Rendiconti del Seminario Matematico della Università di Padova

Alireza Abdollahi

Bijan Taeri

A condition on a certain variety of groups

Rendiconti del Seminario Matematico della Università di Padova, tome 104 (2000), p. 129-134

<http://www.numdam.org/item?id=RSMUP_2000__104__129_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 104 (2000)

A Condition on a Certain Variety of Groups.

ALIREZA ABDOLLAHI (*) - BIJAN TAERI (*)

ABSTRACT - Let $a_1, \ldots, a_n$ be nonzero integers whose greatest common divisor is d. We prove that an infinite group G is of finite exponent dividing d if and only if for every n infinite subsets $X_1, \ldots, X_n$ of G there exist $x_1 \in X_1, \ldots, x_n \in X_n$ such that $x_1^{a_1} \ldots x_n^{a_n} = 1$.

1. Introduction.

Let $\mathcal{V}$ be a variety of groups defined by the law $v(x_1, \ldots, x_n) = 1$. We say that G is a $\mathcal{V}^{\#}$-group if for every n infinite subsets $X_1, \ldots, X_n$ of G there exist $x_1 \in X_1, \ldots, x_n \in X_n$ such that $v(x_1, \ldots, x_n) = 1$.

In [4] P.S. Kim, A.H. Rhemtulla and H. Smith posed the following question: for which variety $\mathcal{V}$, is every infinite $\mathcal{V}^{\#}$-group a $\mathcal{V}$-group?

There exist positive answers for the variety $\mathcal{A}$ of abelian groups defined by the law $[x, y] = 1$ (see [7]), the variety $\mathcal{Q}_2$ defined by the law $[x, y]^2 = 1$ (see [5]), the varieties $\mathcal{E}_2$ and $\mathcal{E}_3$ of 2-Engel and 3-Engel groups defined by the laws $[x, y, y] = 1$ and $[x, y, y, y] = 1$, respectively (see [10] and [11]), the variety $\mathcal{B}_n$ of groups of exponent dividing n, defined by the law $x^n = 1$ (see [6]), the variety $\mathcal{N}_k$ of nilpotent groups of nilpotency class at most k, defined by the law $[x_1, \ldots, x_{k+1}] = 1$ (see [6]), the variety $\mathcal{A}_3$ of 3-abelian groups defined by the law $(xy)^3 y^{-3} x^{-3} = 1$

(*) Indirizzo degli AA.: Department of Mathematics, University of Isfahan, Isfahan 81744, Iran; Department of Mathematics, University of Technology of Isfahan, Isfahan, Iran.

We would like to express our appreciation to our supervisor Professor A. Mohammadi Hassanabadi, who first drew our attention to the subject, and his encouragement and numerous helpful comments. We also would sincerely like to thank the Referee for improving the original result of the paper.

(see [1]) and, the variety generated by the law $x_1 \dots x_n = 1$ (see [3]).

Now let $\mathfrak{V}$ and $\mathfrak{W}$ be varieties of groups defined by different laws $v(x_1, \dots, x_n) = 1$ and $w(y_1, \dots, y_m) = 1$, respectively. C. Delizia in [2] posed the question of whether $\mathfrak{V}^{\#} = \mathfrak{W}^{\#}$ when $\mathfrak{V} = \mathfrak{W}$, and for $k > 1$, he investigated the question for the varieties $\mathfrak{V} = \mathfrak{V}_k$ (the variety defined by the law $[x_1, \dots, x_k, x_1] = 1$) and $\mathfrak{W} = \mathcal{N}_k$.

For nonzero integers $\alpha_1, \dots, \alpha_n$ with the greatest common divisor d, we have considered the question for the varieties $\mathfrak{V}$ defined by the law $x_1^{\alpha_1} \dots x_n^{\alpha_n} = 1$ and $\mathcal{B}_d$; proving

THEOREM. $\mathfrak{V}^{\#} = \mathcal{B}_d^{\#}$.

2. Proofs.

LEMMA 1. *Let $\mathfrak{V}$ be the variety defined by the law $w(x_1, \dots, x_n) = 1$. If G is an infinite FC-group in $\mathfrak{V}^{\#}$, then $G \in \mathfrak{V}$.*

PROOF. Let $g_1, \dots, g_n \in G$, then $|G : \bigcap_{i=1}^{n} C_G(g_i)| \leq \prod_{i=1}^{n} |G : C_G(g_i)| < \infty$. Thus $\bigcap_{i=1}^{n} C_G(g_i)$ is infinite. Let A be an infinite abelian subgroup of $\bigcap_{i=1}^{n} C_G(g_i)$. Then $A \in \mathfrak{V}$ by Lemma 3 of [3]. Now consider the infinite sets $g_1 A, \dots, g_n A$. By the property $\mathfrak{V}^{\#}$ there exist $a_1, \dots, a_n \in A$, such that $w(g_1 a_1, \dots, g_n a_n) = 1$. So

$$1 = w(g_1, \dots, g_n) w(a_1, \dots, a_n) = w(g_1, \dots, g_n) = 1,$$

and $G \in \mathfrak{V}$. ■

From now on, for nonzero constant integers $\alpha_1, \dots, \alpha_n$ and any nonzero integer α we denote the varieties defined by the laws $w = x_1^{\alpha_1} \dots x_n^{\alpha_n}$ and $w = x^{\alpha}$, respectively, by $\mathfrak{V}$ and $\mathcal{B}_{\alpha}$. Also we denote the greatest common divisor of $\alpha_1, \dots, \alpha_n$ by d.

LEMMA 2. *Let $G \in \mathfrak{V}^{\#}$ be an infinite group. If the set $T = \{g \in G \mid g^d = 1\}$ is infinite, then $G \in \mathfrak{V}$. In particular an infinite group in $\mathfrak{V}^{\#} \setminus \mathfrak{V}$ can not have an infinite subgroup in $\mathfrak{V}$.*

PROOF. Let X be an infinite subset of G. Let $1 \leq j \leq n$ be fixed. By considering the infinite subsets $X_i = T$, $i = 1, \dots, j-1, j+1, \dots, n$ and $X_j = X$, and using the property $\mathfrak{V}^{\#}$, we find the elements $g_i \in T$, $i = 1, \dots, j-1, j+1, \dots, n$ and $x \in X$ such that

$g_1^{\alpha_1} \dots g_{j-1}^{\alpha_{j-1}} x^{\alpha_j} g_{j+1}^{\alpha_{j+1}} \dots g_n^{\alpha_n} = 1$. But $g_i^{\alpha_i} = 1$, so we have $x^{\alpha_j} = 1$. Thus $G \in \mathcal{B}_{\alpha_j}^{\#}$, for all $j = 1, \dots, n$. Since $\mathcal{B}_{\alpha_j}^{\#} = \mathcal{B}_{\alpha_j} \cup \mathcal{F}$, we have $G \in \mathcal{B}_{\alpha_j}$, for all $j = 1, \dots, n$. So $G \in \mathcal{V}$, as required. ■

We define the class $\mathcal{V}(\infty)$ including all groups G in which any infinite subset X contains distinct elements $x_1, \dots, x_n$ such that $x_1^{\alpha_1} \dots x_n^{\alpha_n} = 1$. Also, if $k \geq n$ is an integer, we define the class $\mathcal{V}(k)$ including all groups G in which any subset X with $|X| = k$, contains distinct elements $x_1, \dots, x_n \in X$ such that $x_1^{\alpha_1} \dots x_n^{\alpha_n} = 1$. It is clear that, $\mathcal{V}(k) \subseteq \mathcal{V}(\infty)$ and $\mathcal{V}^{\#} \subseteq \mathcal{V}(\infty)$.

LEMMA 3. *Let $G \in \mathcal{V}(\infty)$ be an infinite group. Then $|H : C_H(g^d)|$ is finite, for any $g \in G$, and any infinite subgroup H of G.*

PROOF. Let H be an infinite subgroup of G and suppose, for a contradiction, that there is an element $g \in G$, such that $|H : C_H(g^d)|$ is infinite. Let T be a right transversal to $C_H(g^d)$ in H. List the elements of T as $t_1, t_2, \dots$ under some well ordering $\leq$, so that $t_i < t_j$ if $i < j$.

Consider the set $T^{(n)}$ of all n-element subsets of T. For each $t \in T^{(n)}$, list the elements $t_{i_1}, \dots, t_{i_n}$ of t in ascending order given by $\leq$, and write $\hat{t} = (t_{i_1}, \dots, t_{i_n})$. Create $n! + 1$ sets, one U_σ for each permutation σ of the set $\{1, 2, \dots, n\}$ and V: For each $t \in T^{(n)}$, $\hat{t} = (t_{i_1}, \dots, t_{i_n})$ put $t \in U_\sigma$ if

$$(g^{t_{i_{\sigma(1)}}})^{\alpha_1} \dots (g^{t_{i_{\sigma(n)}}})^{\alpha_n} = 1 ,$$

and put $t \in V$ if $t \notin U_\sigma$, for any σ.

By Ramsey's Theorem, there exists an infinite subset $T_0 \subseteq T$, such that $T_0^{(n)} \subseteq U_\sigma$, for some σ, or $T_0^{(n)} \subseteq V$. Suppose, for a contradiction, that $T_0^{(n)} \subseteq V$, then the set $\{g^x \mid x \in T_0\}$, is infinite. By the property $\mathcal{V}(\infty)$, there exist n distinct elements $t_1, \dots, t_n \in T_0$, such that $(g^{t_1})^{\alpha_1} \dots (g^{t_n})^{\alpha_n} = 1$, but then $t = \{t_1, \dots, t_n\}$ lies in some U_σ, a contradiction.

Thus $T_0^{(n)} \subseteq U_\sigma$, for some σ. Moreover, by restricting the order $\leq$ to T_0 we may assume that $T_0 = \{t_1, t_2, \dots\}$ and $t_i < t_j$ if $i < j$. Hence for any $i_1 < i_2 < \dots < i_n$, we have

$$(g^{t_{i_{\sigma(1)}}})^{\alpha_1} \dots (g^{t_{i_{\sigma(n)}}})^{\alpha_n} = 1 .$$

Now consider a sequence

$$i_1 < i_2 \dots < i_{n-1} < i_n < i_{n+1} < i_{n+2} < i_{n+3} < \dots < i_{2n+1},$$

of $2n+1$ indices and suppose that $k \in \{1, 2, \ldots, n\}$ is fixed and let $\sigma(s)=k$. Define a sequence $j_1 < \ldots < j_n$ as follows: the first $k-1$ elements are $j_1 = i_1, \ldots, j_{k-1} = i_{k-1}$ (if $k=1$ this is empty), $j_k = i_{n+1}$, and the last $n-k$ elements are $j_{k+1} = i_{n+3}, \ldots, j_n = i_{2n-k+2}$ (if $k=n$ this is empty). Thus

$$(g^{t_{i_{n+1}}})^{\alpha_s} = ((g^{t_{i_{\sigma(1)}}})^{\alpha_1} \ldots (g^{t_{i_{\sigma(s-1)}}})^{\alpha_{s-1}})^{-1}((g^{t_{i_{\sigma(s+1)}}})^{\alpha_{s+1}} \ldots (g^{t_{i_{\sigma(n)}}})^{\alpha_n})^{-1}.$$

Now if we put $j_k = i_{n+2}$ in the above sequence, we have

$$(g^{t_{i_{n+2}}})^{\alpha_s} = ((g^{t_{i_{\sigma(1)}}})^{\alpha_1} \ldots (g^{t_{i_{\sigma(s-1)}}})^{\alpha_{s-1}})^{-1}((g^{t_{i_{\sigma(s+1)}}})^{\alpha_{s+1}} \ldots (g^{t_{i_{\sigma(n)}}})^{\alpha_n})^{-1},$$

and thus $(g^{t_{i_{n+1}}})^{\alpha_s} = (g^{t_{i_{n+2}}})^{\alpha_s}$. Therefore $t_{i_{n+1}} t_{i_{n+2}}^{-1} \in C_H(g^{\alpha_s})$. Now if k runs through the set $\{1, 2, \ldots, n\}$, so does s. Thus

$$t_{i_{n+1}} t_{i_{n+2}}^{-1} \in \bigcap_{s=1}^{n} C_H(g^{\alpha_s}) = C_H(g^d),$$

a contradiction. ■

It is natural to ask whether every infinite $\mathfrak{V}(\infty)$-group belongs to $\mathfrak{V}$.

The answer, in general, is negative. In fact, there is an infinite group satisfying the condition $\mathfrak{V}(k)$ for some positive integer k which does not belong to $\mathfrak{V}$.

For example, we can take a group K (with $m=2$ and n to be a sufficiently large prime, $n > 10^{10}$) in Theorem 31.5 in [8]. This group K is given by all relations of the form $A^{n^2}=1$ and $A^n = B^n$, where A and B run through a special set of words. The group K is 2-generated and of exponent n^2, and it is a central extension of $B(2, n)$ (this is the 2-generated free Burnside group of exponent n) by a cyclic central subgroup $\langle z \rangle$ of order n.

Thus K is infinite, since $B(2, n)$ is infinite, and $K^n = \langle z \rangle$ is a finite (central) cyclic subgroup of K. Now we show that the set X of all elements of order n in K coincides with $\langle z \rangle$ and thus is finite. For this purpose we need to show that all non-central elements of K have order n^2. Take an element x outside $Z(K) = \langle z \rangle$. It follows from the construction of K that x is conjugate in K to a product $A^k z^t$, where $A^n = z$. Then $(A^k z^t)^n = (A^n)^k (z^n)^t = z^k \neq 1$, since k is not divided by n. Thus x is of order n^2 as required.

Let, now $\mathcal{O}$ be the variety defined by the law $x_1^n \ldots x_n^n = 1$. Then $K \in \mathcal{O}(n^2)$. For, let $x_1, \ldots, x_{n^2}$ be distinct elements of K. Thus $x_1^n, \ldots, x_{n^2}^n$

are in K^n and so there exists $\{y_1, \ldots, y_n\} \subset \{x_1, \ldots, x_{n^2}\}$ such that $y_1^n = = \ldots = y_n^n = c \in Z(K) = \langle z \rangle$. Therefore $y_1^n \ldots y_n^n = c^n = 1$, since $|K| = = |Z(K)| = n$.

COROLLARY 4. *Let $G \in \mathfrak{V}^{\#}$ be an infinite group, then $G^d = \langle g^d \mid g \in G \rangle$ is finite.*

PROOF. By Lemma 3, $|G : C_G(g^d)|$ is finite, for all $g \in G$. Thus $G^d \leqslant \leqslant F$, the FC-center of G. If F is infinite then $F \in \mathfrak{V}$, by Lemma 1, and so $G \in \mathfrak{V} = \mathcal{B}_d \cup \mathcal{F}$ by Lemma 2, and thus $G^d = 1$. If F is finite, then G^d is also finite, as required. ■

PROOF OF THE THEOREM. Let G be an infinite group in $\mathfrak{V}^{\#}$ and assume $G \notin \mathfrak{V}$. Then G^d is finite, by Corollary 4. Write $H := C_G(G^d)$. Then G/H is finite and $a^d \in Z(H)$, for any $a \in H$. We show that $H/FC(H)$ has exponent 2 where $FC(H)$ is the FC-centre of H. Let $a \in H$, $a^2 \notin FC(H)$. Then the set $X = \{a^g \mid g \in H\}$ is infinite. Let $X_1 = X_2 = \ldots = X_n = X$. Then, since $G \in \mathfrak{V}^{\#}$, there exist $g_1, \ldots, g_n \in H$ such that

$$(a^{g_1})^{\alpha_1}(a^{g_2})^{\alpha_2} \ldots (a^{g_n})^{\alpha_n} = 1 = a^{\alpha_1} a^{\alpha_2} \ldots a^{\alpha_n}$$

since $a^d \in Z(H)$. Write $\alpha_i = d\beta_i$ for all $i \in \{1, \ldots, n\}$. Then

$$(*) \qquad (a^d)^{\beta_1 + \beta_2 + \ldots + \beta_n} = 1 .$$

Now let $Y = \{(a^2)^g \mid g \in H\}$, then Y is infinite, since $a^2 \notin FC(H)$. For $X_1 = Y$, $X_2 = X_3 = \ldots = X_n = X$, there exist $h_1, \ldots, h_n \in H$ such that $(a^{2h_1})^{\alpha_1}(a^{h_2})^{\alpha_2} \ldots (a^{h_n})^{\alpha_n} = 1$. Then $a^{2\alpha_1 + \alpha_2 + \ldots + \alpha_n} = 1$ and

$$(**) \qquad (a^d)^{2\beta_1 + \beta_2 + \ldots + \beta_n} = 1 .$$

From $(*)$ and $(**)$, we get $(a^d)^{\beta_1} = 1$, and arguing similarly with $X = = X_1 = \ldots = X_{i-1} = X_{i+1} = \ldots = X_n$ and $X_i = Y$, we obtain $(a^d)^{\beta_i} = 1$ for any i in the set $\{1, \ldots, n\}$. Then $a^d = 1$, since $\gcd(\beta_1, \beta_2, \ldots, \beta_n) = 1$. But from $a^d = 1$, we have $(a^g)^d = 1$, for any g in H and, by Lemma 2, the set $\{a^g \mid g \in H\}$ is finite, that is $a \in FC(H)$, a contradiction.

Therefore $H/FC(H)$ is a locally finite group, $FC(H)$ is finite by Lemmas 1 and 2, and so H is locally finite. Then H can not be infinite by Lemma 2. Thus H is finite which forces G to be finite. This is a contradiction, which completes the proof. ■

REFERENCES

[1] A. ABDOLLAHI, *A characterization of infinite 3-abelian groups*, Arch. Math. (Basel), **73** (1999), pp. 104-108.
[2] C. DELIZIA, *On groups with a nilpotence condition on infinite subsets*, Algebra Colloq., **2** (1995), pp. 97-104.
[3] G. ENDIMIONI, *On a combinatorial problem in varieties of groups*, Comm. Algebra, **23** (1995), pp. 5297-5307.
[4] P. S. KIM - A. H. RHEMTULLA - H. SMITH, *A characterization of infinite metabelian groups*, Houston J. Math., **17** (1991), pp. 429-437.
[5] P. LONGOBARDI - M. MAJ, *A finiteness condition concerning commutators in groups*, Houston J. Math., **19** (1993), pp. 505-512.
[6] P. LONGOBARDI - M. MAJ - A. RHEMTULLA, *Infinite groups in a given variety and Ramsey's theorem*, Comm. Algebra, **20** (1992), pp. 127-139.
[7] B. H. NEUMANN, *A problem of Paul Erdös on groups*, J. Austral. Math. Soc. (Series A), **21** (1976), pp. 467-472.
[8] A. YU. OL'SHANSKII, *Geometry of defining relations in groups*, Kluwer Academic Publishers (1991).
[9] D. J. S. ROBINSON, *A course in the theory of groups*, Springer-Verlag, New York (1982).
[10] L. S. SPIEZIA, *A property of the variety of 2-Engel groups*, Rend. Sem. Mat. Univ. Padova, **91** (1994), pp. 225-228.
[11] L. S. SPIEZIA, *A characterization of third Engel groups*, Arch. Math. (Basel), **64** (1995), pp. 369-373.

Manoscritto pervenuto in redazione il 10 maggio 1999