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Groups in which Certain Equations have Many Solutions.

Gérard Endimioni (*)

1. Introduction.

Let \( w(x_1, \ldots, x_n) \) be a word in the free group of rank \( n \) and let \( G \) be a group. There are several ways to mean that the equation \( w(x_1, \ldots, x_n) = 1 \) has «many» solutions in \( G \). Here we adopt a combinatorial point of view and we define the class of groups \( \mathcal{V}_\infty(w) \) like this:

A group \( G \) belongs to \( \mathcal{V}_\infty(w) \) if and only if every infinite subset of \( G \) contains \( n \) (distinct) elements \( x_1, \ldots, x_n \) such that \( w(x_1, \ldots, x_n) = 1 \).

Following a question of P. Erdős, this class appeared in a paper of B. H. Neumann [6], where he proved that if \( w(x_1, x_2) = [x_1, x_2] \), then \( \mathcal{V}_\infty(w) \) coincide with the class of central-by-finite groups. Since this first paper, several authors have studied \( \mathcal{V}_\infty(w) \). For example, characterizations of finitely generated soluble groups of \( \mathcal{V}_\infty(w) \) are known when \( w(x_1, x_2) = [x_1, x_2, x_2] \) [5] or when \( w(x_1, x_2) = [x_1, x_2, x_2, x_2] \) [1].

In this paper, we consider the word \( w(x_1, \ldots, x_n) = x_1^{a_1}x_2^{a_2}\ldots x_n^{a_n} \), where \( a_1, a_2, \ldots, a_n \) are nonzero integers. Related to this question, but with a stronger condition, A. Abdollahi and B. Taeri proved that if for every \( n \) infinite subsets \( X_1, \ldots, X_n \) of an infinite group \( G \), there exist elements \( x_1 \in X_1, \ldots, x_n \in X_n \) such that \( x_1^{a_1}\ldots x_n^{a_n} = 1 \), then \( x_1^{a_1}\ldots x_n^{a_n} = 1 \) is a law in \( G \) [2]. On the other hand, by using a construction of Ol’shanskii, these authors showed that for any sufficiently large prime \( n \), there exists

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an infinite group in $\mathcal{V}_{\infty}(x_1^n \ldots x_n^n)$ in which $x_1^n \ldots x_n^n = 1$ is not a law (we shall see other examples in the next section). The aim of this paper is to characterize the groups of $\mathcal{V}_{\infty}(x_1^{a_1} \ldots x_n^{a_n})$.

2. - Results.

We denote by $\mathcal{F}$ the class of finite groups and by $\mathcal{B}_e$ the variety of groups satisfying the law $x^e = 1$ (for a given integer $e$).

Let $m$ be a positive integer. By analogy with $\mathcal{V}_{\infty}(w)$, we define the class $\mathcal{V}_m(w)$ in the following way: a group $G$ belongs to $\mathcal{V}_m(w)$ if and only if every $m$-element subset of $G$ contains $n$ distinct elements $x_1, \ldots, x_n$ such that $w(x_1, \ldots, x_n) = 1$. Clearly, the classes $\mathcal{V}_m(w)$ and $\mathcal{F}$ are included in $\mathcal{V}_{\infty}(w)$.

From now on, we put $w(x_1, \ldots, x_n) = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$, where $\alpha_1, \ldots, \alpha_n$ are nonzero given integers; we write $\alpha$ for the greatest common divisor of these integers. Observe that the variety defined by the law $w(x_1, \ldots, x_n) = 1$ is equal to the variety $\mathcal{B}_\alpha$. Here we prove that as one might expect, the classes $\mathcal{V}_{\infty}(w)$ and $\mathcal{B}_\alpha$ do not coincide but are relatively close:

**Theorem.** Let $G$ be an infinite group. The following assertions are equivalent:

(i) $G \in \mathcal{V}_{\infty}(w)$;

(ii) $G \in \mathcal{B}_{(\alpha_1 + \ldots + \alpha_n)} \cap (\mathcal{F}\mathcal{B}_\alpha)$;

(iii) $G \in \mathcal{V}_m(w)$ for some positive integer $m$.

It follows immediately:

**Corollary 1.** We have the equalities

$$\mathcal{V}_{\infty}(w) = \bigcup_{m > 0} \mathcal{V}_m(w) = \mathcal{F} \cup (\mathcal{B}_{(\alpha_1 + \ldots + \alpha_n)} \cap (\mathcal{F}\mathcal{B}_\alpha)).$$

For example, for any fixed integer $e \geq 2$, denote by $H$ a cyclic group of order $e^2$ and by $K$ the direct product of infinitely many cyclic groups of order $e$. Let $G$ be the direct product of $H$ and $K$. It is easy to see directly that $G$ belongs to $\mathcal{V}_{e + 1}(x_1^e x_2^{-e})$ and so to $\mathcal{V}_{\infty}(x_1^e x_2^{-e})$ (also it is a consequence of our theorem above). However, $x_1^e x_2^{-e} = 1$ is not a law in $G$. No-
tice that contrary to the example given in [2] and quoted above, $G$ is not finitely generated. In fact, when $\alpha$ is "small," finitely generated groups in $\mathcal{V}_\alpha(w)$ are finite. More precisely, since the Burnside problem has a positive answer when the exponent belongs to $\{1, 2, 3, 4, 6\}$ (that is, every group of $\mathcal{B}_\alpha$ is locally finite when $\alpha \in \{1, 2, 3, 4, 6\}$), we may state:

**Corollary 2.** If $\alpha \in \{1, 2, 3, 4, 6\}$, every finitely generated group in $\mathcal{V}_\alpha(w)$ is finite.

Also, notice that $\mathcal{V}_\alpha(w) = \mathcal{F}$ if $\alpha = 1$; this improves Corollary 2 of [3].

3. – Proofs.

We start with a key result for the proof of the theorem:

**Lemma 1.** Let $n$ be a positive integer and let $a_1, \ldots, a_n$ be elements of an infinite group $G$. Let $\alpha_1, \ldots, \alpha_n$ be nonzero integers. Suppose that $G$ contains an infinite subset $E$ satisfying the following property: each infinite subset $E' \subset E$ contains $n$ (distinct) elements $x_1, \ldots, x_n$ such that $\alpha_1 x_1^{\alpha_1} \cdots a_n x_n^{\alpha_n} = 1$. Then:

(i) there exist an infinite subset $F \subset E$ and elements $c_1, \ldots, c_n$ of $G$ such that, for each $i \in \{1, \ldots, n\}$, we have $x^{\alpha_i} = c_i$ for all $x \in F$;

(ii) there exists an element $c$ of $G$ such that $x^c = c$ for all $x \in F$, where $\alpha = \gcd(\alpha_1, \ldots, \alpha_n)$.

**Proof.** (i) We argue by induction on $n$. First suppose that $n = 1$. It follows from hypothesis of the lemma that the set $\{x \in E | a_1 x^{\alpha_1} \neq 1\}$ is finite. Thus we can conclude by taking $F = \{x \in E | a_1 x^{\alpha_1} = 1\}$ and $c_1 = a_1^{-1}$. Now suppose that the result is true for $n - 1$ ($n > 1$). For any set $X$, we denote by $P_n(X)$ the set of subsets of $X$ containing $n$ elements and by $S_n$ the set of all permutations of $\{1, \ldots, n\}$. Let $E_1$ be the set of subsets $\{x_1, \ldots, x_n\} \in P_n(E)$ such that $a_1 x_{\sigma(1)}^{\alpha_1} \cdots a_n x_{\sigma(n)}^{\alpha_n} = 1$ for some permutation $\sigma \in S_n$. Put $E_2 = P_n(E) \setminus E_1$. By Ramsey's Theorem, there exists an infinite subset $X \subset E$ such that $P_n(X) \subset E_1$ or $P_n(X) \subset E_2$. However, the second inclusion is in contradiction with the hypothesis of the
lemma, so $P_n(X) \subseteq E_1$. Let $\{y_1, \ldots, y_{n-1}\}$ be a fixed element of $P_{n-1}(X)$. Then, for each $y = y_n$ in $X \setminus \{y_1, \ldots, y_{n-1}\}$, choose a permutation $f(y) = \sigma$ of $\{1, \ldots, n\}$ such that $a_1 y_{\sigma(1)}^{a_1} \cdots a_n y_{\sigma(n)}^{a_n} = 1$ and consider the mapping $f : X \setminus \{y_1, \ldots, y_{n-1}\} \to S_n$. By the pigeonhole principle, there exists a permutation $\sigma$ of $S_n$ such that $f^{-1}(\sigma)$ is infinite; put $k = \sigma^{-1}(n)$. Then, for all $y$ in $f^{-1}(\sigma)$, we have $a_1 y_{\sigma(1)}^{a_1} \cdots a_k y_{\sigma(k)}^{a_k} \cdots a_n y_{\sigma(n)}^{a_n} = 1$. Therefore, the elements $y_1, \ldots, y_{n-1}$ being fixed in $X$, $y_{\sigma(k)}^{a_k}$ is constant on $f^{-1}(\sigma)$. Put $c_k = y_{\sigma(k)}^{a_k}$ for $y \in f^{-1}(\sigma)$. Clearly, it follows from the hypothesis of the lemma that each infinite subset $E' \subseteq f^{-1}(\sigma)$ contains $n - 1$ distinct elements $x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n$ such that

$$a_1 x_1^{a_1} \cdots a_{k-1} x_{k-1}^{a_{k-1}} a_k^{x_k} x_{k+1}^{a_{k+1}} \cdots a_n x_n^{a_n} = 1 \quad \text{(with } a_{k+1} = a_k c_k a_{k+1})$$

if $k < n$, and such that

$$a_1' x_1^{a_1} x_2^{a_2} \cdots x_{n-1}^{a_{n-1}} x_n^{a_n} = 1 \quad \text{(with } a_1' = a_n c_n a_1)$$

if $k = n$. By induction, there exist an infinite subset $F \subseteq f^{-1}(\sigma)$ and elements $c_1, \ldots, c_{k-1}, c_{k+1}, \ldots, c_n$ of $G$ such that, for each $i \in \{1, \ldots, k-1, k+1, \ldots, n\}$, we have $x^{a_i} = c_i$ for all $x \in F$. Since $x^{a_{k+1}} = c_k$ for all $x \in F$, the property is proved.

(ii) Let $\beta_1, \ldots, \beta_n$ be integers such that $\alpha = \beta_1 a_1 + \cdots + \beta_n a_n$. For all $x \in F$, we have $x^\alpha = x^{\beta_1 a_1} \cdots x^{\beta_n a_n} = c_1^{\beta_1} \cdots c_n^{\beta_n}$, as required.

Recall that in the following, we have

$$w(x_1, \ldots, x_n) = x_1^{a_1} \cdots x_n^{a_n} \quad \text{and} \quad \alpha = \gcd(a_1, \ldots, a_n)$$

Furthermore, we put $a_i' = a_i a_i^{-1}$ for $i = 1, \ldots, n$.

**Lemma 2.** For each group $G \in \mathcal{G}_\infty(w)$, the set $\{x^\alpha\}_{x \in G}$ is finite.

**Proof.** Since the result is trivial if $G$ is finite, we can assume that $G$ is infinite. Suppose that the set $\{x^\alpha\}_{x \in G}$ is infinite. Clearly, in this case, there exists an infinite subset $E \subseteq G$ such that $x^\alpha \neq y^\alpha$ for each pair $\{x, y\}$ of elements of $E$. By applying Lemma 1(ii) to $G$ (with $a_1 = \cdots = a_n = 1$), we obtain a contradiction.

**Lemma 3.** Let $G$ be a group in $\mathcal{G}_\infty(w)$. Suppose that the set $C = \{x \in G | x^\alpha = c\}$ is infinite for some $c \in G$. Then $c a_1 + \cdots + a_n = 1$. 

PROOF. There exist $n$ elements $x_1, \ldots, x_n \in C$ such that $w(x_1, \ldots, x_n) = 1$. Since

$$w(x_1, \ldots, x_n) = x_1^{a_1} \cdots x_n^{a_n} = c_1^{a_1} \cdots c_n^{a_n} = c_1^{a_1 + \cdots + a_n},$$

we obtain $c_1^{a_1 + \cdots + a_n} = 1$. 

PROOF OF THE THEOREM. (i)$\rightarrow$(ii). Let $G$ be an infinite group in $\mathcal{V}_\infty(w)$. By Lemma 2, the set $\{x^a\}_{x \in G}$ is finite. Clearly, this implies that $G$ is periodic. Thus, by Dicman's Lemma, the subgroup generated by $\{x^a\}_{x \in G}$ is finite and so $G$ belongs to $\mathcal{F}_{B_a}$.

Now consider an element $c_i$ in the set $\{x^a\}_{x \in G} = \{c_1, \ldots, c_i\}$ and put $C_i = \{x \in G | x^a = c_i\}$. For all $x \in C_i$, we have

$$x^{a_1 + \cdots + a_n} = x^{a(c_1^{a_1 + \cdots + a_n})} = c_i^{a_1^{a_1 + \cdots + a_n}}.$$ 

It follows from Lemma 3 that $x^{a_1 + \cdots + a_n} = 1$ whenever $C_i$ is infinite. Since $C_1, \ldots, C_i$ is a partition of $G$, the set $\{x \in G | x^{a_1 + \cdots + a_n} \neq 1\}$ is finite. This implies that $G$ belongs to the class $\mathcal{V}_\infty(x^{a_1 + \cdots + a_n})$. In fact, as it is observed in [4], $\mathcal{V}_\infty(x^{a_1 + \cdots + a_n}) = \mathcal{F} \cup B(a_1 + \cdots + a_n)$ and so $G \in \mathcal{B}(a_1 + \cdots + a_n)$.

(ii)$\rightarrow$(iii). Let $H$ be a normal subgroup of $G$ such that $H \in \mathcal{F}$ and $G/H \in \mathcal{B}_n$. Put $m = 1 + (n - 1) |H : \{1\}|$ and show that $G$ belongs to $\mathcal{V}_m(w)$. Let $E$ be a subset of $G$ containing $m$ elements. The function $x \mapsto x^a$ maps each element of $E$ into an element of $H$; thus there exists an element $c \in H$ such that the set $\{x \in E | x^a = c\}$ contains at least $n$ elements. Consider $n$ distinct elements $x_1, \ldots, x_n \in \{x \in E | x^a = c\}$. We have:

$$w(x_1, \ldots, x_n) = x_1^{a_1} \cdots x_n^{a_n} = c_1^{a_1} \cdots c_n^{a_n} = c^{a_1 + \cdots + a_n} = x_1^{a_1 + \cdots + a_n} = 1,$$

for $G \in \mathcal{B}(a_1 + \cdots + a_n)$. Thus we have proved that $G$ belongs to $\mathcal{V}_m(w)$. Since clearly (iii) implies (i), the proof is complete.

We finish with a question of combinatorial nature:

Suppose that $G$ is an infinite group in $\mathcal{V}_\infty(w)$, where $w$ is now an arbitrary word. Does $G$ belong to $\mathcal{V}_m(w)$ for some integer $m$?
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