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The Gross-Koblitz formula revisited


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The Gross-Koblitz Formula Revisited.

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The formula in question gives an explicit value of Gauss sums using the $p$-adic gamma function of Morita. We give here an elementary proof of this formula (valid for all primes). Let me thank L. van Hamme who stimulated me to find such a proof, and A. Junod who helped me to understand [2], which has been my starting point.

1. Preliminary comments on numeration.

Let $q = p^f$ $(f \geq 1)$ be a power of a prime $p$. Each affine map

$$x \mapsto a + qx : \mathbb{Z}_p \rightarrow \mathbb{Z}_p \quad (a \in \mathbb{Z}_p)$$

has a unique fixed point

$$a_\ast = \frac{a}{1 - q} = a + aq + aq^2 + \ldots = a + q(a + aq + aq^2 + \ldots)_\ast.$$

When $a$ is an integer in the interval $0 \leq a < q$, say with $p$-adic expansion

$$a = a_0 + a_1 p + \ldots + a_{f-1} p^{f-1} \quad (0 \leq a_j < p),$$

the fixed point of the corresponding affine transformation has a periodic $p$-adic expansion given by $a + aq + aq^2 + \ldots$ (period of length $f$). Let us write

$$a_\ast = a_0 + p(a_1 + a_2 p + \ldots + a_{f-1} p^{f-2} + a_0 p^{f-1} + \ldots) = a_0 + pa'_\ast.$$

We recognize in $a'_\ast$ the fixed point of the affine map corresponding to

$$a' = a_1 + a_2 p + \ldots + a_{f-1} p^{f-2} + a_0 p^{f-1},$$

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and we observe that $a'$ is obtained from $a$ by a cyclic permutation of its digits. Iterating the procedure, we can write

$$a'_* = a_1 + pa''_*, \quad a''_* = \frac{a''}{1 - q}, \ldots$$

In this way, we obtain a cycle of integers in the interval $\{0, \ldots, q-1\}$

$$a', a'', \ldots, a^{(f-1)}, a^{(f)} = a$$

having $p$-adic expansions obtained by cyclic permutations from that of $a$.

2. $p$-adic extensions of quotients of factorials.

For any prime $p$ and $0 \leq a < p$, the relation

$$(*) \quad \frac{(a + pn)!}{p^n n!} = \frac{(a + pn)!}{(p1)(p2) \ldots (pn)} = (-1)^a + pm + 1 \Gamma_p(a + pn + 1)$$

shows that

$$n \mapsto (-1)^{pn} \frac{(a + pn)!}{p^n n!}$$

has a continuous extension $\mathbb{Z}_p \to \mathbb{Z}_p^\times \subset \mathbb{Q}_p$ given by

$$x \mapsto (-1)^a + pm + 1 \Gamma_p(a + px + 1).$$

This simply follows from the definition of the $\Gamma_p$-function by Morita.

Let us generalize this observation to the case of quotients $m \mapsto (a + qm)!/m!$ when $0 \leq a < q = p^f (f \geq 1)$.

We can introduce the $p$-adic expansion of $a$, say

$$a = a_0 + a_1 p + \ldots + a_{f-1} p^{f-1},$$

and write

$$a + qm = a_0 + p(a_1 + \ldots + a_{f-1} p^{f-2} + p^{f-1}m).$$

Put $n_0 = a + qm = a_0 + pn_1$ and successively

$$n_0 = a_0 + a_1 p + p^2 n_2, \quad n_1 = a_1 + pn_2, \quad \text{etc.}$$

hence with

$$n_1 = a_1 + \ldots + a_{f-1} p^{f-2} + p^{f-1}m = a_1 + p(a_2 + \ldots + a_{f-1} p^{f-3} + p^{f-2}m), \quad \text{etc.}$$
Let us write a telescopic product \((n_0 = a + qm, n_f = m)\)

\[
\frac{(a + qm)!}{m!} = \frac{n_0!}{n_1!} \frac{n_1!}{n_2!} \cdots \frac{n_{f-1}!}{n_f!}
\]

\[
= \frac{(a_0 + pn_1)!}{n_1!} \frac{(a_1 + pn_2)!}{n_2!} \cdots \frac{(a_{f-1} + pm)!}{m!}
\]

\[
= \pm p^{n_1} \Gamma_p(a_0 + pn_1 + 1) \cdot p^{n_2} \Gamma_p(a_1 + pn_2 + 1) \cdots p^{n_{f-1} + 1} \Gamma_p(n_{f-1} + 1),
\]

Recalling \((*)\) in the form

\[
\frac{(a + qm)!}{p^{n_1 + \ldots + n_f} m!} = \pm \prod_{a_0 + pn_1 = n_0} \Gamma_p(a_i + pn_i + 1) = \pm \prod_{0 \leq i < f} \Gamma_p(n_i + 1).
\]

we see that the precise sign is \((-1)^{a_0 + pn + 1} p^n \Gamma_p(a + pn + 1)\). Moreover, the sum \(\sigma = n_1 + \ldots + n_f\) may be computed as follows:

\[
n_1 = \left\lfloor \frac{a}{p} \right\rfloor + p^{f-1}m
\]

\[
n_2 = \left\lfloor \frac{a}{p^2} \right\rfloor + p^{f-2}m
\]

\[
\ldots = \ldots
\]

\[
n_{f-1} = \left\lfloor \frac{a}{p^{f-1}} \right\rfloor + pm
\]

\[
n_f = \frac{q - 1}{p - 1} m
\]

\[
\sigma = \text{ord}_p a! + \frac{q - 1}{p - 1} m
\]
so that \( n_0 + \ldots + n_{f-1} = n_0 + \sigma - n_f = a + (q-1)m + \sigma. \) Hence

\[
\frac{(a + qm)!}{p^\sigma m!} = (-1)^{\sigma + (q-1)m + a + \sigma} \prod_{0 \leq i < f} \Gamma_p(n_i + 1),
\]

\[
\frac{(a + qm)!}{(-p)^m m!} = (-1)^{\sigma + (q-1)m + a} \prod_{0 \leq i < f} \Gamma_p(n_i + 1).
\]

**THEOREM 1.** For a fixed power \( q = p^f (f \geq 1) \) of \( p \), the functions

\[
m \mapsto \frac{(a + qm)!}{(-p)^{\sigma} m!} \quad (0 \leq a < q)
\]

admit continuous extensions \( \mathbb{Z}_p \to \mathbb{Q}_p \) given by

\[
x \mapsto (-1)^{(q-1)m + f + a} (-p)^{\text{ord}_p \alpha} \prod_{0 \leq i < f} \Gamma_p\left(\frac{a}{p^i}\right) + p^{f-i} x + 1 \quad \blacksquare
\]

When the prime \( p \) is odd, \( q-1 \) is even and \( (-1)^{(q-1)m} = +1 \). Hence this sign is relevant only if \( p = 2 \) in which case it is \( \varepsilon(m) = (-1)^m \): let \( \varepsilon \) denote the character sign having kernel \( 2\mathbb{Z}_2 \)

\[
\varepsilon(x) = \begin{cases} +1 & \text{if } x \in 2\mathbb{Z}_2 \\ -1 & \text{if } x \in 1 + 2\mathbb{Z}_2. \end{cases}
\]

We shall be interested in the inverse of the preceding functions. Thus we define continuous functions \( G_a : \mathbb{Z}_p \to \mathbb{Q}_p \) \((0 \leq a < q)\) with

\[
G_a(x) = \varepsilon(x)(-1)^{f+a}\prod_{0 \leq i < f} \Gamma_p(n_i(x) + 1),
\]

\[
G_a(m) = (-p)^{\frac{q-1}{f} - m} \frac{m!}{(a + qm)!} \quad (m \geq 0)
\]

\((\varepsilon = 1 \text{ if } p \neq 2)\). Let us use the Legendre formula to simplify the preceding expressions. When \( p \geq 3 \) is odd, \( \varepsilon = 1 \) and

\[
\frac{1}{\Gamma_p(n_i(x) + 1)} = (-1)^{1+a_i} \Gamma_p(-n_i(x)).
\]
Moreover \( \sum a_i \equiv \sum a_i p^i \equiv a \mod 2 \), so that

\[
G_a(x) = \frac{1}{(-p)^{\text{ord}_p a!}} \prod_{0 \leq i < f} \Gamma_p(-n_i(x)).
\]

This formula is also true when \( p = 2 \), because the Legendre formula is now

\[
\frac{1}{\Gamma_2(n_i(x) + 1)} = (-1)^{1+a_i+a_{i+1}} \Gamma_2(-n_i(x)),
\]

and the product leads to an exponent of \(-1\) equal to

\[
f + (a_0 + a_1) + (a_1 + a_2) + \ldots + (a_{f-1} + x_0) \equiv f + a_0 + x_0 \mod 2.
\]

Since \( \varepsilon(x) = (-1)^{x_0} \) we have \( \varepsilon(x)(-1)^{f + a}(-1)^{f + a_0 + x_0} = 1 \) and there only remains

\[
G_a(x) = \frac{1}{(-2)^{\text{ord}_2 a!}} \prod_{0 \leq i < f} \Gamma_2(-n_i(x)).
\]

3. Mahler coefficients of the functions \( G_a \).

Let us choose a nonzero root \( \pi \in C_p \) of \( X + \frac{1}{p} X^p = 0 \). We have

\[
\pi^{\text{ord}_p a!} = \pi^{a - S_p(a)},
\]

so that

\[
\pi^a G_a(x) = \pi^{S_p(a)} \prod_{0 \leq i < f} \Gamma_p(-n_i(x))
\]

for all primes \( p \). This expression is especially simple at the fixed point \( x = a_* \) of the map \( x \mapsto a + qx \), since in this case

\[
n_i(x) = n_i(a_*) = a_i + a_{i+1} + p + \ldots = \frac{a^{(i)}}{1-q}
\]

are obtained by a cyclic permutation from \( a_* \)

\[
\pi^a G_a(a_*) = \pi^{S_p(a)} \prod_{0 \leq i < f} \Gamma_p\left(\frac{a^{(i)}}{q-1}\right).
\]

It turns out that the Mahler coefficients of the functions \( G_a \) are linked to
the coefficients of the Dwork exponential

\[ \Theta_q(T) = e^{\pi(T - T^q)} = \sum_{n \geq 0} A_n T^n = 1 + \pi T + T^2(...). \]

**Theorem 2.** For \(0 \leq a < q\), the Mahler expansion of \(G_a: N \to \mathbb{Q} \subset \mathbb{Q}_p\) is

\[ G_a(x) = \sum_{k \geq 0} \frac{A_{a+kq}}{\pi^{a+k}} \binom{x}{k}, \]

\[ \tilde{G}_a(x) = \pi^a G_a(x) = \sum_{k \geq 0} \frac{A_{a+kq}}{\pi^k}(x)_k = A_a + \frac{A_{a+q}}{\pi} x + \ldots \]

The proof of this result obviously involves some formal manipulations of power series. These are made easier if we use the **Atkin operators** (*). Let us recall their definition and formal properties. The operator \(U_q\) is defined on formal Laurent series by

\[ f = \sum a_n T^n \mapsto U_q(f) = \sum a_qn T^n. \]

Obviously

\[ T^j U_q(f) = U_q(T^{qj} f), \quad g(T) U_q(f) = U_q(g(T^q) f). \]

For example, replacing \(f\) by \(e^{\pi T} f\) and letting \(g = e^{-\pi T}\), we find

\[ e^{-\pi T} U_q(e^{\pi T} f) = U_q(e^{-\pi T^q} e^{\pi T} f) = U_q(\Theta_q(T) f). \]

This is the reason for the appearance of the Dwork exponential in this context. Observe that the action of the Atkin operator forgets all coefficients \(a_i\) having an index \(i\) not multiple of \(q\) e.g. \(U_q\left( \sum_{n > q} a_n T^n \right) = U_q\left( \sum_{n \geq 0} a_n T^n \right)\), and also

\[ U_q\left( \sum_{n \geq q} a_n T^n \right) = U_q\left( \sum_{n \geq 0} a_n T^n \right) \quad (0 \leq a < q). \]

We shall use twice this observation in the next computation (and indicate it by a «!» on the concerned equality).

(*) Also called «Dwork \(\psi\)-operators» or «Hecke» operators.
Proof of Theorem 2. Let us recall the Boole relation linking the values of a function $f$ to its Mahler coefficients $c_k(f)$

$$e^{-T} \sum_{m \geq 0} f(m) \frac{T^m}{m!} = \sum_{k \geq 0} c_k(f) \frac{T^k}{k!}.$$ 

Take $f = Ga$ and replace the indeterminate $T$ by $\pi T$

$$e^{-\pi T} \sum_{m \geq 0} G_a(m) \frac{\pi^m T^m}{m!} = \sum_{k \geq 0} c_k(G_a) \frac{(\pi T)^k}{k!}.$$ 

Let us now compute the left-hand side, recalling that $(-p)^{\frac{1}{p-1}} = \pi$

$$e^{-\pi T} \sum_{m \geq 0} G_a(m) \frac{\pi^m T^m}{m!}$$

$$= e^{-\pi T} \sum_{m \geq 0} \frac{\pi^{(q-1)m}}{(a + qm)!} \frac{\pi^m T^m}{m!} = e^{-\pi T} \sum_{m \geq 0} \frac{\pi^{qm} \eta^h}{(a + qm)!} \frac{T^m}{m!}$$

$$= e^{-\pi T} U_q \left( \sum_{m \geq 0} \frac{\pi^m T^m}{(a + qm)!} \right) = e^{-\pi T} U_q \left( \sum_{n \geq 0} \frac{\pi^n T^n}{n!} \right)$$

$$= e^{-\pi T} U_q \left( \sum_{n \geq 0} \frac{\pi^n T^n}{\pi^a} \right) = e^{-\pi T} U_q \left( e^{\pi T} \frac{T^{-a}}{\pi^a} \right)$$

$$= U_q \left( \Theta_q(T) \frac{T^{-a}}{\pi^a} \right) = U_q \left( \sum_{n \geq 0} \frac{A_n}{\pi^a} T^{-a} \right)$$

$$= U_q \left( \sum_{n \geq -a} \frac{A_n + a}{\pi^a} T^n \right) = U_q \left( \sum_{n \geq 0} \frac{A_{n-a}}{\pi^a} T^n \right)$$

$$= \sum_{k \geq 0} \frac{A_{a+kq}}{\pi^a} T^k = \sum_{k \geq 0} \frac{A_{a+kq}}{\pi^{a+k} \eta^h} \frac{(\pi T)^k}{k!}.$$ 

This proves the announced formula. ■

Comment. Note that the coefficients $A_n$ of the expansion of the Dwork exponential $\Theta_q$ depend on the power $q = p^f$ and the choice of root $\pi$ such that $\pi^{p-1} = -p$. If we replace $\pi$ by another choice $\zeta \pi$ where $\zeta^{p-1} = 1$, the coefficient $A_n$ is replaced by $\zeta^n A_n$. Since $\zeta = \zeta^p = \ldots = \zeta^q$
implies $\zeta^k = \zeta^{q^k}$, we see that the coefficients $\frac{A_{n + qk}}{n^{qk}}$ are unchanged. On the other hand, these coefficients belong to $Q_p$ simply since they are Mahler coefficients of a $Q_p$-valued continuous function.

4. Gauss sums.

The Gross-Koblitz formula concerns the Gauss sums

$$- \sum_{\varepsilon^q = \varepsilon \not= 0} \varepsilon^{-a} \Theta_q(\varepsilon) = - \sum_{\varepsilon^q = 1} \varepsilon^{-a} \sum_{n \geq 0} A_n \varepsilon^n = - \sum_{n \geq 0} A_n \sum_{\varepsilon^q = 1} \varepsilon^{n-a}$$

(the sign "−" is chosen in order to give it the value +1 when $a = 0$). The sum on roots of unity is $q - 1$ if $n - a$ is a multiple of $q' = q - 1$ and is 0 otherwise. If $a = q - 1 = q'$, we have to take into account the value $k = -1$. Let us assume that $0 \leq a < q'$, so that only the values $k \geq 0$ occur

$$- \sum_{\varepsilon^q = \varepsilon \not= 0} \varepsilon^{-a} \Theta_q(\varepsilon) = (1 - q) \sum_{k \geq 0} A_{a + kq'}.$$

The above Mahler series involve the coefficients of the Dwork exponential having indices in arithmetic progressions of ratio $q$, whereas we are looking for a summation formula for these coefficients with indices in an arithmetic progression of ratio $q' = q - 1$. Here is a link between the two.

**Lemma.** We have $nA_n = \pi A_{n-1}$ ($1 \leq n < q$), $nA_n = \pi (A_{n-1} - qA_{n-q})$ ($n \geq q$).

**Proof.** We differentiate the defining identity

$$\sum_{n \geq 0} A_n T^n = \Theta_q(T) = e^{\pi(T - T^q)}$$

$$\sum_{n \geq 0} nA_n T^{n-1} = \Theta_q(T)' = e^{\pi(T - T^q)}(\pi - q\pi T^{q-1})$$

$$\sum_{n \geq 0} nA_n T^{n-1} = \Theta_q(T)(\pi - q\pi T^{q-1}) = \sum_{n \geq 0} A_n T^n(\pi - q\pi T^{q-1}).$$

The identification of the coefficients of $T^{n-1}$ leads to the result. ■
Let us define functions $\tilde{G}_a$ for all integers $a \geq 0$ by

$$\tilde{G}_a(x) = \sum_{k \geq 0} \frac{A_{a+qk}}{\pi^k} (x)_k = A_a + \frac{A_{a+q}}{\pi} x + \frac{A_{a+2q}}{\pi^2} x(x-1) + \ldots$$

This definition extends the preceding one (given only for $a = a_q$), but let us emphasize that when $a \geq q$, these functions are not simply given by products of $\Gamma_p$ as in the previous case.

**Theorem 3.** For $a \geq 0$, $a_* = \frac{a}{1-q}$, and $q' = q-1$ we have

$$(1-q) \sum_{0 \leq k < N} A_{a+qk'x} = \tilde{G}_a(a_*) - \tilde{G}_a + Nq'(a_* - N) \quad (N \geq 1).$$

**Proof.** The crucial case is $N = 1$:

$$\tilde{G}_a(a_*) - \tilde{G}_{a+q}(a_* - 1) = (1-q) A_a.$$

To compute $\tilde{G}_a(x) - \tilde{G}_{a+q}(x-1)$, we first transform its second term

$$\tilde{G}_{a+q}(x-1) = \sum_{k \geq 0} A_{a+qk} \frac{(x-1)_k}{\pi^k}.$$

Since $a + q' + kq = a + (k+1)q - 1 = n - 1$, we can use the relation (lemma)

$$A_{n-1} = \frac{n}{\pi} A_n + qA_{n-q} \quad (n \geq q),$$

to bring back the sequence of indices into arithmetic progressions of ratio $q$

$$\tilde{G}_{a+q}(x-1) = \sum_{k \geq 0} \left[ \frac{a + (k+1)q}{\pi} A_{a+(k+1)q} + qA_{a+kq} \right] \frac{(x-1)_k}{\pi^k}.$$

Hence $\tilde{G}_a(x) - \tilde{G}_{a+q}(x-1)$ is equal to

$$A_a + \sum_{k \geq 1} A_{a+kq} \frac{(x-1)_{k-1}}{\pi^k} (x-a-kq) - \sum_{k \geq 0} qA_{a+kq} \frac{(x-1)_{k-1}}{\pi^k} (x-k).$$
A miracle happens when $x$ is equal to the fixed point $a_*$:

$$a_* - a - kq = q(a_* - k),$$

so that all terms compensate except $k = 0$, whence the first formula in the theorem. Summing up consecutive expressions and noting that $(\alpha + \sqrt{q})_* = a_* - 1$, we obtain a telescopic sum

$$\tilde{G}_{\alpha} (a_*) - \tilde{G}_{\alpha+Nq'} (a_* - N) = (1 - q) \sum_{0 \leq k < N} A_{a + kq}. \quad \blacksquare$$

More generally,

$$\frac{x - \alpha - kq}{x - k} - q = \frac{x - \alpha - kq - qx + qk}{x - k} = \frac{x - (\alpha + qx)}{x - k},$$

and remembering $a = (1 - q) a_*$

$$\frac{x - \alpha - kq}{x - k} - q = \frac{x - \alpha_* + q\alpha_* - qx}{x - k} = (x - \alpha_*) \frac{1 - q}{x - k},$$

hence the more general formula

$$\tilde{G}_{\alpha}(x) - \tilde{G}_{\alpha+q'}(x-1) = (1 - q) A_{\alpha} + (x - \alpha_*) \frac{1 - q}{\pi} \sum_{k \geq 0} \frac{A_{a + (k + 1)q}}{\pi^k} (x - 1)_k.$$  

It is well known that the Dwork exponential converges in a ball of radius $> 1$, hence $A_n \to 0 \ (n \to \infty)$ so that we may go to the limit

$$ (1 - q) \sum_{k \geq 0} A_{a + kq}, \quad \text{(Gauss-Dwork sum)} $$

$$= \tilde{G}_{\alpha}(a_*) - \lim_{N \to \infty} \tilde{G}_{\alpha+Nq'}(a_* - N).$$

The limit vanishes in view of the following lemma since $a_* - N \in \mathbb{Z}_p$.

**Lemma.** We have $\| \tilde{G}_{\alpha} \| \to 0 \ (\alpha \to \infty)$. More precisely

$$\| \tilde{G}_{\alpha} \| \leq \begin{cases} \|a^{\alpha/q}_p \| & \text{if } p \geq 3 \\ \|a^{(\alpha-q)/2q}_p \| & \text{if } p = 2. \end{cases}$$
PROOF. The norm used here is the sup norm on the unit ball, so that
\[ \|\tilde{G}_\alpha\| \leq \sup_{k \geq 0} \left| \frac{A_\alpha + kq}{\pi^k/k!} \right| \]
(the Mahler theorem states that this is in fact an equality, provided that the sup norm is taken on the unit ball of \( C_p \)). But
\[ \left| \frac{\pi^k}{k!} \right| = \frac{r_p^k}{\prod_{\text{ord}_p k!}} = \frac{r_p^{k-(k-S_p(k))}}{r_p^{S_p(k)}}. \]
On the other hand, the Dwork series \( \Theta_q(T) = e^{q(T-T^q)} = \sum A_n T^n \) is bounded by 1 on the ball of radius \( |p|^{1-p/q} > 1 \)
\[ |A_n| |p|^{n-1/p} \leq 1, \quad |A_n| \leq |p|^{n-1/p} = r_p^{n(p-1)^2/pq}. \]
This leads to
\[ \left| \frac{A_\alpha + kq}{\pi^k/k!} \right| \leq \frac{r_p^{(\alpha + kq)(p-1)^2/pq}}{r_p^{S_p(k)}}. \]

(1) Case \( p \geq 3 \) is odd. In this case, we use the minoration \( r_p^{S_p(k)} \geq r_p^k \) of the denominator. The exponent of \( r_p \) is easily estimated
\[ (\alpha + kq) \frac{(p-1)^2}{pq} - k = \alpha \frac{(p-1)^2}{pq} + k \left( \frac{(p-1)^2}{p} - 1 \right). \]
As \( p \geq 3 \), \( \frac{p-1}{p} > \frac{1}{2} \) and
\[ (\alpha + kq) \frac{(p-1)^2}{pq} - k \geq \alpha \frac{p-1}{2q} + k \frac{p-3}{2} \geq \alpha \frac{1}{q}. \]
Hence this exponent of \( r_p \) is greater or equal to \( \alpha/q \) whence the first assertion.

(2) Case \( p = 2 \). The preceding minoration of the denominator is not precise enough to lead to the result. This is why we keep
scrupulously the exponent $S_2(k)$ and have now to estimate

\[(\alpha + kq) \frac{(p-1)^2}{pq} - S_p(k) = (\alpha + kq) \frac{1}{2q} - S_2(k) = \frac{\alpha}{2q} + \frac{k}{2} - S_2(k).\]

But the following table shows $k - S_2(k) > -1/2$ (it is a simple exercise to prove it formally) which finishes the proof. ■

Summing up, we have obtained the main result.

**Theorem 4** (Gross-Koblitz). For $0 \leq a < q - 1$ $(q = p^f, f \geq 1)$, we have

\[\sum_{\varepsilon^{q-1}} \varepsilon^{-a} \Theta_q(\varepsilon) = \pi^{S_p(a)} \prod_{0 \leq i < f} \Gamma_p \left( \frac{a^{(i)}}{q-1} \right)\]

where the integers $0 \leq a^{(i)} < q - 1$ have $p$-adic expansions obtained by cyclic permutation from that of $a$, and $S_p(a)$ is the sum of digits of $a$ in base $p$.

Since the values of $\Gamma_p$ are $p$-adic units, we deduce the following result.

**Corollary 1** (Stickelberger). For $0 \leq a < q$, the $p$-adic absolute value of the Gauss sum $\sum_{\varepsilon^{q-1}} \varepsilon^{-a} \Theta_q(\varepsilon)$ is

\[|\pi^{S_p(a)}| = r_p^{S_p(a)} = \left| p \right|_p^{S_p(a)}.\]

**Corollary 2.** When $p \equiv 1 \mod n$, the values of $\Gamma_p$ at the rational numbers $\frac{a}{n}$ are algebraic numbers. More precisely

\[\Gamma_p \left( \frac{a}{n} \right) \in Q(\mu_{np}, n\sqrt{-p}).\]
PROOF. By the functional equation of $\Gamma_p$, it is enough to establish this when $0 \leq m < n$. If we write $p - 1 = ln$ and $\frac{m}{n} = \frac{lm}{p - 1}$, we can use the Gross-Koblitz formula for $q = p$ and $a = lm$. ■

APPENDIX 1. For an odd prime $p \geq 3$, the Legendre relation for $\Gamma_p$ is

$$\Gamma_p(x) \Gamma_p(1 - x) = (-1)^{R(x)}$$

where $R(x) \in \{1, \ldots, p\}$ is in the class of $x \mod p$. Let us write it in the equivalent form

$$\Gamma_p(-x) \Gamma_p(x + 1) = (-1)^{R(-x)} = (-1)^{p - x_0} = (-1)^{1 + x_0} \quad (x = x_0 + x_1p + \ldots)$$

For $p = 2$ and $x = x_0 + x_12 + x_22^2 + \ldots$, we have

$$\Gamma_2(x) \Gamma_2(1 - x) = (-1)^{1 + x_1},$$

$$\Gamma_2(-x) \Gamma_2(x + 1) = (-1)^{1 + x_0 + x_1}.$$ One way of unifying the two cases consists in writing

$$\Gamma_p(-x) \Gamma_p(x + 1) = (-1)^{1 + x_0 + (p - 1)x_1}.$$ 

APPENDIX 2. It is well known that the $\Theta_q(\varepsilon) \in C_p$ are $p$th roots of unity (Dwork’s theorem). We can observe

$$\Theta_q(T) = \Theta_p(T) \Theta_p(T^p) \ldots \Theta_p(T^{q/p})$$

$$= 1 + \pi(T + T^p + \ldots + T^{q/p}) + \ldots$$

so that

$$\Theta_q(\varepsilon) \equiv 1 + \pi(\varepsilon + \varepsilon^p + \ldots + \varepsilon^{q/p}) \mod \pi^2$$

$$\Theta_q(t(x)) = \xi^{t(x)/p}, \quad \xi = \Theta_p(1) \quad (t: \text{Teichmüller})$$

and the Gauss sums considered here are precisely Gauss sums for the field $F_q$.

APPENDIX 3. The Atkin operators still satisfy

$$U_q(f)(T^q) = \sum a_qT^{q^n} = q^{-1}\sum_{\xi \in \mu_q} f(\xi T)$$

(often used for $q = p$). On the other hand, the operator $\delta = T(d/dT)$ is the
degree operator: it sends $T^n$ onto $nT^n$ hence

$$\delta = T \frac{d}{dT} : \sum_{n \geq 0} a_n T^n \mapsto \sum_{n \geq 0} n a_n T^n.$$  

From this, the relation $U_q \circ \delta = q(\delta \circ U_q)$ immediately follows

$$U_q \circ \delta \left( \sum_{n \geq 0} a_n T^n \right) = \sum_{n \geq 0} q n a_{qn} T^n = q \sum_{n \geq 0} n a_{qn} T^n = q \delta \circ U_q \left( \sum_{n \geq 0} a_n T^n \right).$$

REFERENCES


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