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Rigid Meromorphic Foliations on Complex Surfaces.

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Introduction.

We are interested in the problem of existence and density of foliations without algebraic leaves. Here we give a construction (see Theorem 1.1) of singular meromorphic foliations without algebraic leaves on every smooth projective surface. In sections 2 and 3 we consider the related problem of «rigidity» or «persistency» of a singular meromorphic foliation on a compact complex surface X. We study the case of a foliation coming from a fibration, i.e. from a morphism $X \to B$ with B smooth curve. In section 2 we study the case of a surface with Kodaira dimension $-\infty$, $X \neq \boldsymbol{P}^2$ and give (see Theorem 2.1) another proof of the theorem proved in [15]. In section 3 we consider the case of an elliptic fibration.

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1. Foliations without algebraic leaves.

Recall that a meromorphic foliation by curves on a smooth complex manifold M is given by a non zero morphism $\boldsymbol{i}: L \to TM$ with L line bundle on M. Of course, if $\dim(M) = 2$ this is a codimension 1 meromorphic foliation with singularities on M. We will call «foliation» any meromorphic foliation with singularities. The singular set $\mathrm{Sing}(F)_{\mathrm{red}}$ (or just $\mathrm{Sing}(F)$) of the foliation F is the set of points of M where $\boldsymbol{i}$ drops rank. The foliation is called *saturated* if $\boldsymbol{i}$ drops rank at most in codimension 2.

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If M is a surface and F is saturated we have an exact sequence

$$0 \to L \to TM \to I_Z \otimes K_M^{-1} \otimes L^{-1} \to 0 \tag{1}$$

with $\dim(Z) = 0$, $Z := \mathrm{Sing}(F)$ with its scheme structure (see e.g. [8]); if $\dim(M) > 2$ or one is interested in foliations on singular surfaces, the best background material on saturated subsheaves is probably contained in the first section of [12].

We may move the foliation either varying L or fixing L and choosing a nearby non proportional section of $H^0(M, TM \otimes L^{-1})$. Note that in every small deformation of the foliation F the algebraic, numerical and topological equivalence class of the line bundle L remain constant. Hence if $F \subset M$ is a curve, we have $\deg(L_t | F) = \deg(L|F)$ for all t.

For the theory of deformations of singular foliations, see [7] or [6] or [16] or [17] or [2]. For the particular case of deformations of foliations by curves, see [8] and [9]. Hence we will say that a singular meromorphic foliation is *rigid* if every flat deformation of it parametrized by a reduced space is trivial.

Let M be a complex projective surface. In this section we give a construction of families of singular meromorphic foliations on M with large dimension in which the set of foliations without algebraic leaves is dense. We will prove the following result.

THEOREM 1.1. *Let M be a smooth complex projective surface. Fix a very ample line bundle R on M. Set $x := h^0(M, R)$. For every integer $r \geqslant 4$ the moduli space of saturated singular meromorphic foliations associated to a non zero* map $R^{\otimes(-r)} \to TM$ *contains a Zariski open subset of a projective space of dimension* $\max\{3(x-3), r^2+6r+8\}$ *in which the set of foliations without any algebraic leaf is dense in the euclidean topology and a Zariski open non-empty subset of a projective space of dimension* $3(x-3)$ *formed by foliations without any algebraic leaf.*

PROOF. Recall that the Grassmannian $G(3, x)$ of 3-dimensional subspaces of $\boldsymbol{C}^x$ has dimension $3(x-3)$. Consider M embedded by R in the projective space $|R| \cong \boldsymbol{P}^{x-1}$ and let $p: M \to \boldsymbol{P}^2$ be a general projection. Hence $R \cong p^*(O_{\boldsymbol{P}}(1))$. Call $A \subset M$ (resp. $D \subset \boldsymbol{P}^2$) the ramification locus (resp. the discriminant divisor). Fix a singular meromorphic saturated foliation by curves G on $\boldsymbol{P}^2$ of degree $r \geqslant 4$. By a form of the Bertini theorem (see e.g. [14]) there is $g \in \mathrm{Aut}(\boldsymbol{P}^2)$ such that $g(D)$ is transversal to

G outside finitely many points. Taking $g^*(G)$ instead of G we may assume that the discriminant D is transversal to G outside finitely many points. Let ω be the meromorphic 1-form inducing G and let E be the foliation induced by $p^*(\omega)$. In general E may be non saturated. Let F be the saturation of E. Note that for every algebraic leaf T of F on $M \backslash A$, the closure of $p(T)$ is an algebraic leaf of G. Hence every algebraic leaf of F is either contained in the counterimage of an algebraic leaf of G or it is contained in A. Since the discriminant divisor D is transversal to G outside finitely many points, there is no algebraic leaf of F contained in A and $E = F$ is saturated. By a theorem of Jouanolou ([13], Ch. 4, Th. 1.1) every euclidean neighborhood of G contains foliations without algebraic leaves. Fixing the projection p, we obtain a Zariski open subset U of a projective space of dimension $r^2 + 6r + 8$ in which the set of foliations without any algebraic leaf is dense in the euclidean topology. Viceversa, fixing any such foliation G of degree r on $\boldsymbol{P}^2$ and varying the projections we find a Zariski open dense subset of $G(3, x)$ parametryzing one to one foliations without algebraic leaves. Indeed, to check that the parametrization is one to one it is sufficient to look at the singularities of the pull-backs of G arising as counterimages of the singularities of G. By a theorem of Gomez-Mont and Kempf ([10]) every degree r non-degenerate (i.e. such that all its singularities have multiplicity one) foliation on $\boldsymbol{P}^2$ is uniquely determined by the set of its singularities. By [13], part 2) of Th. 2.3 at p. 87, a Zariski open non empty subset of U parametrizes non-degenerate foliations. ∎

Usually under the assumptions of Theorem 1.1 the integer x is small with respect to r and hence $3(x-3)$ is much smaller than $r^2 + 6r + 8$. A family of exceptional cases is given taking $R \cong M^{\otimes m}$ with $M \in \operatorname{Pic}(X)$, M very ample, and m very large. This family of examples is interesting only for the last assertion of Theorem 1.1, because usually $h^0(X, M)$ is much smaller than mr.

2. Rigid and ruled fibrations.

In this section we will study the meromorphic foliations with singularities on a smooth projective surface X with Kodaira dimension $-\infty$, $X \neq \boldsymbol{P}^2$. This is the class of all surfaces with a morphism $u: X \to B$, B smooth curve with general fiber isomorphic to $\boldsymbol{P}^1$ (a ruling of X). We will say that such a surface is ruled; we will say that X is geometrically ruled if all the fibers of u are smooth (hence isomorphic to $\boldsymbol{P}^1$). Some authors call birationally ruled our general set up and call ruled surfaces only the geometrically ruled surfaces. Any such X is obtained from a geometrically ruled surface, Y, with a finite number of blowing ups; the surface Y and the morphism $\pi: X \to Y$ is uniquely determined if B has genus $g > 0$. The case of a geometrically ruled surface was considered in [8]. As a consequence of our analysis we will prove Theorem 2.1 below, i.e. we will give another proof of the theorem proved in [14]. The local analysis of what happens to a holomorphic foliation making a blowing up (strict transform of the foliation) was made in [9], § 6. We will fix the following notations. Let g be the genus of B and $t \geqslant 0$ the number of blowing ups whose composition gives π. We will identify divisors and line bundles and often use the additive notation for both. Let $v: Y \to B$ be the ruling of Y. As a base for the Neron Severi group $NS(Y)$ of divisors of Y (i.e. divisors modulo numerical equivalence) we will give the classes h and f with $h^2 = 0$, $h \cdot f = 1$, $f^2 = 0$, f class of a fiber of v, h class of a section, up to multiples of f (i.e. there may not be any effective curve with numerical class h and, even if there is one, it may consist of an irreducible section plus a few fibers). We will denote by $-e$ the minimal self-intersection of a section of v; by a theorem of Nagata we have $e \geqslant -g$. Call H (resp. F) the total transform of h (resp. f) on X; hence $H^2 = 0$, $H \cdot F = 1$ and $F^2 = 0$; F will denote also a general fiber of u (hence a general fiber of v). As a base of the Neron Severi group $NS(X)$ of X we will take H, F and the following divisors E_i, $1 \leqslant i \leqslant t$, with $H \cdot E_i = F \cdot E_i = 0$, $K_X \cdot E_i = E_i^2 = -1$ for all t. Decompose π into t blowing ups and call $\pi(i)$: $X(i) \to Y$, $0 \leqslant i \leqslant t$, the composition of the first i of these blowing ups; assume to have defined E_j, $j \leqslant i$, for some $i < t$ as a class on $X(i)$; as classes E_a on $X(i+1)$ takje the total transform of the classes E_a on $X(i)$ for $a \leqslant i$ and the class of the exceptional divisor of the blowing up $X(i+1) \to (X(i)$ as class of E_{i+1}. Note that every E_j is effective, but may be reducible.

The case of singular foliations on Y was studied in detail in [8]. Call G the foliation of fibration type induced by the ruling of X and let L'' be the associated saturated line subsheaf of TX. Note that the ruling (and hence the foliation) is unique except in the cases $g = 0$, $e = 0$, $t = 0$ or 1, in which there are exactly two rulings. For every smooth fiber $F \cong \boldsymbol{P}^1$ we have $(\deg(L''|F) = 2$. We claim that the numerical equivalence class of L'' is $2H - \sum_{1 \leqslant i \leqslant t} E_i$. To check the claim, use [8], Lemma 1.4, for the case $y = X$, the local analysis of the behaviour of tangent bundles on surfaces by blowing ups made in [9], § 4, and the fact that $c_1(TX) - L''$ is numerically equivalent to $(2-2g)F$. Consider a small deformation $\{G_t\}_{t \in \Delta}$, Δ the unit disc of $\boldsymbol{C}$, of G with X fixed. Note that in any small deformation of a foliation by curves the numerical equivalence class of the saturated line subsheaf of TX remains constant. Let L_t'' be the line bundle corresponding to L'' for the foliation G_t. Since $c_1(TX) - L''$ is numerically equivalent to the pull-back of a line bundle on the curve B (the base of the ruling) for a general fiber, F, of the ruling u we have $\deg(L_t'' \mid F) = 0$. Since $F \cong \boldsymbol{P}^1$, $L_t'' \mid F$ is trivial. Hence F is a leaf of G_t. Thus G is persistent, giving another proof of the following theorem proved in [14].

THEOREM 2.1. *On every smooth projective surface with Kodaira dimension* $-\infty$ *except the projective plane there is a singular meromorphic foliation (the foliation induced by a ruling) which is rigid.*

There is an inclusion between (saturated) singular foliations by curves in Y and X ([9], § 6); with the terminology of [9], § 6, the foliation on X corresponding to a foliation A on Y is called the strict transform of A. As in [9], Def. 2.5, we will give the following definition of foliation on X of Riccati type.

DEFINITION 2.2. A saturated foliation on X induced by an inclusion $L \to TX$ is called a *Riccati foliation* if there is $M \in \mathrm{Pic}(B)$ with $c_1(L) = c_1(u^*(M))$.

Fix a Riccati foliation F on X. Since we have $H^2(X, \boldsymbol{O}_X) = 0$, the exponential sequence

$$0 \to \boldsymbol{Z} \to \boldsymbol{O}_X \to \boldsymbol{O}_X^* \to 0$$

shows that numerical equivalence, algebraic equivalence and topological equivalence of line bundles on X coincide. Furthermore, $\mathrm{Pic}^0(X) \cong u^*(\mathrm{Pic}^0(B))$. Hence in the definition 2.2 of Riccati foliation we may assume $L \cong u^*(M)$. Since $u = v \circ \pi$, there is $L' \in \mathrm{Pic}(Y)$ with $\pi^*(L') \cong L$. Let $U \subset Y$ be the Zariski open subset of Y with card $(Y \setminus U)$ finite and such that $\pi | \pi^{-1}(U)$: $\pi^{-1}(U) \to U$ is an isomorphism. The restriction to $\pi^{-1}(U)$ of the Riccati foliation F induced a singular holomorphic foliation G on U. However, a priori this singular holomorphic foliation does not extend to a singular meromorphic foliation on all X. Since it is sufficient to check the integrability condition for a foliation defined by a meromorphic 1-form on a Zariski open dense subset, to obtain the extension of G to Y (as singular foliation) it is sufficient to show the existence of $L'' \in \mathrm{Pic}(Y)$ and of a map $\boldsymbol{i}$: $L'' \to TY$ such that $\boldsymbol{i} \mid U$ induces G. Since $\mathrm{codim}(Y \backslash U) = 2$, for every $M \in \mathrm{Pic}(Y)$ we have $h^0(Y, TY \otimes M) = h^0(U, (TY \otimes M) \mid U)$. Hence it is sufficient to find $L'' \in \mathrm{Pic}(Y)$ and $\boldsymbol{r}$: $L'' \mid U \to TU$ inducing G. We claim that we may take L' as such line bundle L''. Indeed by the definition of U and L' the morphism $\pi \mid \pi^{-1}(U)$ induces an isomorphism of $\pi^{-1}(U)$ onto U and $\pi^*(L'' \mid U) = L$, $\pi^*(TU) = T\pi^{-1}(U)$. Since $u = v \circ \pi$ we have $L' \in v^*(\mathrm{Pic}(B))$. Thus every Riccati foliation is the strict transform of a Riccati foliation on the geometrically ruled surface Y. Such foliations on Y are studied in [8], § 2.

3. Rigid and elliptic fibrations.

In this section we consider the meromorphic singular foliation induced by an elliptic fibration π: $X \to B$ with B smooth curve of genus $g \geq 0$. Hence the general fiber of π is a smooth elliptic curve. The main difference with respect to the case of a ruling considered in section 2 is that now the general fiber of the fibration is not simply connected. Let $T_{X/B}$ be the relative tangent sheaf of π (see e.g. [18], pages 408-409). We assume that the following exact sequence

$$0 \to L \to TX \to N \otimes \boldsymbol{I}_Z \to 0 \tag{2}$$

with $L \cong T_{X/B}$ and $N^{-1} \cong K_X \otimes L$ defines the foliation F induced by π.

Note that the fibration π of X is rigid as fibration and that the irreducible component of the Hilbert scheme $\mathrm{Hilb}(X)$ of X containing a smooth fiber F of the fibration is given by the fibers of the fibration π. Hence the foliation induced by π is rigid if the following two conditions are satisfied:

($a1$) Every nearby foliation is induced by the same line bundle L.

($a2$) We have $h^0(X, TX \otimes L^{-1}) = 1$.

REMARK 3.1. Assume that the elliptic fibration is relatively minimal. Then $L = T_{X/B} \cong \pi^*(A) \otimes \boldsymbol{O}(\Sigma(1 - m_i) F_i)$ where the sum Σ is over all multiple fibers F_i's, F_i has multiplicity m_i and $A \in \mathrm{Pic}(B)$, $\deg(A) = -\chi(\boldsymbol{O}_X)$, $N = \pi^*(A')$ with $\deg(A') = 2(1 - g(B))$ ([3], p. 162).

Now consider the restriction of (2) to a smooth fiber F of the fibration π. By the adjunction formula we obtain the following exact sequence

$$0 \to \boldsymbol{O}_F \to TX \mid F \to \boldsymbol{O}_F \to 0\,. \tag{3}$$

Two cases are possible: either (3) splits or not. We will call the fibration of *indecomposable type* if the exact sequence (3) does not split. Assume that (3) does not split. By Atiyah's classification of vector bundles on an elliptic curve ([1]), this is equivalent to the fact that $TX|F$ is isomorphic to tyhe unique indecomposable rank 2 vector bundle of F with trivial determinant. This implies $h^0(F,(TX|F)\otimes D)=0$ for every $D\in \mathrm{Pic}^0(F)$ with $D\neq \boldsymbol{O}_F$ and $h^0(F, TX|F)=1$. Thus for every nearby foliation F_t induced, say, by $s_t\colon L_t\to TX$, $s_t|F=s|F$ is uniquely determined, i.e. the tangent direction of the foliation F_t at any point of F is the same as the one for F, i.e. F is a leaf of F_t. Thus $F_t=F$ and F is rigid. Hence we have proved the following result.

PROPOSITION 3.2. *The foliation induced by an elliptic fibration of indecomposable type is rigid.*

Here is another case in which F is rigid.

PROPOSITION 3.3. *Assume that the elliptic fibration π is relatively minimal. Assume $\chi(\boldsymbol{O}_X)<2g-2$, i.e. $c_2<12(2g-2)$. Then the foliation F induced by π is rigid.*

PROOF. Let $\{L_t\}_{t\in\Delta}$, Δ the unit disk of $\boldsymbol{C}$, be the family of saturated rank 1 subsheaves of TX associated to a small deformation of the foliation F. Hence L_t is numerically equivalent to L. We have $h^0(X, \mathrm{Hom}(L_t, N))=0$ by the numerical assumptions on A and A' with $L=T_{X/B}\cong\pi^*(A)\otimes\boldsymbol{O}(\Sigma(1-m_i)F_i)$ and $N:=\pi^*(A')$. Thus $L_t\cong L$ for all t. Since $h^0(X, \mathrm{Hom}(L,N))=0$ by the numerical assumptions, we have $h^0(X, TX\otimes L^{-1})=1$. Thus the foliation F is rigid. ■

Here we will consider the case of hyperelliptic surfaces (also called bielliptic surfaces). For the classification of these surfaces, see [5], p. 36-37, or [3], p. 148 and 189, or [4], pp. 113-114, or [11], pp. 585-590.

THEOREM 3.4. *Let X be a surface birational to a hyperelliptic surface. Let F be the foliation induced by the elliptic fibration $\pi\colon X\to B$ given by the Albanese map and G the foliation induced by the unique elliptic pencil $m\colon X\to \boldsymbol{P}^1$. Then F is rigid. If X is minimal, then G is rigid.*

PROOF. First assume X minimal. By [3], p. 148 and 168, the fibration π has $g(B)=1$, $\chi(\boldsymbol{P}_X)=0$, $h^0(X, \Omega^1_X)=1$ and $Z=\emptyset$. By [11], p. 585, π is smooth. Hence, taking $A, A' \in \mathrm{Pic}(B)$ with $L \cong \pi^*(A')$ and $N \cong \pi^*(A)$, we have $\deg(A) = \deg(A') = 0$. Since $c_2(X) = 0$ we see that also the fibration m induces an exact sequence (2) with $Z = \emptyset$; the only difference is that now L is not of theform $m^*(A')$, because there is the contribution of the multiple fibers (which are the only singular fibers of the fibration m). Let $\{L_t\}_{t \in \Delta}$ be the family of saturated rank 1 subsheaves of TX associated to a small deformation of the foliation F (or the foliation G). By (2) we have $h^0(X, \mathrm{Hom}(L_t, N)) = 0$ if L_t and N are not isomorphic. Since $\deg(L_t | F) = \deg(N | F) = 0$ for every fiber F of π and for a general fiber F of m, we obtain that L is constant in such small deformation of F (or G). We have $h^0(X, TX \otimes L^{-1}) = 1$ unless $L \cong N$ and $TX \cong L \oplus L$. Thus in order to obtain a contradiction we may assume $TX \cong L \oplus L$. Thus Ω^1_X is the direct sum of two isomorphic line bundles. Hence $h^0(X, \Omega^1_X)$ is even, contradiction. Now we drop the assumption of minimality of X. We use the notations of section 2 for the exceptional divisors. We use that on the minimal model the fibration π is smooth. As in the case of the Riccati foliations on ruled surfaces considered at the end of section 2, now $L = \pi^*(A) \otimes (-\Sigma E_i)$, $N = \pi^*(A')$ with $\deg(A) = \deg(A') = 0$. As in the case of the Riccati foliations we see that every small deformation of F comes from a small deformation of the foliation of fibration type on the minimal model of X. Hence we conclude. ■

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