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# Asymptotic behavior of solutions of Schrödinger inequalities on unbounded domains of nilpotent Lie groups

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# Asymptotic Behavior of Solutions of Schrödinger Inequalities on Unbounded Domains of Nilpotent Lie Groups.

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## 1. Introduction.

The aim of this note is to present a technique which allows to find asymptotic behavior at infinity, for solutions of a wide class of equations.

Let $\mathcal{G} = \bigoplus_{j=1}^{m} \mathcal{G}_j$ be a stratified nilpotent Lie algebra of vector fields and $H = (\mathbb{R}^N, \circ)$ be its associated homogeneous Lie group. Let $\{X_1, \ldots, X_n\}$ be a basis of $\mathcal{G}_1$ and $\mathcal{L}$ be the differential operator

$$\mathcal{L} = \sum_{j=1}^{n} X_j^2 .$$

Moreover we denote by $S_{\text{loc}}$ the $\mathcal{L}$-natural local Sobolev space. We shall assume that $H$ has homogeneous dimension $Q \geq 3$. We work in this setting since we need the existence of a fundamental solution of $-\mathcal{L}$ of the type $\Gamma \sim cd^{2-Q}$, where $d$ is the natural «distance» on $H$. We refer to section 2 for more precise definitions and additional notation.

The simplest example of this kind of operators is the classical Laplacian $\mathcal{L} = \Delta$ on $H = (\mathbb{R}^N, +)$, for $N = Q \geq 3$. The simplest non-abelian example is the Kohn Laplacian $\mathcal{L} = \Delta_{\mathbb{H}^k}$ on the Heisenberg group $H = \mathbb{H}^k = (\mathbb{R}^{2k+1}, \circ)$, with homogeneous dimension $Q = 2k + 2$.

We consider a nonnegative weak solution $u \in S_{\text{loc}}(\Omega)$ of the Schrödinger-type inequality

$$-\mathcal{L}u \leq Vu \tag{1.1}$$

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in an unbounded domain $\Omega$ and we obtain an asymptotic behavior of $u$ at infinity, starting from its $L^p$ properties. The potential $V$ will be supposed to belong to the space $L^q(\Omega)$ for all $q$ in a neighborhood of $Q/2$, i.e. to the set

$$(1.2)\qquad L^{]Q/2[}(\Omega) := \{v \in L^{Q/2}(\Omega) \mid \exists q_1, q_2 : q_1 < \frac{Q}{2} < q_2, \quad v \in L^{q_1}(\Omega) \cap L^{q_2}(\Omega)\}.$$

Our technique relies on the use of some remarkable representation formulas on the $d$-balls and is inspired to a work of Simader [S] and one of Citti-Garofalo-Lanconelli [CGL].

We first consider inequality (1.1) in an exterior domain $\Omega$ (i.e. $\Omega = H \setminus F$ with $F$ a compact subset of $H$). The following theorem is the main result of this note.

THEOREM 1.1. *Let $\Omega$ be an exterior domain of $H$ and let $V \in L^{]Q/2[}(\Omega)$. If $u \in S_{\mathrm{loc}}(\Omega)$ is a nonnegative weak solution of*

$$-\mathcal{L}u \leq Vu \quad \text{in } \Omega$$

*such that $u \in L^p(\Omega)$ for a $p \in [Q/(Q-2), +\infty[$, then*

$$u(\xi) = O\left(\frac{1}{d(\xi)^{Q/p}}\right), \qquad \text{as } d(\xi) \to +\infty .$$

*If moreover there exists $p_1 \in [1, Q/(Q-2)[$ such that $u \in L^{p_1}(\Omega) \cap L^{Q/(Q-2)}(\Omega)$, then*

$$u(\xi) = O\left(\frac{1}{d(\xi)^{Q/p_1}}\right), \qquad \text{as } d(\xi) \to +\infty .$$

REMARK 1.2. Theorem 1.1 holds true also removing the hypothesis on the nonnegativity of $u$ if we replace the inequality $-\mathcal{L}u \leq Vu$ with $|\mathcal{L}u| \leq |Vu|$. In particular the Theorem holds for the equations $-\mathcal{L}u = Vu$, with $u$ that may change sign.

We emphasize that in Theorem 1.1 no assumption is made about the boundary values of $u$. We also remark that $1/d^{Q/p} \in L^p_{\mathrm{weak}}(H)$; hence our result can be considered optimal.

In Theorem 1.1 we deal with exterior domains since we need to write representation formulas on $d$-balls and allow to go to infinity both the center and the radius of the balls. This limitation can be overcome if we are able to find an auxiliary function $w \geq u$ in $\Omega$ of the type $w = \Gamma * f$, with $f \leq |V| u$ in $H$. This idea allows to get asymptotic behavior for solutions of Dirichlet problems related to (1.1) on arbitrary unbounded domains. In particular we derive the following theorem.

THEOREM 1.3. *Let $\Omega$ be an (arbitrary) unbounded open subset of $H$ and let $V \in L^{]Q/2[}(\Omega)$. If $u$ is a nonnegative classical solution of*

$$\begin{cases} -\mathcal{L}u \leq Vu & \text{in } \Omega \\ u = 0 & \text{in } \partial\Omega \\ u(\xi) \to 0 & \text{as } d(\xi) \to +\infty \end{cases}$$

*then*

$$u(\xi) = O\left(\frac{1}{d(\xi)^s}\right) \quad \text{as } d(\xi) \to +\infty, \quad \forall s < Q-2 .$$

REMARK 1.4. Theorem 1.3 holds true also removing the hypothesis on the nonnegativity of $u$ if we replace the inequality $-\mathcal{L}u \leq Vu$ with $|\mathcal{L}u| \leq |Vu|$. In particular the Theorem holds for the equations $-\mathcal{L}u = Vu$, with $u$ that may change sign.

We remark that in Theorem 1.3 we do not assume any a priori summability of $u$. We also remark that the «limit» behavior $O(1/d(\xi)^{Q-2})$ is the one of the fundamental solution of $-\mathcal{L}$.

The results of this note can be applied to semilinear equations of the type $-\mathcal{L}u = u^q$, whenever we know that $u$ belongs to the suitable $L^p$ spaces. Indeed if $u$ is a weak solution in a (global) Sobolev space, say $u \in S_0^{1,2}$ ($\hookrightarrow L^p$ for every $p \in [2, 2Q/(Q-2)]$), both the condition on $u$ and on the potential $V = u^{q-1}$ could be automatically satisfied. For example, if $1 + 4/Q < q < (Q+2)/(Q-2)$, then we immediately obtain that $V = u^{q-1}$ belongs to the class $L^{]Q/2[}$.

Moreover our results can be used as a starting point in order to obtain nonexistence theorems in unbounded domains. An example of application is given in [LU1] where we find asymptotic behavior of nonnegative weak solutions to the critical semilinear Dirichlet problem

$$\begin{cases} -\Delta_{\mathbb{H}^k} u = u^{(Q+2)/(Q-2)} & \text{in } \Omega \\ u \in S_0^1(\Omega) \end{cases} \tag{1.3}$$

and we also prove some related nonexistence results (see also [U1]). We stress that the Sobolev space $S_0^1$ considered in [LU1]-[U1] is not embedded in $L^2$; hence a solution $u$ of (1.3) belongs a priori to $L^p$ only for $p = 2Q/(Q-2)$. Actually in [LU1] we prove that such a solution belongs to $L^p$ for every $p \in ]Q/(Q-2), +\infty]$ (and so $V = u^{(Q+2)/(Q-2)-1} \in L^{]Q/2[}$) but this result is highly nontrivial, in particular for $Q/(Q-2) < p < 2Q/(Q-2)$.

Other examples of application to nonexistence results for semilinear equations on the Heisenberg group will be given in the forthcoming papers [LU2] and [U2]. In this respect we point out that Liouville-type theorems for semilinear $\Delta_{\mathbb{H}^k}$-inequalities on some unbounded domains have been recently proved by Birindelli-Capuzzo Dolcetta-Cutrì [BCC]. Finally we quote that asymptotic behavior for capacitary problems on exterior domains of groups of Heisenberg type has been treated by Danielli-Garofalo [DG].

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## 2. Notation and definitions.

Let $(\mathcal{G}, [.])$ be a stratified nilpotent real Lie algebra and let $(H, \circ)$ be its simply connected associated Lie group. We can identify $H$ with $\mathbb{R}^N$, for a suitable $N \in \mathbb{N}$, and $(\mathcal{G}, [.])$ with the Lie algebra of $\circ$-left-invariant vector fields on $\mathbb{R}^N$, with the usual Lie bracket law $[X, Y] = XY - YX$. We denote by $\{\delta_\lambda\}_{\lambda > 0}$ the group of dilations naturally associated to $H$ and by $Q$ the homogeneous dimension of $H$. Let $\mathcal{G} = \bigoplus_{j=1}^{m} \mathcal{G}_j$ be the stratification of $\mathcal{G}$ and let $\{X_1, \dots, X_n\}$ be a basis of $\mathcal{G}_1$. We shall deal with the differential operator

$$\mathcal{L} = \sum_{j=1}^{n} X_j^2 .$$

We recall that $X_1, \dots, X_n$ generate the whole $\mathcal{G}$ by the assumption $\mathcal{G}$ stratified. Moreover, if we set

$$\nabla_{\mathcal{L}} = (X_1, \dots, X_n)$$

then $\nabla_{\mathcal{L}}$ and $\mathcal{L}$ are homogeneous, w.r.t. the dilations $\delta_\lambda$, of degree one and of degree two respectively.

Throughout the paper we will make the assumption

$$Q \geqslant 3$$

on the homogeneous dimension of $H$. We remark that, if $Q \leqslant 3$, then it is necessarily $\mathcal{G} = \mathcal{G}_1$, $(H, \circ)$ is simply $(\mathbb{R}^Q, +)$ and $\mathcal{L}$ is the Laplace operator $\Delta$ on $\mathbb{R}^Q$. If $Q \geqslant 3$, there exists a homogeneous norm $|\cdot|_{\mathcal{L}}$ on $H$ such that, setting

$$d_\xi(\eta) = |\eta^{-1} \circ \xi|_{\mathcal{L}},$$

a fundamental solution of $-\mathcal{L}$ with pole at $\xi$ is given by

$$\Gamma_\xi = \frac{c_Q}{d_\xi^{Q-2}} \tag{2.1}$$

where $c_Q$ is a suitable positive constant depending only on $Q$ (see [G]). We set

$$d(\xi, \eta) = d_\xi(\eta), \qquad \Gamma(\xi, \eta) = \Gamma_\xi(\eta).$$

Moreover we will often write $d(\eta)$ instead of $d_0(\eta)$ and $\Gamma(\eta)$ instead of $\Gamma_0(\eta)$. We recall that a homogeneous norm on $H$ is a function $|\cdot| : H \to$ $\to [0, +\infty[$ such that $|\cdot| \in C^\infty(H \setminus \{0\}) \cap C(H)$, $|\xi| = 0$ iff $\xi = 0$, $|\xi| =$ $= |\xi^{-1}|$ $(= |-\xi|)$ and

$$|\delta_\lambda \xi| = \lambda |\xi|. \tag{2.2}$$

Moreover any homogeneous norm satisfies the following triangle inequality

$$|\xi \circ \eta| \leqslant c(|\xi| + |\eta|)$$

for a suitable $c \geqslant 1$. Hence there exists $\beta \geqslant 1$ such that

$$d(\xi, \eta) \leqslant \beta(d(\xi, \zeta) + d(\zeta, \eta)) \tag{2.3}$$

for every $\xi, \eta, \zeta \in H$. Moreover, since $|\xi|_{\mathcal{L}} = |\xi^{-1}|_{\mathcal{L}}$, it is also

$$d(\xi, \eta) = d(\eta, \xi).$$

We denote the $d$-balls on $H$ by

$$B_d(\xi, r) = \{\eta \in H \,|\, d(\xi, \eta) < r\}.$$

Since the Lebesgue measure is a Haar measure on $H$ we have

$$|B_d(\xi, r)| = |B_d(0, r)| = r^Q |B_d(0, 1)|.$$

Moreover on $H$ the following polar coordinates formula holds:

$$\int_H f(d(\xi))\, d\xi = Q|B_d(0, 1)| \int_0^{+\infty} f(\varrho)\, \varrho^{Q-1} d\varrho .$$

In particular it follows that, for every $s \in ]0, Q[$,

$$(2.4) \qquad \frac{1}{d_\xi^s} \in L^p(B_d(\xi, 1)) \cap L^q(H \backslash B_d(\xi, 1)), \quad \text{for } 1 \leq p < \frac{Q}{s} < q \leq +\infty.$$

We also remark that

$$(2.5) \qquad |\nabla_{\mathcal{L}} d_0| \in L^\infty(H)$$

since $\nabla_{\mathcal{L}} d_0$ is homogeneous of degree zero w.r.t. the dilations $\delta_\lambda$ and then

$$\sup_{\xi \neq 0} |(\nabla_{\mathcal{L}} d_0)(\xi)| = \sup_{\xi \neq 0} |(\nabla_{\mathcal{L}} d_0)(\delta_{1/d_0(\xi)} \xi)| = \max_{d_0(\eta) = 1} |(\nabla_{\mathcal{L}} d_0)(\eta)|.$$

More details on nilpotent Lie algebras and homogeneous groups can be found, for example, in [FH], [F] and [RS].
If $\Omega$ is an open subset of $H$, we denote by $S(\Omega)$ the Sobolev space of the functions $u \in L^2(\Omega)$ such that $\nabla_{\mathcal{L}} u \in L^2(\Omega)$. The norm in $S(\Omega)$ is given by

$$\|u\|_{S(\Omega)} = \left( \int_\Omega |\nabla_{\mathcal{L}} u|^2 + u^2 \right)^{1/2}.$$

Moreover we denote by $S_{\text{loc}}(\Omega)$ the set of those $u \in L^2_{\text{loc}}(\Omega)$ such that $\varphi u \in S(\Omega)$ for every $\varphi \in C_0^\infty(\Omega)$. Let $V \in L^{]Q/2[}(\Omega)$; a function $u \in S_{\text{loc}}(\Omega)$ is called a weak solution of

$$-\mathcal{L} u \leq Vu \quad \text{in } \Omega$$

if

$$\int_\Omega \langle \nabla_{\mathcal{L}} u, \nabla_{\mathcal{L}} \varphi \rangle \leq \int_\Omega Vu\varphi \qquad \forall \varphi \in C_0^\infty(\Omega), \quad \varphi \geq 0.$$

We remark that in the definition above $Vu \in L^1_{\text{loc}}$ since $u \in S_{\text{loc}} \subseteq L^{Q/(Q-2)}_{\text{loc}}$ and $V \in L^{Q/2}$. We also remark that every classical solution of $-\mathcal{L}u \leq Vu$ is a weak solution in our definition, since $X_j^* = -X_j$ for $j = 1, \ldots, n$.

## 3. Proof of the main theorems.

The following representation formula plays a basic role in the proof of Theorem 1.1. Let $\Omega$ be an open subset of $H$ and $u \in S_{\text{loc}}(\Omega)$; then for a.e. $\xi \in \Omega$ and every $r > 0$ such that $\overline{B_d(\xi, r)} \subseteq \Omega$, we have

$$u(\xi) = (M_r u)(\xi) + \frac{Q}{r^Q} \int_0^r \varrho^{Q-1} \left( \int_{B_d(\xi, \varrho)} \langle \nabla_{\mathcal{L}} \Gamma_\xi, \nabla_{\mathcal{L}} u \rangle \right) d\varrho \tag{3.1}$$

where $M_r$ is the mean value operator defined by

$$(M_r u)(\xi) = \frac{c}{r^Q} \int_{B_d(\xi, r)} |\nabla_{\mathcal{L}} d_\xi|^2 u \tag{3.2}$$

and $c$ is a suitable positive constant. This formula has been proved in [CGL], Proposition 2.3. We have only replaced the $\mathcal{L}$-balls $\Omega_r(\xi) = \{\Gamma_\xi > 1/r\}$ of [CGL] with our $d$-balls and used the «explicit» expression of $\Gamma$ (2.1). We remark that the integral in the right-hand side of (3.1) is finite since (using (2.5))

$$|\langle \nabla_{\mathcal{L}} \Gamma_\xi, \nabla_{\mathcal{L}} u \rangle(\eta)| \leq |\nabla_{\mathcal{L}} \Gamma_\xi| |\nabla_{\mathcal{L}} u|(\eta) \leq c d(\xi, \eta)^{1-Q} |\nabla_{\mathcal{L}} u(\eta)| \in L^1_{\text{loc}, \eta}$$

for a.e. $\xi \in \Omega$ by Tonelli's Theorem, since $d(\xi, \eta)^{1-Q} \in L^1_{\text{loc}, \xi}$ (see (2.4)) and $|\nabla_{\mathcal{L}} u(\eta)| \in L^1_{\text{loc}, \eta}$.

We define $r: H \to [0, +\infty[$,

$$r(\xi) = \frac{d(\xi)}{4\beta^3} \tag{3.3}$$

where $\beta$ has been introduced in (2.3).

LEMMA 3.1. *Let $\Omega$ be an exterior domain of $H$, let $V \in L^{]Q/2[}(\Omega)$ and let $u \in S_{\text{loc}}(\Omega)$ be a nonnegative weak solution of*

$$-\mathcal{L}u \leq Vu \quad \text{in } \Omega$$

*such that $u \in L^p(\Omega)$ for a $p \in [1, +\infty[$. If there exist $s \in ]0, Q/p]$ and two* *positive constant* $R$, $M$ *such that*

$$\int_{B_d(\xi,\,2\beta r(\xi))\setminus B_d(\xi,\,r(\xi))} \Gamma_\xi |V| u \leq \frac{M}{r(\xi)^s}, \qquad \text{for } d(\xi) > R, \tag{3.4}$$

*then*

$$u(\xi) = O\left(\frac{1}{d(\xi)^s}\right), \qquad \text{as } d(\xi) \to +\infty.$$

PROOF. For every exponent $t \in ]1, +\infty[$ we shall denote by $t' = t/(t-1)$ the conjugate exponent of $t$. Since $V \in L^{]Q/2[}(\Omega)$ there exist $q_1$, $q_2$ such that $1 < q_1 < Q/2 < q_2 < +\infty$ and

$$V \in L^{q_1}(\Omega) \cap L^{q_2}(\Omega). \tag{3.5}$$

Moreover $q_2' < (Q/2)' = Q/(Q-2) < q_1'$ and then

$$\Gamma \in L^{q_2'}(B_d(0,1)) \cap L^{q_1'}(H \setminus B_d(0,1)) \tag{3.6}$$

by means of (2.1) and (2.4).

For every $\xi \in H$ and $r > 0$ such that $\overline{B_d(\xi, r)} \subseteq \Omega$, we define (see (3.2))

$$M_r(\xi) = (M_r u)(\xi),$$

$$N_r(\xi) = \int_{B_d(\xi,\,r)} \Gamma_\xi |V|$$

and

$$I_r(\xi) = \int_{B_d(\xi,\,r)} \Gamma_\xi |V| u = \int_{B_d(\xi,\,r)} \Gamma(\xi,\eta)\,|V|(\eta)\,u(\eta)\,d\eta.$$

We remark that $I_r(\xi) < +\infty$ a.e. by Tonelli's Theorem, since $\Gamma(\xi,\eta) \in L^1_{\text{loc},\,\xi}$ and $(Vu)(\eta) \in L^1_{\text{loc},\,\eta}$ because $u \in S_{\text{loc}} \subseteq L^{Q/(Q-2)}_{\text{loc}}$ and $V \in L^{Q/2}$. Moreover

$$\begin{aligned} N_r(\xi) &\leq \|\Gamma_\xi\|_{L^{q_2'}(B_d(\xi,1))} \|V\|_{L^{q_2}(B_d(\xi,r))} + \|\Gamma_\xi\|_{L^{q_1'}(H\setminus B_d(\xi,1))} \|V\|_{L^{q_1}(B_d(\xi,r))} \\ &\leq c(\|V\|_{L^{q_2}(B_d(\xi,r))} + \|V\|_{L^{q_1}(B_d(\xi,r))}) \end{aligned} \tag{3.7}$$

by (3.6). In particular (3.5) and (3.7) give

$$\sup_{\{\xi,\, r \mid \overline{B_d(\xi, r)} \subseteq \Omega\}} N_r(\xi) \leqslant c\,. \tag{3.8}$$

By (2.5), (3.2) and the assumption $u \in L^p(\Omega)$ we can estimate also $M_r$ and get

$$M_r(\xi) \leqslant \frac{c}{r^Q} \int_{B_d(\xi, r)} u \leqslant \frac{c}{r^Q} \|u\|_p \|1\|_{L^{p'}(B_d(\xi, r))} = \frac{c}{r^Q} r^{Q/p'} = \frac{c}{r^{Q/p}}\,. \tag{3.9}$$

Let us now recall the representation formula (3.1) for $u$. Since $-\mathcal{L}u \leqslant$ $\leqslant Vu$ weakly in $\Omega$, it is not difficult to see that

$$\int_{B_d(\xi, \varrho)} \langle \nabla_{\mathcal{L}} \Gamma_\xi, \nabla_{\mathcal{L}} u \rangle \leqslant \int_{B_d(\xi, \varrho)} \left( \Gamma_\xi - \frac{c_Q}{\varrho^{Q-2}} \right) |V| u$$

(one only needs to approximate $(\Gamma_\xi - c_Q/\varrho^{Q-2})$ with a suitable sequence of functions $C_0^\infty(\Omega)$). Then we get

$$u(\xi) \leqslant M_r(\xi) + \frac{Q}{r^Q} \int_0^r \varrho^{Q-1} \left( \int_{B_d(\xi, \varrho)} \left( \Gamma_\xi - \frac{c_Q}{\varrho^{Q-2}} \right) |V| u \right) d\varrho \tag{3.10}$$

$$\leqslant M_r(\xi) + \frac{Q}{r^Q} \int_0^r \varrho^{Q-1} \left( \int_{B_d(\xi, r)} \Gamma_\xi |V| u \right) d\varrho$$

$$= M_r(\xi) + I_r(\xi) \leqslant \frac{c}{r^{Q/p}} + I_r(\xi)$$

(by (3.9)). Hence, if $\overline{B_d(\xi, 2\beta r)} \subseteq \Omega$ (see (2.3)), we have

$$I_r(\xi) \leqslant \int_{B_d(\xi, r)} \Gamma_\xi(\eta) |V|(\eta) \left( \frac{c}{r^{Q/p}} + I_r(\eta) \right) d\eta \tag{3.11}$$

$$\leqslant \frac{c}{r^{Q/p}} N_r(\xi) + \int_{B_d(\xi, r)} \Gamma_\xi |V| I_r \leqslant \frac{c}{r^{Q/p}} + \int_{B_d(\xi, r)} \Gamma_\xi |V| I_r$$

(by (3.8)). Moreover, if $\overline{B_d(\xi, 3\beta^2 r)} \subseteq \Omega$, then

$$(3.12)\quad \int_{B_d(\xi,r)} \Gamma_\xi |V| I_r = \int_{B_d(\xi,r)} \Gamma(\xi,\eta)|V|(\eta)\left(\int_{B_d(\eta,r)} \Gamma(\eta,\zeta)|V|(\zeta)u(\zeta)\,d\zeta\right)d\eta$$

$$= \int_{B_d(\xi,2\beta r)} |V|(\zeta)u(\zeta)\left(\int_{B_d(\xi,r)\cap B_d(\zeta,r)} \Gamma(\xi,\eta)\Gamma(\eta,\zeta)|V|(\eta)\,d\eta\right)d\zeta$$

$$\leqslant c \sup_{\zeta\in B_d(\xi,2\beta r)} N_r(\zeta) \int_{B_d(\xi,2\beta r)} \Gamma_\xi |V| u$$

since, setting $A = B_d(\xi, r) \cap B_d(\zeta, r)$, $A_1 = \{\eta \in A \,|\, d(\eta, \zeta) \leqslant d(\xi, \zeta)/(2\beta)\}$ and $A_2 = A \setminus A_1$, we have (note that $d(\xi, \eta) \geqslant d(\xi, \zeta)/(2\beta)$ for every $\eta \in A_1$)

$$\int_A \Gamma_\xi \Gamma_\zeta |V| = \int_{A_1} \Gamma_\xi \Gamma_\zeta |V| + \int_{A_2} \Gamma_\xi \Gamma_\zeta |V|$$

$$\leqslant c\Gamma(\xi, \zeta)\left(\int_{A_1} \Gamma_\zeta |V| + \int_{A_2} \Gamma_\xi |V|\right)$$

$$\leqslant c\Gamma(\xi, \zeta)(N_r(\zeta) + N_r(\xi)).$$

From now on we will take $r = r(\xi)$ (see (3.3)); in this way $\overline{B_d(\xi, 3\beta^2 r)} \subseteq H \setminus B_d(0, r) \subseteq \Omega$, for large $d(\xi)$. Using this fact we obtain

$$\sup_{\zeta\in B_d(\xi,2\beta r(\xi))} N_{r(\xi)}(\zeta) \leqslant c(\|V\|_{L^{q_2}(H\setminus B_d(0,r(\xi)))} + \|V\|_{L^{q_1}(H\setminus B_d(0,r(\xi)))}) \xrightarrow{d(\xi)\to+\infty} 0$$

by means of (3.7) and (3.5). Hence there exists $R_0 > R$ such that, for $d(\xi) > R_0$, we have (see (3.12))

$$(3.13)\quad \int_{B_d(\xi,r(\xi))} \Gamma_\xi |V| I_{r(\xi)} \leqslant \frac{1}{2} \int_{B_d(\xi,2\beta r(\xi))} \Gamma_\xi |V| u$$

$$= \frac{1}{2} I_{r(\xi)}(\xi) + \frac{1}{2} \int_{B_d(\xi,2\beta r(\xi))\setminus B_d(\xi,r(\xi))} \Gamma_\xi |V| u$$

$$\leqslant \frac{1}{2} I_{r(\xi)}(\xi) + \frac{M}{2r(\xi)^s}$$

by assumption (3.4). From (3.11) and (3.13) we finally get, for $d(\xi) > R_0$,

$$\frac{1}{2} I_{r(\xi)}(\xi) \leq \frac{c}{r(\xi)^{Q/p}} + \frac{c}{r(\xi)^s}\,.$$

This estimate and (3.10) allow us to conclude that

$$u(\xi) = O\left(\frac{1}{d(\xi)^s}\right), \qquad \text{as } d(\xi) \to +\infty$$

since $r(\xi) = d(\xi)/(4\beta^3)$ and $s \leq Q/p$. ■

PROOF OF THEOREM 1.1. To prove the first part of the statement we only need to obtain (3.4) for $s = Q/p$ and then use Lemma 3.1. If $p = Q/(Q-2)$ we have (for large $d(\xi)$)

$$\int\limits_{B_d(\xi, 2\beta r(\xi)) \setminus B_d(\xi, r(\xi))} \Gamma_\xi |V| u = \int\limits_{B_d(\xi, 2\beta r(\xi)) \setminus B_d(\xi, r(\xi))} \frac{c_Q}{d_\xi^{Q-2}} |V| u \leq \frac{c}{r(\xi)^{Q-2}} \int\limits_H |V| u$$

$$\leq \frac{c}{r(\xi)^s} \|V\|_{Q/2} \|u\|_{Q/(Q-2)}$$

and then (3.4) holds.

We now suppose $p > Q/(Q-2)$. For every exponent $t \in ]1, +\infty[$ we shall denote by $t' = t/(t-1)$ the conjugate exponent of $t$. Since $V \in L^{]Q/2[}(\Omega)$ there exists $q < Q/2$ such that $Q/(Q-2) = (Q/2)' < q' < p$ and $V \in L^q(\Omega)$. Moreover $Q - 2 - s = Q - 2 - (Q/p) > 0$. Hence

$$\frac{Q-2-s}{Q} q' > \frac{Q-2-Q/p}{Q-2} = 1 - \frac{Q/(Q-2)}{p} > 1 - \frac{q'}{p} = \frac{1}{(p/q')'}$$

and then (see (2.4))

$$\frac{1}{d_0^{(Q-2-s)q'}} \in L^{(p/q')'}(H \setminus B_d(0,1))\,.$$

We can now obtain (3.4). For large $d(\xi)$ we have

$$\int_{B_d(\xi,2\beta r(\xi))\setminus B_d(\xi,r(\xi))} \Gamma_\xi |V| u \leqslant \frac{c}{r(\xi)^s} \int_{B_d(\xi,2\beta r(\xi))\setminus B_d(\xi,r(\xi))} |V| \frac{u}{d_\xi^{Q-2-s}}$$

$$\leqslant \frac{c}{r(\xi)^s} \|V\|_q (\|u^{q'}\|_{p/q'} \|d_\xi^{-(Q-2-s)q'}\|_{L^{(p/q')'}(H\setminus B_d(\xi,1))})^{1/q'}$$

$$\leqslant \frac{c}{r(\xi)^s} \|d_0^{-(Q-2-s)q'}\|^{1/q'}_{L^{(p/q')'}(H\setminus B_d(0,1))} \leqslant \frac{c}{r(\xi)^s}\,.$$

Let us now prove the second part of the theorem. We know that $u \in L^{p_1} \cap L^{Q/(Q-2)}$ for a $p_1 \in [1, Q/(Q-2)[$. By means of Lemma 3.1 we only need to prove that (3.4) holds for $s = Q/p_1$. For every $t \in ]0, Q/p_1[$ we set

$$\sigma(t) = Q - 2 + t - \frac{p_1 t(Q-2)}{Q}$$

and we claim that (3.4) holds for $s = \sigma(t)$ if (3.4) holds for $s = t$. Indeed, by Lemma 3.1, if (3.4) holds for $s = t$ then

$$u(\xi) = O\left(\frac{1}{d(\xi)^t}\right), \qquad \text{as } d(\xi) \to +\infty\,; \tag{3.14}$$

therefore, for large $d(\xi)$,

$$\int_{B_d(\xi,2\beta r(\xi))\setminus B_d(\xi,r(\xi))} \Gamma_\xi |V| u \leqslant \frac{c}{r(\xi)^{Q-2}} \|V\|_{Q/2} \left( \int_{B_d(\xi,2\beta r(\xi))} u^{Q/(Q-2)-p_1} u^{p_1}\right)^{(Q-2)/Q}$$

$$\leqslant \frac{c}{r(\xi)^{Q-2}} \|u\|^{1-(p_1(Q-2))/Q}_{L^\infty(H\setminus B_d(0,r(\xi)))} \|u\|_{p_1}^{p_1(Q-2)/Q} \leqslant \frac{c}{r(\xi)^{\sigma(t)}}$$

(by means of (3.14) and (3.3)).

Since $u \in L^{Q/(Q-2)}(\Omega)$, (3.4) holds for $s = Q-2$ (see the beginning of the proof); moreover if we set

$$\begin{cases} t_1 = Q-2 \\ t_{k+1} = \sigma(t_k) \end{cases}$$

it is easy to see that $t_k \nearrow (Q/p_1)$. Henceforth (3.4) holds for every $s \in$

$\in ]0, Q/p_1[$. In particular if we choose $q < Q/2$ such that $V \in L^q(\Omega)$ and we set

$$\tau = \frac{Q/p_1 - Q + 2}{1 - p_1/q'}$$

we have $0 < \tau < Q/p_1$ and then (3.4) holds for $s = \tau$. Hence, by Lemma 3.1,

$$u(\xi) = O\left(\frac{1}{d(\xi)^{\tau}}\right), \qquad \text{as } d(\xi) \to +\infty. \tag{3.15}$$

We are now able to prove (3.4) for $s = Q/p_1$. In fact we have (for large $d(\xi)$)

$$\int_{B_d(\xi, 2\beta r(\xi)) \setminus B_d(\xi, r(\xi))} \Gamma_\xi |V| u \leqslant \frac{c}{r(\xi)^{Q-2}} \|V\|_q \left( \int_{B_d(\xi, 2\beta r(\xi))} u^{q'-p_1} u^{p_1} \right)^{1/q'}$$

$$\leqslant \frac{c}{r(\xi)^{Q-2}} \|u\|^{1-p_1/q'}_{L^\infty(H \setminus B_d(0, r(\xi)))} \|u\|^{p_1/q'}_{p_1}$$

$$\leqslant \frac{c}{r(\xi)^{Q-2+\tau(1-p_1/q')}} = \frac{c}{r(\xi)^{Q/p_1}}$$

(by means of (3.15) and (3.3)). ∎

PROOF OF THEOREM 1.3. We set

$$f = \begin{cases} |V| u & \text{in } \Omega \\ 0 & \text{in } H \setminus \Omega \end{cases}$$

and we define $w: H \to \mathbb{R}$,

$$w(\xi) = (\Gamma * f)(\xi) = \int_H \Gamma(\xi^{-1} \circ \eta) \, f(\eta) \, d\eta .$$

Then $w \geqslant 0$ and $-\mathcal{L} w = f$ weakly in $H$. In particular $u \leqslant w$ in $\partial\Omega \cup \{\infty\}$ and $-\mathcal{L} u \leqslant |V| u = -\mathcal{L} w$ in $\Omega$. Hence

$$0 \leqslant u \leqslant w \quad \text{in } \Omega \tag{3.16}$$

by the weak maximum principle for $\mathcal{L}$. Moreover if $t \in ]1, Q/2[$ and $f \in$

$\in L^t(H)$ then

$$w \in L^{(1/t - 2/Q)^{-1}}(H) \tag{3.17}$$

and

$$\nabla_{\mathcal{L}} w \in L^{(1/t - 1/Q)^{-1}}(H) \tag{3.18}$$

(see [RS], Proposition B, p. 264).

Since $V \in L^{]Q/2[}(\Omega)$ there exists $q_1 \in ]1, Q/2[$ such that $V \in L^{q_1}(\Omega) \cap \cap L^{Q/2}(\Omega)$. We now fix $q \in ]q_1, Q/2[$ and we set $q' = q/(q-1)$ and

$$\varepsilon = \frac{1}{q} - \frac{2}{Q}.$$

If $p \in ]q', +\infty]$ and $u \in L^p(\Omega)$ then $f \in L^t(H)$ for $t = (1/q + 1/p)^{-1}$, since

$$\int_\Omega |Vu|^t \leq \| |V|^t \|_{q/t} \|u^t\|_{p/t} = \|V\|_q^t \|u\|_p^t.$$

Moreover $t \in ]1, q] \subseteq ]1, Q/2[$ and then we get $w \in L^{(1/t - 2/Q)^{-1}}(H) = = L^{(1/p + \varepsilon)^{-1}}(H)$ (see (3.17)) and also $u \in L^{(1/p + \varepsilon)^{-1}}(\Omega)$ (see (3.16)). We know a priori that $u \in L^\infty(\Omega)$; hence we can iterate the process above and get $w \in L^{q'}(H)$. Since $q \in ]q_1, Q/2[$ is arbitrary, we finally obtain

$$w \in L^p(H) \qquad \forall p \in \left] \frac{Q}{Q-2}, +\infty \right[. \tag{3.19}$$

Moreover, using (3.18) one can easily see that $w \in S_{\text{loc}}(H)$. Hence $w$ is a nonnegative weak solution of

$$\begin{cases} -\mathcal{L}w \leq |V|w & \text{in } H \\ w \in S_{\text{loc}}(H) \end{cases}$$

(see (3.16)). It follows that

$$w(\xi) = O\left(\frac{1}{d(\xi)^s}\right) \quad \text{as } d(\xi) \to +\infty, \qquad \forall s < Q-2,$$

by means of (3.19) and Theorem 1.1. Again by (3.16) we finally get our statement. ■

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