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BV as a Dual Space.

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Abstract - Let $\mathcal{C}$ be a field of subsets of a set I. It is well known that the space FA of all the finitely additive games of bounded variation on $\mathcal{C}$ is the norm dual of the space of all simple functions on $\mathcal{C}$. In this paper we prove that the space BV of all the games of bounded variation on $\mathcal{C}$ is the norm dual of the space of all simple games on $\mathcal{C}$. This result is equivalent to the compactness of the unit ball in BV with respect to the vague topology.

1. Introduction.

Let $\mathcal{C}$ be a field of subsets of a set I. It is well known that the space FA of all the finitely additive games of bounded variation on $\mathcal{C}$, equipped with the total variation norm, is isometrically isomorphic to the norm dual of the space of all simple functions on $\mathcal{C}$, endowed with the sup norm (see [2] p. 258).

In this paper we establish a parallel result for the space BV of all the games of bounded variation on $\mathcal{C}$. To this end, we first introduce simple games on $\mathcal{C}$: the ones that assign non zero worth only to a finite number of elements of $\mathcal{C}$. Then we show that BV, equipped with the total variation norm, is isometrically isomorphic to the norm dual of the space of all simple games, endowed with a suitable norm. A first proof of this result builds on the compactness of the unit ball in BV with respect to the vague topology, proved by Marinacci [4].

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Another contribution of this paper is showing that the duality of BV is indeed equivalent to the vague compactness of the unit ball. In order obtain this result, we provide a second *direct* proof of the dual nature of BV, and then observe that the vague topology can be viewed as the weak* topology induced by duality.

The paper is organized as follows. After a brief section of preliminaries, in Section 3 we use compactness to obtain duality. In Section 4, we go the other way around: we prove duality and obtain compactness as a corollary. Section 5 clarifies the connections between the two approaches.

2. Preliminaries.

This section contains a few, well known, definitions and properties. We follow the notation of cooperative game theory, instead of the more common one of measure theory, since the spaces of set functions we consider are widely used in that field (see, e.g. [1]).

Let $\mathcal{C}$ be a field of subsets of a non empty set I. A set function $v: \mathcal{C} \to \mathbb{R}$ is a *game* if $v(\emptyset) = 0$. A game on $\mathcal{C}$ is *monotone* if $v(A) \leqslant v(B)$ whenever $A \subseteq B$.

A *chain* $\{S_i\}_{i=0}^n$ in $\mathcal{C}$ is a finite strictly increasing sequence

$$\emptyset = S_0 \subset S_1 \subset \ldots \subset S_{n-1} \subset S_n = I$$

of elements of $\mathcal{C}$. BV is the set of all games such that

$$\|u\| = \sup \left\{ \sum_{i=1}^{n} |u(S_i) - u(S_{i-1})| : \{S_i\}_{i=0}^n \text{ is a chain in } \mathcal{C} \right\} < \infty .$$

A game in BV is said to be *of bounded variation*. The pair $(BV, \|\cdot\|)$ is a Banach space, generated by all monotone games (see [1] pp. 14, 26-28).

Since BV is a (proper) subspace of $\mathbb{R}^{\mathcal{C}}$, it inherits a topology from the product topology of $\mathbb{R}^{\mathcal{C}}$. This is the weak topology generated by the projection functionals

$$\begin{aligned} p_A : \; BV &\to \mathbb{R} \\ u &\mapsto u(A) \end{aligned}$$

where $A \in \mathcal{C}$. A net $\{u_a\}$ converges to u in this topology iff $u_a(A) \to u(A)$ for all $A \in \mathcal{C}$ (we write $u_a \xrightarrow{\mathcal{C}} u$). This topology is called *vague topology* (for the analogy with the vague topology on the set of probability measures).

3. From compactness to duality.

In this section, we start by observing that the unit ball in BV is compact in the vague topology, and we use this fact to show that BV is a dual space.

THEOREM 1 (MARINACCI). *The unit ball* $U(BV) = \{u \in BV : \|u\| \leq 1\}$ *is compact w.r.t. the vague topology.*

PROOF. See [4]. The proof is based on the Tychonoff Theorem. ■

Let X be the space of all the games which are non zero only on a finite number of elements of $\mathcal{C}$, the *simple games*. The pair (BV, X) is dual w.r.t. the following functional

$$\begin{aligned} D:\ BV \times X &\to \mathbb{R} \\ (u, x) &\mapsto \sum_{A \in \mathcal{C}} u(A)\, x(A)\,. \end{aligned}$$

That is:

- D is bilinear,
- if $D(u, x) = 0$ for all u, then $x = 0$,
- if $D(u, x) = 0$ for all x, then $u = 0$.

Therefore, X can be interpreted as a total subspace of the algebraic dual of BV by identifying $x \in X$ with the linear functional

$$\begin{aligned} D_x:\ BV &\to \mathbb{R} \\ u &\mapsto D(u, x)\,. \end{aligned}$$

For this reason, in the rest of this section, we will write $\langle u, x \rangle$ instead of $D(u, x)$. Some useful properties of X are stated in the following lemma, whose easy proof is omitted.

LEMMA 2. *X is a total subspace of the norm dual BV' of BV, and*

$$\|x\|_{BV'} = \max_{\|u\|=1} \Big(\sum_{A \in \mathcal{C}} u(A)\, x(A) \Big)$$

for all $x \in X$.

Moreover, the topology $\sigma(BV, X)$ coincides with the vague topology.

Given a game u of bounded variation, let $J_u : X \to \mathbb{R}$ be the functional defined by

$$J_u(x) = \langle u, x \rangle$$

for all $x \in X$.

THEOREM 3. *Let X' be the norm dual of $(X, \|\cdot\|_{BV'})$. The operator*

$$\begin{aligned} J : \ BV &\to X' \\ u &\mapsto J_u \end{aligned}$$

is an isometric isomorphism from BV onto X'. Moreover, J is a vague-weak homeomorphism.*

PROOF. The operator J is well defined, linear, injective and $\|J\| \leqslant 1$. Let $U(BV)$ be the unit ball in BV, $U(X')$ be the unit ball in X'. For the Goldstine-Weston density Lemma $J(U(BV))$ is weak* dense in $U(X')$ (see, e.g., [3] p. 126).

Consider the vague topology on BV and the weak* topology on X'. Let $\{u_a\}$ be a net in BV. We have that $u_a \xrightarrow{\mathcal{C}} u$ iff $u_a \xrightarrow{\sigma(BV, X)} u$ iff $\langle u_a, x \rangle \to \to \langle u, x \rangle$ for all $x \in X$ iff $J_{u_a}(x) \to J_u(x)$ for all $x \in X$ iff J_{u_a} weak* converges to J_u (briefly $J_{u_a} \xrightarrow{w^*} J_u$). Thus J is vague-weak* continuous. Therefore, $J(U(BV))$ is weak* compact in X' and

$$J(U(BV)) = U(X').$$

Since $\|J\| \leqslant 1$, then $\|J_u\| \leqslant \|u\|$, for all $u \in BV$. Suppose that there exists $u \in BV$ such that $\|J_u\| < \|u\|$. Then $u \neq 0$ and $0 < \|J_{u/\|u\|}\| = \varrho < 1$, set $w = \dfrac{u}{\|u\|}$. Since $\|J_{w/\varrho}\| = 1$, it follows that $J_{w/\varrho} \in U(X')$, but $\dfrac{w}{\varrho} \notin U(BV)$ and J is injective, which is absurd. Thus $\|J_u\| = \|u\|$, for all $u \in BV$ and J is an isometry.

If $x' \in X' - \{0\}$ then there exists $u \in U(BV)$ s.t. $J_u = \dfrac{x'}{\|x'\|}$, hence $J_{\|x'\| u} = x'$. That is J is surjective. Since, for every net $\{u_a\}$ in BV, $u_a \xrightarrow{\mathcal{C}} u$ iff $J_{u_a} \xrightarrow{w^*} J_u$, then J is a vague-weak* homeomorphism. ∎

4. From duality to compactness.

In this section the opposite approach is adopted, we directly prove that *BV* is a dual space and use this fact to provide an alternative proof that the unit ball in *BV* is compact w.r.t. the vague topology, thus obtaining the equivalence of the two results.

We define the game $e_A\colon \mathcal{C} \to \mathbb{R}$ by

$$e_A(B) = \begin{cases} 1 & \text{if } B = A \\ 0 & \text{otherwise} \end{cases}$$

for all $A \in \mathcal{C} - \{\emptyset\}$, and $e_\emptyset = 0$. Being $x = \sum_{A \in \mathcal{C}} x(A)\, e_A$ for all $x \in X$, we have $X = \langle e_A\colon A \in \mathcal{C} \rangle$.

For each chain $\Omega = \{S_i\}_{i=0}^n$ in $\mathcal{C}$, define a seminorm on X by

$$\|x\|_\Omega = \max_{0 \leq k \leq n} \left| \sum_{i=k}^n x(S_i) \right| \tag{1}$$

for all $x \in X$. Let $X_\Omega = \langle e_A\colon A \in \Omega \rangle$. If $x \in X_\Omega$, we say that x *depends on the chain* Ω.

For all $x \in X$, set

$$\|x\| = \inf \sum_{l=1}^L \|x_l\|_{\Omega_l}$$

where the inf is taken over all finite decompositions $x = \sum_{l=1}^L x_l$ in which x_l depends on the chain Ω_l and $\|\cdot\|_{\Omega_l}$ is defined as in (1) for all $l = 1, \ldots, L$.

Lemma 4. *The function* $\|\cdot\|\colon X \to \mathbb{R}$ *is a norm on X.*

Proof. It is easy to prove that $\|\cdot\|$ is a seminorm. We just show that $x \neq 0$ implies $\|x\| \neq 0$. If $x \in X_\Omega$ and $A \in \Omega$ then

$$\|x\|_\Omega \geq \frac{1}{2} |x(A)|\,.$$

In fact,

$$|x(A)| = \left| \sum_{i\colon S_i \supseteq A} x(S_i) - \sum_{i\colon S_i \supset A} x(S_i) \right| \leq \left| \sum_{i\colon S_i \supseteq A} x(S_i) \right| + \left| \sum_{i\colon S_i \supset A} x(S_i) \right| \leq 2\|x\|_\Omega .$$

Let $x \in X - \{0\}$. There exists $A \in \mathcal{C}$ such that $x(A) \neq 0$. If $x = \sum_{l=1}^L x_l$,

$x_l \in X_{\Omega_l}$ for all $l = 1, \ldots, L$, then $A \in \Omega_l$ for some l (otherwise $x(A) = 0$). Hence,

$$|x(A)| = \left| \sum_{l : A \in \Omega_l} x_l(A) \right| \leqslant \sum_{l : A \in \Omega_l} |x_l(A)|,$$

and so

$$\sum_{l=1}^{L} \|x_l\|_{\Omega_l} \geqslant \sum_{l : A \in \Omega_l} \|x_l\|_{\Omega_l} \geqslant \sum_{l : A \in \Omega_l} \frac{1}{2} |x_l(A)| \geqslant \frac{1}{2} |x(A)|,$$

then $\|x\| \geqslant \frac{1}{2} |x(A)| > 0$. ■

Given a linear continuous functional $f : X \to \mathbb{R}$, define the game G_f as follows

$$G_f(A) = f(e_A)$$

for all $A \in \mathcal{C}$.

THEOREM 5. *Let X' be the norm dual of $(X, \|\cdot\|)$. The operator*

$$\begin{aligned} G : \ X' &\to BV \\ f &\mapsto G_f \end{aligned}$$

is an isometric isomorphism from X' onto BV. Moreover, G is a weak-vague homeomorphism.*

Notice that, together with the Alaoglu Theorem (see, e.g., [3] p. 70), the above result immediately yields Theorem 1, that is, the compactness of the unit ball $U(BV)$ in the vague topology.

PROOF. We first show that if $\Omega = \{S_i\}_{i=0}^{n}$ is a chain in $\mathcal{C}$ then

$$\sum_{k=1}^{n} |G_f(S_k) - G_f(S_{k-1})| \leqslant \|f\|,$$

which implies that $G_f \in BV$ and $\|G_f\| \leqslant \|f\|$.

Define $x \in X_\Omega$ by

$$\begin{aligned} x(S_n) &= \operatorname{sgn}(f(e_{S_n}) - f(e_{S_{n-1}})), \\ x(S_n) + x(S_{n-1}) &= \operatorname{sgn}(f(e_{S_{n-1}}) - f(e_{S_{n-2}})), \\ &\ldots \\ x(S_n) + x(S_{n-1}) + \ldots + x(S_1) &= \operatorname{sgn}(f(e_{S_1}) - f(e_{S_0})), \\ x(S_0) &= 0, \end{aligned}$$

Obviously $\|x\|_\Omega \leqslant 1$, so that $\|x\| \leqslant 1$. Thus

$$\begin{aligned}
\|f\| &\geqslant f(x) = \sum_{j=0}^{n} f(e_{S_j})\, x(S_j) = \\
&= f(e_{S_0}) \sum_{k=0}^{n} x(S_k) + \sum_{j=1}^{n} \left[(f(e_{S_j}) - f(e_{S_{j-1}})) \sum_{k=j}^{n} x(S_k) \right] = \\
&= \sum_{j=1}^{n} [(f(e_{S_j}) - f(e_{S_{j-1}}))\, \operatorname{sgn}(f(e_{S_j}) - f(e_{S_{j-1}}))] = \\
&= \sum_{j=1}^{n} |f(e_{S_j}) - f(e_{S_{j-1}})| = \sum_{j=1}^{n} |G_f(S_j) - G_f(S_{j-1})|\,.
\end{aligned}$$

Then G is well defined and obviously linear and injective.

Given $u \in BV$, we can define f_u on X by

$$f_u(x) = \sum_{A \in \mathcal{C}} u(A)\, x(A),$$

for all $x \in X$. Trivially, f_u is linear.

If x depends on $\Omega = \{S_j\}_{j=0}^{n}$, then

$$\begin{aligned}
f_u(x) &= \sum_{j=0}^{n} u(S_j)\, x(S_j) = \\
&= u(S_0) \sum_{k=0}^{n} x(S_k) + \sum_{j=1}^{n} \left[(u(S_j) - u(S_{j-1})) \sum_{k=j}^{n} x(S_k) \right] = \\
&= \sum_{j=1}^{n} \left[(u(S_j) - u(S_{j-1})) \sum_{k=j}^{n} x(S_k) \right] \leqslant \\
&\leqslant \sum_{j=1}^{n} \left[|u(S_j) - u(S_{j-1})| \left| \sum_{k=j}^{n} x(S_k) \right| \right] \leqslant \\
&\leqslant \sum_{j=1}^{n} [\,|u(S_j) - u(S_{j-1})|\, \|x\|_\Omega] \leqslant \\
&\leqslant \|x\|_\Omega \sum_{j=1}^{n} |u(S_j) - u(S_{j-1})| \leqslant \|x\|_\Omega \|u\|\,.
\end{aligned}$$

If $x = \sum_{l=1}^{L} x_l$ with $x_l \in X_{\Omega_l}$ for all $l = 1, 2, \dots, L$, then

$$f_u(x) = \sum_{l=1}^{L} f_u(x_l) \leqslant \sum_{l=1}^{L} \|u\|\, \|x_l\|_{\Omega_l} \leqslant \|u\| \sum_{l=1}^{L} \|x_l\|_{\Omega_l},$$

and so

$$\begin{aligned}
f_u(x) &\leqslant \inf \left\{ \|u\| \sum_{l=1}^{L} \|x_l\|_{\Omega_l} : x = \sum_{l=1}^{L} x_l,\ x_l \in X_{\Omega_l} \right\} = \\
&= \|u\|\, \|x\|\,.
\end{aligned}$$

We conclude that $f_u \in X'$, $G_{(f_u)} = u$ and G is onto. For all $u \in BV$, $f_u = G_u^{-1}$ and $\|G_u^{-1}\| = \|f_u\| \leq \|u\|$. Therefore, for all $f \in X'$, $\|f\| = \|G_{(G_f)}^{-1}\| \leq \|G_f\|$ and G is an isometry.

Finally, let $\{f^a\}$ be a net in X'. We have that $f^a \xrightarrow{w^*} f$ iff $f^a(x) \to f(x)$ for all $x \in X$ iff $f^a(e_A) \to f(e_A)$ for all $A \in \mathcal{C}$ iff $G_{f^a}(A) \to G_f(A)$ for all $A \in \mathcal{C}$ iff $G_{f^a} \xrightarrow{\mathcal{C}} G_f$. Hence, G is a weak*-vague homeomorphism. ■

We conclude the section by observing that Theorem 5 corrects Theorem 1.1 in [5]: it can be shown with a counterexample that the norm used there does not lead to an isometry.

5. Summing up.

In sections 3 and 4 we have given two independent proofs that BV is the norm dual of X, when X is endowed with the norms $\|\cdot\|_{BV'}$ and $\|\cdot\|$, respectively. The two approaches are indeed one the mirror image of the other. In fact:

PROPOSITION 6. *The norm $\|\cdot\|_{BV'}$ on X coincides with the norm $\|\cdot\|$ on X. Moreover, $J = G^{-1}$.*

PROOF. Let $(X', \|\cdot\|')$ be the norm dual of $(X, \|\cdot\|)$. We have, for all x,

$$\begin{aligned} \|x\|_{BV'} &= \sup\left\{ \left| \sum_{A \in \mathcal{C}} u(A)\, x(A) \right| : u \in BV,\ \|u\| = 1 \right\} = \\ &= \sup\{ |f_u(x)| : u \in BV,\ \|u\| = 1 \} = \\ &= \sup\{ |f(x)| : f \in X',\ \|f\|' = 1 \} = \\ &= \|x\|\,. \end{aligned}$$

In particular the norm dual of $(X, \|\cdot\|_{BV'})$ and the norm dual of $(X, \|\cdot\|)$ are the same space X'. For all $u \in BV$ and $x \in X$,

$$J_u(x) = \langle u, x \rangle = \sum_{A \in \mathcal{C}} u(A)\, x(A) = f_u(x) = G_u^{-1}(x)\,,$$

that is $J_u = G_u^{-1}$ for all $u \in BV$ and $J = G^{-1}$. ■

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