$, we may assume that $H(a)$ is a division algebra of dimension at most 4 over its center $Z(H(a))$. It follows that there exists $m \geq 1$ such that $\delta^m(r) = 0$, for any $r \in H(a)$, that is $\delta$ is a nil of bounded index on $H(a)$.

By [9, lemma 1.1.9] there exists $z \in Z(H(a))$ such that $a - z$ is nilpotent and so $a - z = 0$, because $H(a)$ is a division ring. Hence $a \in Z(H(a))$. Therefore, for any $r \in H(a)$, $0 = d([r, a]) = d(ra - ar) = [r, d(a)]$, that is $d(a) \in Z(H(a))$. ■
Lemma 3.3. Let $R$ be a domain. For any $x, y \in I$ there exists $m = m(x, y) \geq 1$ such that $C_R(\{x, y\}^m) = \{r \in R : r(x, y)^m \neq 0\}$ is invariant under derivation $d$, that is $d(C_R(\{x, y\}^m)) \subset C_R(\{x, y\}^m)$.

Proof. If $\text{char. } R > 0$ then, as we said in Remark 1, our assumption about the derivation $d$ implies that for any $x, y \in R$ there exists $m = m(x, y) \geq 1$ such that $d([x, y]^m) = 0$. For any $r \in C_R(\{x, y\}^m)$ we have

$$0 = d([x, y]^m, r) = [x, y]^m, d(r)$$

that is $d(r) \in C_R(\{x, y\}^m)$.

Now let $\text{char. } R = 0$. For any $r \in C_R(\{x, y\}^m)$ one has

$$0 = d([x, y]^m, r) = [d([x, y]^m), r] + [x, y]^m, d(r)].$$

Since by previous lemma $d([x, y]^m) \in Z(H([x, y]^m))$ then $[x, y]^m, d(r)] = 0$, that is $d(r) \in C_R(\{x, y\}^m)$.

The last step in our proof is the following:

Proposition 3.2. Let $Q$ be a domain, then $R$ satisfies $S_4(x_1, x_2, x_3, x_4)$.

Proof. First we show that for all $x, y \in I$ one has:

$$[[x, y], d([x, y])]^2, [x, y] = 0.$$  

In fact given $x, y \in I$, by previous lemma there exists an integer $m = m(x, y) \geq 1$ such that $d(C_R(\{x, y\}^m)) \subset C_R(\{x, y\}^m)$, and of course we can assume $[x, y] \neq 0$. We denote $A = C_R(\{x, y\}^m)$, therefore $[x, y]^m$ is a non-zero element of $Z(A)$ and $I \cap A$ is a non-zero two-sided ideal of $A$. By localizing $A$ at $Z(A)$ we obtain a domain $D$ whose center is a field containing $[x, y]^m$, moreover $D = \{rz^{-1} : r \in A, z \in Z(A) - \{0\}\}$. As we said in Remark 2 $d$ extends uniquely to a derivation on $D$, which we will also denote $d$ and moreover $d$ satisfies our assumption on $D$ with respect to the Lie ideal $L$ which is the localization of $[I \cap A, A] \subset [I, R] = L$.

Of course $[x, y]$ is invertible in $D$, therefore by lemma 3.1 and main result in [5], either $d([x, y]) = \alpha \{x, y\}$, for some $\alpha \in Z(A)$ or $D$ satisfies $S_4(x_1, x_2, x_3, x_4)$.

In the first case $[[[x, y], d([x, y])] = 0$ and a fortiori

$$[[[x, y], d([x, y])]^2, [x, y] = 0.$$
In the second case $D$ is a division algebra of dimension at most 4 over its center. Moreover we know that in this case, for any $a, b \in D$, $[a, b]^2 \in Z(D)$.

This implies $[[x, y], d([x, y])]^2 \in Z(A)$, because $[x, y] \in A \subseteq D$ and $d([x, y]) \in A \subseteq D$.

In particular the following holds

$$[[[x, y], d([x, y])]^2, [x, y]] = 0.$$ 

Therefore, in any case, we have

$$[[[x, y], [d(x), y] + [x, d(y)]]^2, [x, y]] = 0$$

for all $x, y \in I$.

In other words

$$\phi(x_1, x_2, d(x_1), d(x_2)) = [[[x_1, x_2], [d(x_1), x_2] + [x_1, d(x_2)]]^2, [x_1, x_2]]$$

is a differential identity for $I$.

Because any non-zero two-sided ideal of a prime ring $R$ is also a dense (or rational, see [8] page 50) $R$-submodule of $U$, then, by [16, theorem 2], $\phi(x_1, x_2, d(x_1), d(x_2))$ is a differential identity for $U$.

By theorem 1 of [16] (or theorem 2 in [14]) it follows that either $d$ is an inner derivation of $U$ or $U$ satisfies the polynomial identity

$$\phi(z_1, z_2, z_3, z_4) = [[[z_1, z_2], [z_3, z_2] + [z_1, z_4]]^2, [z_1, z_2]].$$

If $d$ is an inner derivation induced by some $q \in U$ then

$$[[[x, y], [q, [x, y]]]^2, [x, y]] = 0$$

for all $x, y \in U$.

In particular this one holds in $R$ and so $R$ is a GPI-ring [2], its central closure $S = RC$ is a primitive ring having minimal right ideal, moreover, for any minimal idempotent $e = e^2 \neq 0$, $eSe$ is a division algebra finite dimensional over its center $eCe \cong C$ [9, theorem 1.3.2].

Because $S = RC \subseteq Q$ and $Q$ is a domain then $S$ is a domain and so any idempotent element $e$ of $S$ is trivial.

This implies that $S$ is a division algebra finite dimensional over $C$, that is $R$ is a PI-ring and $C$ is the quotient field of $Z(R)$.

It follows that $RC = S = Q = U$. 
Moreover $RC = S = \overline{R} = \{ rz^{-1} : r \in R, z \in Z(R) - \{0\} \}$ by Posner’s theorem and so, for any $u \in \overline{L} = \{ uz^{-1} : u \in L, z \in Z(R) - \{0\} \}$, there exist integers $m, k$ such that $[d(u^m), u^m]_k = 0$ (see Remark 2).

Because $d$ is the inner derivation in $U$ induced by $q \in U$, we obtain that $q \in H_U(L)$, that is either $q \in Z(U)$ or $U$ satisfies $S_4(x_1, x_2, x_3, x_4)$. In this last case we are done because $R \subseteq U$. If $q \in Z(U)$ then $d = 0$ in $U$, and this is a contradiction. Now we have to analyze the only case in which $\phi(z_1, z_2, z_3, z_4)$ is a polynomial identity of $U$. In this case $R \subseteq U$ satisfies the blended component $\left( [[z_1, z_2], [z_3, z_2]]^2, [z_1, z_2] \right)$ of the polynomial identity $\phi(z_1, z_2, z_3, z_4)$.

Since $R$ is prime there exists a field $F$ such that $R$ and $M_k(F)$, the ring of all $k \times k$ matrices over $F$, satisfy the same polynomial identities (see [12]).

Suppose $k \geq 3$. Let $e_{ij}$ the matrix unit with 1 in $(i, j)$ entry and 0 elsewhere.

Let $z_1 = e_{13} + e_{22}$, $z_2 = e_{21} + e_{33}$, $z_3 = e_{32} + e_{31}$. By calculation we obtain

$$[z_3, z_2] = - e_{32}$$

$$[z_1, z_2] = e_{21} - e_{23} + e_{13}$$

$$[[z_1, z_2], [z_3, z_2]] = e_{22} - e_{12} + e_{31} - e_{33}$$

$$([[z_1, z_2], [z_3, z_2]]^2 = e_{22} - e_{12} - e_{32} - e_{31} + e_{33}$$

$$[[[[z_1, z_2], [z_3, z_2]]^2, [z_1, z_2]] = e_{12} - e_{31} \neq 0$$

and this is a contradiction. So $k \leq 2$ and $R$ satisfies $S_4(x_1, x_2, x_3, x_4)$. ■

At this point the proof of our theorem is complete and we state it here again for sake of clearness:

**Theorem 3.1.** Let $R$ be a prime ring with no non-zero nil right ideals, $d$ a non-zero derivation of $R$, $L$ a non-central Lie ideal of $R$. If $d$ satisfies $[d(u^m), u^m]_k = 0$ for all $u \in L$, $m = m(u) \geq 1$, $k = k(u) \geq 1$, then $R$ satisfies $S_4(x_1, x_2, x_3, x_4)$. 
REFERENCES
