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Torsion Groups in Cotorsion Classes.

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ABSTRACT - For torsion-free abelian groups G of arbitrary rank we discuss the class $\mathcal{TC}(G)$ of all torsion groups T satisfying $\mathrm{Ext}(G, T) = 0$, that is the subclass of all torsion groups of the cotorsion class cogenerated by G. The main question we consider is when for such G there exists a subgroup of the rationals $\mathbb{Z} \subseteq R \subseteq \mathbb{Q}$ such that $\mathcal{TC}(G) = \mathcal{TC}(R)$.

Introduction.

Throughout this paper we work in the category Mod-$\mathbb{Z}$ of abelian groups. All terminology used here can be found in [F1], [F2] and [EM].

Cotorsion theories for abelian groups have been introduced by Salce in 1979 [S]. Following his notation we call a pair $(\mathcal{F}, \mathcal{C})$ a cotorsion theory if $\mathcal{F}$ and $\mathcal{C}$ are classes of abelian groups which are maximal with respect to the property that $\mathrm{Ext}(F, C) = 0$ for all $F \in \mathcal{F}$, $C \in \mathcal{C}$.

Salce [S] has shown that every cotorsion theory is cogenerated by a class of torsion and torsion-free groups where $(\mathcal{F}, \mathcal{C})$ is said to be cogenerated by the class $\mathcal{A}$ if $\mathcal{C} = \mathcal{A}^{\perp} = \{X \in \text{Mod-}\mathbb{Z} \mid \mathrm{Ext}(A, X) = 0 \text{ for all } A \in \mathcal{A}\}$ and $\mathcal{F} = {}^{\perp}(\mathcal{A}^{\perp}) = \{Y \in \text{Mod-}\mathbb{Z} \mid \mathrm{Ext}(Y, X) = 0 \text{ for all } X \in \mathcal{A}^{\perp}\}$. Examples for cotorsion theories are: $(\text{Mod-}\mathbb{Z}, \mathcal{D}) = ({}^{\perp}(G^{\perp}), G^{\perp})$ with $G = \bigoplus_{p \in \Pi} \mathbb{Z}(p)$ where $\mathcal{D}$ is the class of all divisible groups and Π is the set of all prime numbers, $(\mathcal{L}, \text{Mod-}\mathbb{Z}) = ({}^{\perp}(\mathbb{Z}^{\perp}), \mathbb{Z}^{\perp})$ where $\mathcal{L}$ is the class of all free groups, and the classical one $(\mathcal{TF}, \mathcal{CO}) = ({}^{\perp}(\mathbb{Q}^{\perp}), \mathbb{Q}^{\perp})$ where $\mathcal{TF}$ is

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the class of all torsion-free groups and $\mathcal{CO}$ is the class of all (classical) cotorsion groups. In view of the last example the classes $\mathcal{F}$ and $\mathcal{C}$ of a cotorsion theory $(\mathcal{F}, \mathcal{C})$ are said to be the torsion-free class and the cotorsion class of this cotorsion theory.

In this paper we shall restrict to cotorsion classes cogenerated by a single torsion-free group G and turn our attention to the subclass of all torsion groups of the cotorsion class cogenerated by G. This class was denoted by $\mathcal{TC}(G)$ and first studied by Wallutis and the author in [SW].

A characterization of these classes is obviously closely related to the solution of the Baer problem (e.g. see [F2]); put into our context it says that $\mathcal{TC}(G)$ is maximal, i.e. $\mathcal{TC}(G) = \mathfrak{T}$, where $\mathfrak{T}$ is the class of all torsion groups, if and only if G is free.

Recall that torsion-free groups of rank-1 can be indentified with subgroups of the rationals $\mathbb{Q}$ and are therefore also called rational groups. Using a result by Salce [S] a full characterization of $\mathcal{TC}(R)$ for some given rank-1 group was obtained in [SW]. Introducing the quasi-reduced type $type^{qr}(R)$ of an rank-1 group R it was shown that $\mathcal{TC}(R) = \mathcal{TC}(R')$ if and only if $type^{qr}(R) = type^{qr}(R')$ for any rank-1 groups R, R'; when representing the type of R by $type(R) = (r_p)_{p \in \Pi}$ the quasi-reduced type of R can be represented by $type^{qr}(R) = (s_p)_{p \in \Pi}$ with $s_p = r_p$ whenever $r_p = 0$ or $r_p = \infty$ and $s_p = 1$ otherwise. This characterization of the classes $\mathcal{TC}(R)$ for rank-1 groups had immediate consequences for completely decomposable groups. Therefore, it is of particular interest to know when the class $\mathcal{TC}(G)$ for a given torsion-free group G is generated by a rational group R, i.e. $\mathcal{TC}(G) = \mathcal{TC}(R)$ for some rational group R.

After we shall have considered some basic facts in section 1 we define the class $\mathcal{R}$ of all torsion-free groups G such that $\mathcal{TC}(G)$ is generated by a rational group, i.e. $\mathcal{R} = \{G \text{ torsion-free} : \mathcal{TC}(G) = \mathcal{TC}(R) \text{ for some rational group } R \subseteq \mathbb{Q}\}$. We will first show in section 2 that all torsion-free groups of finite rank belong to $\mathcal{R}$. In fact we are able to prove that any torsion-free group G of finite rank satisfies $\mathcal{TC}(G) = \mathcal{TC}(\mathrm{OT}(G))$ where $\mathrm{OT}(G)$ denotes the outer-type of G. Recall that the outer-type of G is defined to be the supremum of all rank-1 torsion-free quotients of G. We shall use this result to show that for almost all rational groups R there is an indecomposable almost completely decomposable group G of arbitrarily large rank satisfying $\mathcal{TC}(G) = \mathcal{TC}(R)$. Moreover, we show that if R is divisible by at least one prime p then there is even an indecomposable homogeneous group G of arbitrarily large rank with $\mathcal{TC}(G) = \mathcal{TC}(R)$. Finally, for any rational group R which is non-idempotent (i.e. $\mathrm{End}(R) \not\cong R$) an indecomposable group G of given rank n is constructed such that G is homogeneous of type R and $\mathcal{TC}(G) = \mathcal{TC}(R)$. This shows that Griffith's solution of Baer's problem (see [G]) cannot be generalized to homogeneous groups of non-idempotent type.

In section 3 we consider torsion-free groups G of countable rank and extend the notion of outer-type in a suitable way; OT_G is defined to be the set of all quasi-reduced outer types of finite rank pure subgroups of G. Using this notion a criterion is given when $G \in \mathcal{R}$. A countable torsion-free group G satisfies $\mathcal{TC}(G) = \mathcal{TC}(R)$ for a rational group if and only if the supremum of OT_G exists and OT_G is ∞-closed which means that whenever P is an infinite set of primes and for all $p \in P$ there exists $R \in \mathrm{OT}_G$ such that $\chi_p^R(1) = \infty$, then there exists $R \in \mathrm{OT}_G$ such that $\chi_p^R(1) \neq 0$ for almost all $p \in P$. It turns out that the class $\mathcal{R}$ is no longer closed under direct summands if we restrict to countable groups.

Finally, we prove in section 4 a very technical necessary and sufficient condition for torsion-free groups of arbitrary rank to belong to $\mathcal{R}$. Therefore, we extend the notion of ∞-closed to a condition on the vanishing of certain Ext-groups.

Let us begin with some easy facts.

1. Preliminaries.

In this section we first recall the definition of the class $\mathcal{TC}(G)$ for a given group G as it was given in [SW]. Moreover, we introduce two related classes of groups and recall some easy and known results adjusted to these notations.

For an arbitrary group G the classes $G^{\perp}$ and ${}^{\perp}(G^{\perp})$ are defined by $G^{\perp} = \{X \in \text{Mod-}\mathbb{Z} \mid \mathrm{Ext}(G,X) = 0\}$ and ${}^{\perp}(G^{\perp}) = \{Y \in \text{Mod-}\mathbb{Z} \mid \mathrm{Ext}(Y, X) = 0 \ \forall X \in G^{\perp}\}$, respectively. Due to Salce [S] the pair $({}^{\perp}(G^{\perp}), G^{\perp})$ is called the *cotorsion theory cogenerated by* G. In view of the classical cotorsion theory $(\mathcal{TF}, \mathcal{CO})$ where $\mathcal{CO} = \mathbb{Q}^{\perp}$ is the class of all cotorsion groups and $\mathcal{TF} = {}^{\perp}(\mathbb{Q}^{\perp})$ is the class of all torsion-free groups, we also call $G^{\perp}$ the *cotorsion class cogenerated by* G. As we have seen in the introduction it makes sense to restrict our attention to the cotorsion classes cogenerated by torsion-free groups and furthermore to its subclass of all torsion groups.

Let $\mathfrak{T}$ be the class of all torsion groups. In [SW] the class $\mathcal{TC}(G)$ was defined for any group G as $\mathcal{TC}(G) = G^{\perp} \cap \mathfrak{T}$, the class of all torsion groups belonging to the cotorsion class $G^{\perp}$ cogenerated by G. It is well known that $G^{\perp}$ as well as $\mathfrak{T}$ are closed under epimorphic images and extensions (see [F1] and [S]). Hence we immediately have the following (see also [SW]):

LEMMA 1.1. *For any group G the following are true:*

(i) $\mathcal{TC}(G)$ *is closed under epimorphic images;*

(ii) $\mathcal{TC}(G)$ *is closed under extensions, especially under finite direct sums;*

(iii) *if G is torsion-free, then* $\mathcal{TC}(G)$ *contains all torsion cotorsion groups, i.e.* $\mathcal{TC}(\mathbb{Q}) = \mathcal{CO} \cap \mathfrak{T} \subseteq \mathcal{TC}(G)$.

Moreover, an easy and well-known lemma is sometimes very useful. First recall that the basic subgroup B of a torsion group T is the direct sum $B = \bigoplus_{p \in \Pi} B_p$ of the basic subgroups B_p of the p-components T_p; for each prime p, B_p is a direct sum of cyclic p-groups, B_p is a pure subgroup of T_p and the quotient T_p/B_p is divisible (see [F1]). The proof of the following result is left to the reader (see also [SW]).

LEMMA 1.2. *Let T be a torsion group and* $B \subseteq T$ *a basic subgroup of* T. *Then, for any group* G, T *is an element of* $\mathcal{TC}(G)$ *if and only if B is.*

In [SW] the class $\mathcal{TC}(G)$ was studied for torsion-free abelian groups G, in particular for subgroups R of the rationals $\mathbb{Q}$ and for completely decomposable groups. Recall that the *quasi-reduced type* $\text{type}^{qr}(R)$ of a type R is defined as the subgroup S of $\mathbb{Q}$ such that $\chi_p^S(1) = 1$ if $0 \neq \chi_p^R(1) \neq \infty$ and $\chi_p^S(1) = \chi_p^R(1)$ else (p a prime).

LEMMA 1.3 (Strüngmann-Wallutis, [SW]). *Let R be a rational group with* $\chi(R) = (r_p)_{p \in \Pi}$ *and let* $T = \bigoplus_{p \in \Pi} T_p$ *be a reduced torsion group with p-components* T_p.

Then $\operatorname{Ext}(R, T) = 0$ *if and only if the following conditions are satisfied:*

(i) T_p *is bounded for all p such that* $r_p = \infty$;

(ii) $T_p = 0$ *for almost all p such that* $r_p \neq 0$.

In particular, $\mathcal{TC}(R) = \mathcal{TC}(\text{type}^{qr}(R))$ *and for two rational groups R and S we have* $\mathcal{TC}(R) \subseteq \mathcal{TC}(S)$ *if and only if* $\text{type}^{qr}(R) \geq \text{type}^{qr}(S)$.

It was shown in [SW] that any countable torsion-free abelian group G satisfies $\mathcal{TC}(G) = \mathcal{TC}(C)$ for some completely decomposable group C. Moreover, for a large class of torsion-free groups G it was proved that $\mathcal{TC}(G) = \mathcal{TC}(R)$ for some rational group R. Motivated by these results we introduce the following classes of torsion-free abelian groups.

DEFINITION 1.4. *Let $\mathcal{TF}$ be the class of all torsion-free abelian groups. Then*

(i) $\mathfrak{C} = \{G \in \mathcal{TF}: \mathcal{TC}(G) = \mathcal{TC}(C)$ *for some completely decomposable group* $C\}$;

(ii) $\mathcal{R} = \{G \in \mathcal{TF}: \mathcal{TC}(G) = \mathcal{TC}(R)$ *for some rational group* $R \subseteq \mathbb{Q}\}$;

(iii) $\mathcal{TS} = \{G \in \mathcal{TF}: \mathcal{TC}(G) = \mathcal{TC}(Tst(G))\}$, *where $Tst(G)$ denotes the typeset of G and* $\mathcal{TC}(Tst(G)) = \mathcal{TC}\left(\bigoplus_{R \in Tst(G)} R\right)$.

Clearly $\mathcal{R} \subseteq \mathfrak{C}$ and $\mathcal{TS} \subseteq \mathfrak{C}$ and it was shown in [SW] that any countable torsion-free group belongs to $\mathfrak{C}$ but it is undecidable in ZFC whether or not $\mathfrak{C} = \mathcal{TF}$. Moreover, the following is a collection of the main results in [SW]. Recall that a torsion-free group G is a Butler group if $\mathrm{Bext}^1(G, T) = 0$ for all torsion groups T (for details on the functor Bext^1 and Butler groups see [F3]). Moreover, G satisfies the torsion extension property if for every pure subgroup H of G and every torsion group T the induced homomorphism $\mathrm{Hom}(G, T) \to \mathrm{Hom}(H, T)$ is surjective.

LEMMA 1.5 (Strüngmann-Wallutis, [SW]). *The following hold.*

(i) *the class $\mathcal{TS}$ contains all Butler groups of arbitrary rank;*

(ii) *the class $\mathcal{R}$ contains all torsion-free groups of finite rank satisfying the torsion extension property, hence all Butler groups of finite rank.*

Nevertheless, the next example which can be found in [F2, Example 5 on p. 125] shows that the class $\mathcal{TS}$ is ugly although it contains the nice subclass of all Butler groups.

EXAMPLE 1.6. Let p be a prime and π a p-adic integer, not a rational; say $\pi = s_0 + s_1 p + \ldots + s_n p^n + \ldots$ with $0 < s_i < p$. Moreover, let x_1, x_2 be linearly independent elements and, for each $n \in \mathbb{N}$, let $y_n = p^{-n}(x_1 + (s_0 + s_1 p + \ldots + s_{n-1} p^{n-1})\, x_2)$.

We define G by

$$G = \langle x_1, x_2, y_1, y_2, \ldots, y_n, \ldots\rangle \subseteq \mathbb{Q}^{(p)}x_1 \oplus \mathbb{Q}^{(p)}x_2.$$

It can be easily verified that G is indecomposable and homogeneous of type $\mathbb{Z}$ (see [F2]). Also, G is not a Butler group since homogeneous Butler groups are completely decomposable.

Moreover, we have $\mathcal{TC}(G) = \mathcal{TC}(\mathbb{Q}^{(p)})$ (see [SW, Example 4.5]), hence $G \in \mathfrak{C}$ but $G \notin \mathcal{TS}$, else G would be free by Griffith's solution of the Baer problem (see [G]).

Now put $H = G \oplus \mathbb{Q}^{(p)}$, then $\mathrm{Tst}(H) = \{\mathbb{Z}, \mathbb{Q}^{(p)}\}$ and clearly $\mathcal{TC}(H) = \mathcal{TC}(\mathbb{Q}^{(p)}) = \mathcal{TC}(\mathrm{Tst}(H))$, hence $H \in \mathcal{TS}$.

COROLLARY 1.7. *The class $\mathcal{TS}$ is neither closed under direct summands nor extensions.*

PROOF. In the Example 1.6 the group $H \in \mathcal{TS}$ but its direct summand G does not belong to $\mathcal{TS}$. Moreover, the group G from Example 1.6 shows that $\mathcal{TS}$ is not closed under extensions. ■

Since the group G from Example 1.6 is not a Butler group it follows that $H \in \mathcal{TS}$ is not a Butler group and hence the class $\mathcal{TS}$ strictly contains the class of all Butler groups which answers Question 4.7 from [SW] negatively.

We are now ready for the next section in which we will consider torsion-free groups of finite rank and show that the class $\mathcal{R}$ behaves much nicer than $\mathcal{TS}$.

2. Torsion-free groups of finite rank.

In this section we shall first show that the class $\mathcal{R}$ contains all finite rank torsion-free groups. Let us begin with an easy lemma.

LEMMA 2.1. *The following are equivalent, where $\mathcal{TFF}$ denotes the class of all torsion-free groups of finite rank:*

(i) *$\mathcal{R} \cap \mathcal{TFF}$ is the class of all torsion-free groups of finite rank;*

(i) *$\mathcal{R} \cap \mathcal{TFF}$ is closed under extensions;*

(iii) $\mathcal{R} \cap \mathcal{TFF}$ is closed under subgroups.

PROOF. The implications (i) $\Rightarrow$ (ii) and (i) $\Rightarrow$ (iii) are trivial. To show (iii) $\Rightarrow$ (i) notice that any finite rank torsion-free group can be embedded into its divisible hull which is obviously a member of $\mathcal{R}$. The remaining implication (ii) $\Rightarrow$ (i) is easily seen by induction on the rank and left to the reader. ■

To continue we need Baer's Theorem from [B].

LEMMA 2.2 (Baer, [B]). *Let T be a torsion group and G a torsion-free group such that* $\operatorname{Ext}(G, T) = 0$. *Then the following hold:*

(i) *if $\{p_1, \ldots, p_i, \ldots\}$ is an infinite set of different primes for which $p_i T < T$, then G contains no pure subgroup S of finite rank such that G/S has elements $\neq 0$ divisible by all p_i;*

(ii) *if, for some prime p, the reduced part of the p-component of T is unbounded, then G contains no pure subgroup S of finite rank such that G/S has elements $\neq 0$ divisible by all powers of p.*

Moreover, if G is countable, then (i) *and* (ii) *suffice for* $\operatorname{Ext}(G, T)$ *to be zero.* ■

We obtain the following lemma.

LEMMA 2.3. *Let G be a countable torsion-free group and H a pure subgroup of G of finite rank. If T is a torsion group such that $T \in \mathcal{TC}(G)$, then $T \in \mathcal{TC}(G/H)$.*

PROOF. Let $H \subseteq G$ and T be given as stated such that $T \in \mathcal{TC}(G)$. Assume that $T \notin \mathcal{TC}(G/H)$, then since G/H is torsion-free there exists a pure subgroup K/H of G/H of finite rank such that $(G/H)/(K/H)$ contains an element x violating one of the conditions from Lemma 2.2. But $(G/H)/(K/H) \cong G/K$ is torsion-free, hence G contains a pure subgroup K of finite rank such that G/K contains an element violating one of Baer's conditions. Since G is countable we conclude that $\operatorname{Ext}(G, T) \neq 0$, hence $T \notin \mathcal{TC}(G)$ - a contradiction. ■

THEOREM 2.4. *Let G be a torsion-free group of finite rank. Then $\mathcal{TC}(G) = \mathcal{TC}(R)$ for some rational group R.*

PROOF. We induct on the rank n of G. If G has rank one, then the claim is trivially true. Hence assume G has rank $n > 1$. We choose a pure subgroup H of G of rank $n-1$ and obtain $G/H = S \subseteq \mathbb{Q}$. Now clearly

$$\mathcal{TC}(S) \cap \mathcal{TC}(H) \subseteq \mathcal{TC}(G) \text{ and } \mathcal{TC}(G) \subseteq \mathcal{TC}(H).$$

But by Lemma 2.3 we also have that $\mathcal{TC}(G) \subseteq \mathcal{TC}(S)$, hence we obtain $\mathcal{TC}(G) = \mathcal{TC}(H) \cap \mathcal{TC}(S)$. By induction hypothesis $\mathcal{TC}(H) = \mathcal{TC}(R')$ for some rational group R'. Thus $\mathcal{TC}(G) = \mathcal{TC}(S) \cap \mathcal{TC}(R') = \mathcal{TC}(R)$ where $R = S \cap R' \subseteq \mathbb{Q}$. ■

The next theorem shows that we can choose R in Theorem 2.4 to be the outer type of G. Recall that for a torsion-free group G of finite rank the *outer type* $\mathrm{OT}(G)$ is defined as follows: Let $g_1, \ldots, g_n$ be a maximal linearly independet subset of G and put $H_i = \langle g_1, \ldots, \widehat{g_i}, g_{i+1}, \ldots, g_n \rangle_* \subseteq G$, where $\widehat{g}$ means that the element is missing. Then put $S_i = G/H_i \subseteq \mathbb{Q}$ and define $\mathrm{OT}(G) = \sup\{S_1, \ldots, S_n\}$. The set $\{S_i : i \leqslant n\}$ is called the *cotypeset* of G. Note that for a pure subgroup H of G we have $\mathrm{OT}(G) \geqslant \mathrm{OT}(H)$. Similarly, the *inner type* $\mathrm{IT}(G)$ of G is defined as $\mathrm{IT}(G) = \inf\{\langle g_i \rangle_* : i \leqslant n\}$.

THEOREM 2.5. *Let G be a torsion-free group of finite rank and $\mathrm{OT}(G)$ its outer type. Then $\mathcal{TC}(G) = \mathcal{TC}(\mathrm{OT}(G))$.*

PROOF. By Theorem 2.4 we know that $\mathcal{TC}(G) = \mathcal{TC}(R)$ for some rational group R. We induct on the rank n of G to show that we can choose $R = \mathrm{OT}(G)$. For rational groups S it is trivial since $\mathrm{OT}(S) = S$. Hence assume G is of rank $n > 1$. We let S_i be defined as in the definition of $\mathrm{OT}(G)$. Since G is of finite rank we obtain by Lemma 2.3 $\mathcal{TC}(G) = \mathcal{TC}(R) \subseteq \mathcal{TC}(S_i)$ for every $i \leqslant n$ and therefore $\mathcal{TC}(G) = \mathcal{TC}(R) \subseteq \bigcap_{i \leqslant n} \mathcal{TC}(S_i) = \mathcal{TC}(\mathrm{OT}(G))$ since $\mathrm{OT}(G)$ is the supremum of the types S_i. Now assume that $\mathcal{TC}(R)$ is strictly contained in $\mathcal{TC}(\mathrm{OT}(G))$, then there is $T \in \mathcal{TC}(\mathrm{OT}(G)) \setminus \mathcal{TC}(R)$. Thus $T \in \mathcal{TC}(S_i)$ and hence $T \notin \mathcal{TC}(H_i)$ for any $i \leqslant n$. But $\mathrm{OT}(G) \geqslant \mathrm{OT}(H_i)$, hence $\mathcal{TC}(\mathrm{OT}(G)) \subseteq \mathcal{TC}(\mathrm{OT}(H_i)) = \mathcal{TC}(H_i)$ by induction hypothesis and therefore $T \in \mathcal{TC}(H_i)$ - a contradiction. ■

Using a result due to Warfield (see [W]) we can determine the outer type and hence $\mathcal{TC}(G)$ explicitly for a torsion-free group G of finite rank.

LEMMA 2.6 (Warfield, [W]). *Let G be a torsion-free group of finite rank and let F be a free subgroup of G such that G/F is torsion, i.e.*

$$G/F = \bigoplus_p T_p \ \textit{with}\ T_p = \mathbb{Z}(p^{i_{p,1}}) \oplus \ldots \oplus \mathbb{Z}(p^{i_{p,n_p}})$$

and $0 \leqslant i_{p,1} \leqslant \ldots \leqslant i_{p,n_p} \leqslant \infty$. *Then* $\mathrm{OT}(G) = [(i_{p,n_p})]$ *and* $\mathrm{IT}(G) = [(i_{p,n_1})]$.

We will now show that for almost all types R the class $\mathcal{TC}(R)$ can be realized as $\mathcal{TC}(G)$ for an indecomposable, almost decomposable group of rank n for any natural number n. This proves that the structure of the group G is less effected by $\mathcal{TC}(G)$ than for example by $G^{\perp}$ even for finite rank groups in $\mathcal{TS} \cap \mathcal{R}$.

LEMMA 2.7. *If $\mathbb{Q} \not\cong R$ is a type such that $R \not\cong \mathbb{Q}^{(P)}$ for any set P of primes of cardinality at most two, then there exists for any natural number n an indecomposable, almost decomposable group G of rank n such that $\mathcal{TC}(G) = \mathcal{TC}(R)$.*

PROOF. For $n = 1$ there is nothing to prove taking $G = R$. Hence assume that $n > 1$. We may assume without loss of generality that R is in reduced form, i.e. $R = \mathrm{type}^{qr}(R)$. If $R \not\cong \mathbb{Q}^{(P)}$ for any finite set P of primes, then we can give an explicite example. We will use the construction of an almost completely decomposable group as given in [A, Example 2.2]. Hence it is enough to find incomparable types A_i ($i \leqslant n$) such that the supremum of the A_i's is exactly R. But $R \not\cong \mathbb{Q}^{(P)}$ for any finite set P of primes implies that there are infinitely many primes p such that $\chi_p^R(1) = \infty$ or there are infinitely many primes p such that $\chi_p^R(1) = 1$. Let P be the set of these primes. We divide P in n disjoint infinite subsets P_i and define the types A_i by letting

$$\chi_p^{A_i}(1) = \chi_p^R(1) \ \text{ for } \ p \in P_i$$

$$\chi_p^{A_i}(1) = 0 \ \text{ for } \ p \in \bigcup_{j \neq i} P_j$$

$$\chi_p^{A_i}(1) = \chi_p^R(1) \ \text{ else.}$$

Since $\mathbb{Q} \not\cong R$ we may assume without loss of generality that there is a prime p such that $\chi_p^{A_i}(1) = 0$ for all i. Then clearly the supremum of the A_i's is R and by [A, Example 2.2] there exists an indecomposable almost completely decomposable group G of rank n such that $\mathrm{Tst}(G)$ is the meet closure of $\{A_1, \ldots, A_n\}$, hence Lemma 1.5 implies

$$\mathcal{TC}(G) = \mathcal{TC}(\mathrm{Tst}(G)) = \mathcal{TC}(R).$$

If $R \cong \mathbb{Q}^{(P)}$ for some finite set P of primes of cardinality greater than two, then the existence of the desired almost completely decomposable group follows from [AD, Theorem 1.8]. Divide P into two non-empty sets P_1 and P_2 and choose $p \in P_2$. Put T to be the meet closure of the set $\{\mathbb{Q}^{(P_1)}, \mathbb{Q}^{(P_2)}, \mathbb{Q}^{(p)}\}$, then [AD, Theorem 1.8] gives the existence of an almost completely decomposable group G of any finite rank n such that the critical typeset of G is contained in $\{\mathbb{Q}^{(P_1)}, \mathbb{Q}^{(P_2)}, \mathbb{Q}^{(p)}\}$ and hence $\mathcal{TC}(G) = \mathcal{TC}(R)$. ■

Note that if $R \ncong \mathbb{Q}^{(P)}$ for some set of primes of cardinality less or equal to two, then it is known that an indecomposable almost completely decomposable group G with $\mathcal{TC}(G) = \mathcal{TC}(R)$ must have rank at most two by the structure of the critical typeset (see also Lemma 2.11 and [A, Theorem 2.3]).

If we don't require G to satisfy $\mathcal{TC}(G) = \mathcal{TC}(\mathrm{Tst}(G))$, then we can realize any type $\mathbb{Q} \ncong R$ which is divisible by at least one prime (in order to exclude $\mathbb{Z}$) as the outer type of an indecomposable homogeneous group.

Lemma 2.8. *Let $\mathbb{Q} \ncong R$ be a type such that $pR = R$ for at least one prime, then there exists for any natural number n an indecomposable homogeneous torsion-free group G such that $\mathcal{TC}(G) = \mathcal{TC}(R)$. Moreover, if S is the type of G, then $S \otimes \mathbb{Q}^{(p)} = R$.*

Proof. For $n = 1$ the claim is trivial, hence assume $n > 1$. Fix the prime p for which $pR = R$ and let S be the type identical with R but $\chi_p^S(1) = 0$. We now use a known example (see [F2, Example 5 on p. 125]). Let $H = \bigoplus_{i \leq n} \mathbb{Q}^{(p)} a_i$ be completely decomposable and homogeneous of type $\mathbb{Q}^{(p)}$ and of rank n. Chosse $n-1$ algebraically independent p-adic units $\pi_2, \ldots, \pi_n$ and let $\pi_1 = 1$. For $m = 1, 2, \ldots$ we put

$$x_m = p^{-m}(a_1 + \pi_{2m} a_2 + \ldots + \pi_{nm} a_n) \in H$$

where $\pi_{im} = s_{i0} + s_{i1} p + \ldots + s_{i(k-1)} p^k$ is the $(k-1)$st partial sum of the standard form of π_i. We define A as

$$A = \langle a_1, \ldots, a_n, x_1, \ldots, x_k, \ldots \rangle.$$

Then A is indecomposable and homogeneous of type $\mathbb{Z}$. Moreover, for each subgroup B of rank $n-1$ we have $A/B \cong \mathbb{Q}^{(p)}$. Furthermore, A stays indecomposable if it is tensored by any rank one group L such that $pL \neq \neq L$. We put $G = S \otimes A$ to obtain a torsion-free group of rank n which is homogeneous of type S and for each subgroup C of rank $n-1$ we have $G/C \cong S \otimes \mathbb{Q}^{(p)} \cong R$, hence $\mathrm{OT}(G) = R$ and therefore $\mathcal{T}\mathcal{C}(G) = = \mathcal{T}\mathcal{C}(R)$. ■

For $\mathbb{Q}$ we have a similar realization lemma. Recall that a torsion-free group G of finite rank is called *almost-free* if any proper pure subgroup of G is free.

LEMMA 2.9. *For each natural number n there is an indecomposable almost-free group G of rank n such that $\mathcal{T}\mathcal{C}(G) = \mathcal{T}\mathcal{C}(\mathbb{Q})$.*

PROOF. By a construction due to Corner (see [F2, Exercise 8 on p. 128]) there exists an indecomposable torsion-free group G of rank n such that G is almost-free and each quotient of rank 1 is divisible, hence $\mathrm{OT}(G) = \mathbb{Q}$ and we are done. ■

Note that by the construction in Lemma 2.8 the type of G is as close to R as possible since G homogeneous of type R would imply $\mathrm{IT}(G) = = \mathrm{OT}(G)$ and hence [A, Corollary 1.13] implies that G is completely decomposable. But if we assume that the type R is non-idempotent, then we can even do better. Recall that a type R is *idempotent* if $\mathrm{End}(R) \cong R$.

LEMMA 2.10. *Let R be any non-idempotent type. Then there exists for any natural number n an indecomposable group G of rank n which is homogeneous of type R and $\mathcal{T}\mathcal{C}(G) = \mathcal{T}\mathcal{C}(R)$.*

PROOF. Since R is non-idempotent there exists an infinite set P of primes such that $0 < \chi_p^R(1) < \infty$. Put S to be the type such that $\chi_p^S(1) = = \chi_p^R(1)$ whenever $p \notin P$ and $\chi_p^S(1) = \chi_p^R(1) + 1$ if $p \in P$. We will show that there is a torsion-free group G of rank n such that G is homogeneous of type R and $\mathrm{OT}(G) = S$, hence G is as desired. By an easy argument (see e.g. Schultz [Sch]) it is enough to prove that there exists a torsion-free group H of rank n such that H is homogeneous of type $\mathbb{Z}$ and the outer type of H is Q where $Q = \langle 1/p : p \in P \rangle$. Without loss of generality we may assume that $P = \Pi$. Let $F = \bigoplus_{i \leq n} \mathbb{Z}$ be a free group of rank n. We choose n-tuples $t_i = (a_1^i, a_2^i, \ldots, a_n^i) \in F$ such that

(i) $\gcd(a_1^i, a_2^i, \ldots, a_n^i) = 1$;

(ii) $|a_j^i| \leqslant n$ for all $j \leqslant n$, where $|a|$ denotes the absolute value of a.

Let $\Pi = \{p_i : i \in \omega\}$ be an enumeration of the set of primes such that p_i is the i'th prime number. Put $G = F + \langle t_i/p_i : p_i \in \Pi \rangle$. Then G is a torsion-free group of rank n and it remains to show that G is homogeneous of type $\mathbb{Z}$, indecomposable and has outer type Q. The later is easily seen and hence left to the reader. To prove that G is homogeneous of type $\mathbb{Z}$ let $g = (g_1, g_2, \ldots, g_n) \in G$ (viewed inside its divisible hull) and assume that g is divisible by infinitely many primes. Without loss of generality g is divisible by all $p_i \in \Pi$. Then $g \equiv t_i$ modulo $p_i G$ for all $i \in \omega$. Thus without loss of generality $g_1 - a_1^i \neq 0$ is divisible by p_i. But this can happen only for finitely many i since a_1^i is less or equal to n but p_i is the order of $n * log(n)$, where log denotes the logarithm with basis 10 - a contradiction. Therefore, G is homogeneous of type $\mathbb{Z}$. It remains to prove that G is indecomposable. Assume that $G = G_1 \oplus G_2$ with $G_i \neq 0$ and note that $G/F \cong \bigoplus_{i \in \omega} \mathbb{Z}(p_i)$. Thus $F = F_1 \oplus F_2$ with $F_i = F \cap G_i$ and hence $G/F = G_1/F_1 \oplus G_2/F_2$. But then almost all elements t_i must belong either to G_1 or G_2 which is obviously a contradiction (1). ■

As a corollary we obtain a general version of Griffith's solution of the Baer problem for groups of finite rank.

COROLLARY 2.11. *Let G be a torsion-free group of finite rank and homogeneous of the idempotent type R. Then G is completely decomposable if and only if* $\mathcal{TC}(G) = \mathcal{TC}(R)$.

PROOF. One implication is trivial, hence assume that $\mathcal{TC}(G) = \mathcal{TC}(R)$. Since $\mathcal{TC}(G) = \mathcal{TC}(\mathrm{OT}(G))$ we obtain that the quasi-reduced types of R and $\mathrm{OT}(G)$ are equal by Lemma 1.3 but since R is idempotent this implies that the types R and $\mathrm{OT}(G)$ are equal, hence $\mathrm{IT}(G) = R = \mathrm{OT}(G)$ and the result follows by [A, Corollary 1.13]. ■

COROLLARY 2.12. *Let G be a torsion-free group of finite rank. Then G is free if and only if* $\mathcal{TC}(G) = \mathcal{TC}(\mathbb{Z})$.

PROOF. Again one implication is trivial, hence assume that $\mathcal{TC}(G) =$

(1) The author would like to thank Prof. C. Vinsonhaler for indicating the construction to him.

$= \mathcal{TC}(\mathbb{Z})$. To apply Corollary 2.11 we have to show that G is homogeneous of type $\mathbb{Z}$. But $\mathcal{TC}(G) = \mathcal{TC}(\mathbb{Z})$ implies that $\mathrm{OT}(G) = \mathbb{Z}$, hence G must be homogeneous of type $\mathbb{Z}$. ■

Let us note that Corollary 2.11 may fail if we don't assume that R is idempotent (see Lemma 2.10), hence Griffith's solution of the Baer problem cannot be generalized to homogeneous groups of non-idempotent type.

Our final well-known example shows that even for the class of finite rank torsion-free abelian groups we cannot obtain some kind of Krull-Schmidt theorem using $\mathcal{TC}$.

EXAMPLE 2.13 (Jonsson, [J]). *There are indecomposable torsion-free groups* A_1, A_2 *and* B_1, B_2 *of ranks* $1, 3, 2, 2$, *respectively such that* $A_1 \oplus A_2 = B_1 \oplus B_2$ *but* $\mathcal{TC}(A_i) \neq \mathcal{TC}(B_j)$ *for* $i, j \in \{1, 2\}$.

PROOF. The example is well-known and it is easy to check that the constructed groups A_i and B_i for $i = 1, 2$ are almost completely decomposable, hence members of the class $\mathcal{TS}$. Since the four groups contain types of the form $\mathbb{Q}^{(p)}$ for different primes p it follows immediately that $\mathcal{TC}(A_i) \neq \mathcal{TC}(B_j)$ for $i, j \in \{1, 2\}$. ■

3. Torsion-free groups of countable rank.

In this section we shall consider torsion-free groups which are countable and we will first characterize those which belong to the class $\mathcal{R}$.

LEMMA 3.1. *Let* G *be a countable torsion-free group and* T *a torsion group. Then* $T \in \mathcal{TC}(G)$ *if and only if* $T \in \mathcal{TC}(H)$ *for any finite rank pure subgroup* H *of* G.

PROOF. Clearly $T \in \mathcal{TC}(G)$ implies that $T \in \mathcal{TC}(H)$ for any finite rank pure subgroup H of G. Conversely, assume that $T \in \mathcal{TC}(H)$ for every pure subgroup H of G of finite rank. If $T \notin \mathcal{TC}(G)$, then there exists a pure subgroup H of finite rank of G and an element $x + H \in G/H$ violating one of the two conditions of Baer's Lemma 2.2. Note that here we need the countability of G. We write $\langle x + H \rangle_* \subseteq G/H$ as

$$\langle x + H \rangle_* = H'/H$$

for some pure subgroup H' of G of finite rank. Then Lemma 2.2 implies that $\operatorname{Ext}(H', T) \neq 0$, hence $T \notin \mathcal{TC}(H')$ - a contradiction. ■

For our next result we need the following definition which is a reasonable extension of the notion of outer type to groups of countable rank.

Definition 3.2. *Let G be a countable torsion-free group. Then we define the* outer type class OT_G *of G as*

$$\mathrm{OT}_G = \{\mathrm{type}^{qr}(\mathrm{OT}(H)) : H \text{ a pure subgroup of } G \text{ of finite rank}\}.$$

In the following theorem $\mathcal{TC}(\mathrm{OT}_G)$ means $\mathcal{TC}\Big(\bigoplus_{R \in \mathrm{OT}_G} R\Big)$.

Theorem 3.3. *Let G be a countable torsion-free group. Then $\mathcal{TC}(G) = \mathcal{TC}(\mathrm{OT}_G)$. Hence $G \in \mathcal{R}$ if and only if $\bigoplus_{R \in \mathrm{OT}_G} R \in \mathcal{R}$.*

Proof. By Lemma 3.1 we have that $\mathcal{TC}(G) = \cap \mathcal{TC}(H)$ where H ranges over all pure subgroups of G of finite rank. But by Theorem 2.5 it follows that $\mathcal{TC}(H) = \mathcal{TC}(\mathrm{OT}(H)) = \mathcal{TC}(\mathrm{type}^{qr}(\mathrm{OT}(H)))$ for every pure subgroup H of G of finite rank. Hence

$$\mathcal{TC}(G) = \cap \mathcal{TC}(\mathrm{OT}(H)) = \cap \mathcal{TC}(\mathrm{type}^{qr}(\mathrm{OT}(H))) = \mathcal{TC}(\mathrm{OT}_G)$$

which finishes the proof. ■

An easy corollary is the well-known result of Pontryagin.

Corollary 3.4. *Let G be a countable torsion-free group. Then G is free if and only if each pure finite rank subgroup of G is free.*

Proof. One implication is trival, hence assume that any pure subgroup H of G of finite rank is free. Then $\mathrm{OT}(H) = \mathbb{Z}$, hence by Theorem 3.3 $\mathcal{TC}(G) = \mathcal{TC}(\mathrm{OT}_G) = \mathcal{TC}(\mathbb{Z})$ and thus G is free by Griffith's solution of the Baer problem (see [G]). ■

Theorem 3.3 reduces the question of whether or not a countable torsion-free group G belongs to $\mathcal{R}$ to completely decomposable groups with types in quasi-reduced form. Note that the supremum of infinitely many types need not exist as the following example demonstrates.

EXAMPLE 3.5. *Let Π_i $(i \in \omega)$ be a system of disjoint infinite subsets of the primes Π. Put $R_i = \langle 1/p : p \in \Pi_i \rangle \subseteq \mathbb{Q}$. Then the types R_i $(i \in \omega)$ have no supremum.*

We need another definition.

DEFINITION 3.6. *Let $S = \{R_i \colon i \in I\}$ be a set of types. Then S is called infinity-closed* (∞-closed) *if for any infinite set P of primes such that for all $p \in P$ there exists $i \in I$ such that $\chi_p^{R_i}(1) = \infty$, there exists $i \in I$ such that $\chi_p^{R_i}(1) \neq 0$ for almost all $p \in P$.*

PROPOSITION 3.7. *Let C be completely decomposable with types in quasi-reduced form. Then $C \in \mathcal{R}$ if and only if $S = \sup(\mathrm{Tst}(C))$ exists and* $\mathrm{Tst}(C)$ *is ∞-closed. In this case $\mathcal{TC}(C) = \mathcal{TC}(S)$.*

PROOF. If the supremum S of $\mathrm{Tst}(C)$ exists and is ∞-closed, then it is easily checked that $\mathcal{TC}(S) = \mathcal{TC}(C)$ using that all the types of C are quasi-reduced. Conversely, if $\mathcal{TC}(C) = \mathcal{TC}(R)$ for some rational group R, then we first have to show that the supremum of $\mathrm{Tst}(C)$ exists. Without loss of generality we may assume that R is in quasi-reduced form. Hence Lemma 1.3 implies that $R \geqslant S$ for any type $S \in \mathrm{Tst}(C)$ since all types are quasi-reduced. Now assume that R is not the supremum of $\mathrm{Tst}(C)$. Then there exists a type L such that $L < R$ and $L \geqslant S$ for all types $S \in \mathrm{Tst}(C)$. Therefore there exists

(i) an infinite set of primes P such that $\chi_p^R(1) = 1$ and $\chi_p^L(1) = 0$ for all primes $p \in P$ or

(ii) there exists a prime p such that $\chi_p^R(1) = \infty$ and $\chi_p^L(1) < \infty$.

In both cases we easily obtain a contradiction since $\mathcal{TC}(C) = \mathcal{TC}(\mathrm{Tst}(C))$ and $L \geqslant S$ for every type $S \in \mathrm{Tst}(C)$. Thus $\sup(\mathrm{Tst}(C))$ exists and clearly $\mathrm{Tst}(C)$ must be ∞-closed. ■

COROLLARY 3.8. *Let G be a countable torsion-free group. Then $G \in \mathcal{R}$ if and only if* $\sup(\mathrm{OT}_G)$ *exists and* OT_G *is ∞-closed.* ■

PROOF. The claim follows by Theorem 3.3 and Proposition 3.7. ■

It is now trivial to see that for groups $G \in \mathfrak{C}$ we also have $G \in \mathcal{R}$ if and only if $\sup(\mathrm{Tst}(G))$ exists and $\mathrm{Tst}(G)$ is ∞-closed.

Finally let us remark that in contrast to the finite rank case for countable groups also the behavior of the class $\mathcal{R}$ is as ugly as the behavior of the class $\mathcal{TS}$ as the following examples show.

EXAMPLE 3.9. Let Π_i $(i \in \omega)$ be disjoint infinite sets of primes. Put $R_i = \langle 1/p : p \in \Pi_i \rangle \subseteq \mathbb{Q}$ and let $R = \langle 1/p : p \in \Pi_i (i \in \omega) \rangle \subseteq \mathbb{Q}$. Then the supremum of the types R_i $(i \in \omega)$ does not exist but if we adjoin R, then the supremum of $\{R, R_i : i \in \omega\}$ exists and is equal to R.

EXAMPLE 3.10. *Enumerate all primes by* ω, *e.g.* $\Pi = \{p_i : i \in \omega\}$ *and let* $R_i = \langle 1/(p_i^n) : n \in \omega \rangle \subseteq \mathbb{Q}$ *for* $i \in \omega$. *Moreover, let* $R = \langle 1/p : p \in \Pi \rangle \subseteq \mathbb{Q}$, *then the set* $\{R_i : i \in \omega\}$ *is not* ∞*-closed but the set* $\{R, R_i : i \in \omega\}$ *is* ∞*-closed.*

Thus a set of types which is ∞-closed may contain a subset which is not ∞-closed and the same holds for sets of types such that there supremum exists. We obtain the following corollary which contrasts the finite rank case.

COROLLARY 3.11. *The class of all countable torsion-free groups* $G \in \mathcal{R}$ *is neither closed under direct summands nor epimorphic images nor extensions.*

PROOF. We take the types R_i $(i \in \omega)$ and R from Example 3.9 or from Example 3.10 and put $C = \bigoplus_{i \in \omega} R_i$. Then $C \notin \mathcal{R}$ by Proposition 3.7. But $C \oplus R \in \mathcal{R}$, hence the claim follows. ■

4. Torsion-free groups of arbitrary rank.

In this section we consider torsion-free groups of arbitrary rank and try to generalize the results from Section 2 and Section 3, i.e. we are looking for a characterization of groups $G \in \mathcal{R}$. We don't have a characterization in terms of types any longer as we had in the finite rank and coutable case but we can characterize the groups by some conditions on certain extension groups. Let us first recall a basic theorem from [SW] which characterizes the classes of torsion groups that can appear as $\mathcal{TC}(C)$ for some completely decomposable group C.

THEOREM 4.1 (Strüngmann-Wallutis, [SW]). *Let* $\mathfrak{C}$ *be a class of torsion groups. Then* $\mathfrak{C} = \mathcal{TC}(C)$ *for some completely decomposable group* C *if and only if the following conditions are satisfied:*

(i) $\mathfrak{C}$ *contains all torsion cotorsion groups*;

(ii) $\mathfrak{C}$ *is closed under epimorphic images*;

(iii) $\bigoplus_{n \in \omega} \mathbb{Z}(p^n) \in \mathfrak{C}$ *if and only if* $\mathfrak{C}$ *contains all p-groups for all primes p*;

(iv) *If P is an infinite set of primes then,* $\bigoplus_{p \in P} \mathbb{Z}(p) \in \mathfrak{C}$ *if and only if* $\bigoplus_{p \in P} T_p \in \mathfrak{C}$ *for all p-groups* $T_p \in \mathfrak{C}$;

(v) *If P is an infinite set of primes such that* $\bigoplus_{p \in P} \mathbb{Z}(p) \notin \mathfrak{C}$ *then there exists an infinite subset* P' *of P such that* $\bigoplus_{p \in X} \mathbb{Z}(p) \notin \mathfrak{C}$ *for all infinite* $X \subseteq P'$.

PROOF. See Theorem 3.6 in [SW]. ■

We will now first give a homological characterization of the groups in $\mathcal{TS}$ which clarifies the connection to Butler groups. We therefore define the class $\mathcal{B}(G)$ for a torsion-free group G as $\mathcal{B}(G) = \{T \text{ torsion} : \mathrm{Bext}^1(G, T) = 0\}$.

THEOREM 4.2. *Let G be a torsion-free group. Then* $G \in \mathcal{TS}$ *if and only if* $\mathcal{TC}(\mathrm{Tst}(G)) \subseteq \mathcal{B}(G)$.

PROOF. Let G be a torsion-free group such that $G \in \mathcal{TS}$. Let $C = \bigoplus_{g \in G} \langle g \rangle_*$. We obtain the following exact sequence

$$0 \to K \to C \to G \to 0$$

where the mapping $C \to G$ is the obvious one and K is the corresponding kernel. In fact, this sequence is easily checked to be balanced exact and hence we obtain the induced sequences

$$\mathrm{Hom}(C, T) \to \mathrm{Hom}(K, T) \to \mathrm{Bext}^1(G, T) \to \mathrm{Bext}^1(C, T) = 0$$

and

$$\mathrm{Hom}(C, T) \to \mathrm{Hom}(K, T) \to \mathrm{Ext}(G, T) \to \mathrm{Ext}(C, T)$$

with the same mapping $\mathrm{Hom}(C, T) \to \mathrm{Hom}(K, T)$ for any torsion group T. If $T \in \mathcal{TC}(\mathrm{Tst}(G))$, then $\mathrm{Ext}(C, T) = 0$, hence also $\mathrm{Ext}(G, T) = 0$ and thus the mapping $\mathrm{Hom}(C, T) \to \mathrm{Hom}(K, T)$ is surjective, i.e. $\mathrm{Bext}^1(G, T) = 0$. Hence $\mathcal{TC}(C) \subseteq \mathcal{B}(G)$ as claimed. Conversely, assume that $\mathcal{TC}(C) \subseteq \mathcal{B}(G)$. In order to prove that $G \notin \mathcal{TS}$ we have to show that

$\mathcal{T}\mathcal{C}(C) \subseteq \mathcal{T}\mathcal{C}(G)$ to prove $G \in \mathcal{T}\mathcal{S}$. Therefore let $T \in \mathcal{T}\mathcal{C}(C)$. By assumption we obtain $\mathrm{Bext}^1(G, T) = 0$, hence again the mapping $\mathrm{Hom}(C, T) \to \to \mathrm{Hom}(K, T)$ is surjective and it follows that $\mathrm{Ext}(G, T) \subseteq \mathrm{Ext}(C, T)$ which is zero by the choice of T. Hence $\mathrm{Ext}(G, T) = 0$ and therefore $T \in \mathcal{T}\mathcal{C}(G)$. ■

An immediate corollary is the following. Recall that a torsion-free group G is a B_1-group if $\mathcal{B}(G) = \mathfrak{T}$.

Corollary 4.3 (Strüngmann-Wallutis, [SW]). *Let G be a B_1-group. Then $G \in \mathcal{T}\mathcal{S}$.*

Question 4.4. *To characterize the Butler groups among the groups in $\mathcal{T}\mathcal{S}$ one need to know the following: which is the minimal class of torsion groups $\mathfrak{H}$ such that $\mathfrak{H} \subseteq \mathcal{B}(G)$ forces G to be a Butler group?*

Our next theorem will give a characterization of the classes that can appear as $\mathcal{T}\mathcal{C}(R)$ for some rank-1 group R by adding two more conditions in Theorem 4.1. Therefore we need the following definition.

Definition 4.5. *Let $\mathfrak{C}$ be a class of torsion groups and P_1 and P_2 two subsets of the primes.*

(i) $\mathcal{P} = \{P \subseteq \Pi \ (P \text{ infinite}) \mid \bigoplus_{p \in X} \mathbb{Z}(p) \notin \mathfrak{C} \text{ for all infinite } X \subseteq P\}$;

(ii) *We say that P_1 and P_2 are equivalent ($P_1 \sim P_2$) if their symmetric difference $P_1 \Delta P_2$ is finite;*

(iii) *For $P \in \mathcal{P}$ we let*

$$[P] = \{Q \in \mathcal{P} : P \sim Q\}$$

be the equivalence class of P inside $\mathcal{P}$;

(iv) $\mathcal{M}_{\mathfrak{C}} = \{[P] : P \in \mathcal{P}\}$;

(v) *For $[P], [Q] \in \mathcal{M}_{\mathfrak{C}}$ we define $[P] \subseteq [Q]$ if and only if P is almost contained in Q, i.e. $P \backslash Q$ is finite; hence $(\mathcal{M}_{\mathfrak{C}}, \subseteq)$ is a partially ordered set.*

Note that the definition of $\subseteq$ in (v) is well-defined and so is a finite union of equivalence classes $\bigcup_{i \leqslant n} [P_i] = \left[\bigcup_{i \leqslant n} P_i\right]$. But the union of infinite many equivalence classes cannot be defined independently of the chosen representatives!

THEOREM 4.6. *Let* $\mathfrak{C}$ *be a class of torsion groups. Then* $\mathfrak{C} = \mathcal{T}\mathcal{C}(R)$ *for some rational group* R *if and only if the following conditions are satisfied:*

(i) $\mathfrak{C}$ *contains all torsion cotorsion groups*;

(ii) $\mathfrak{C}$ *is closed under epimorphic images*;

(iii) $\bigoplus_{n\in\omega}\mathbb{Z}(p^n)\in\mathfrak{C}$ *if and only if* $\mathfrak{C}$ *contains all* p*-groups for all primes* p;

(iv) *If* P *is an infinite set of primes then,* $\bigoplus_{p\in P}\mathbb{Z}(p)\in\mathfrak{C}$ *if and only if* $\bigoplus_{p\in P}T_p\in\mathfrak{C}$ *for all* p*-groups* $T_p\in\mathfrak{C}$;

(v) *If* P *is an infinite set of primes such that* $\bigoplus_{p\in P}\mathbb{Z}(p)\notin\mathfrak{C}$ *then there exists an infinite subset* P' *of* P *such that* $\bigoplus_{p\in X}\mathbb{Z}(p)\notin\mathfrak{C}$ *for all infinite* $X\subseteq P'$;

(vi) *If* P *is an infinite set of primes such that* $\mathfrak{C}$ *contains only bounded* p*-groups for* $p\in P$, *then* $\bigoplus_{p\in P}\mathbb{Z}(p)\notin\mathfrak{C}$;

(vii) *The set* $\mathcal{M}_{\mathfrak{C}}$ *contains a unique maximal element.*

PROOF. One implication is almost trivial. If R is a rank-1 group, then obviously conditions (vi) and (vii) are satisfied; for (vii) define $P=\{p\in\Pi:\chi_R(1)\neq 0\}$ to see that $[P]$ is maximal in $\mathcal{M}_{\mathfrak{C}}$. Hence it remains to show the converse implication. As in the proof given in [SW] of Theorem 4.1 define the set

$$M=\left\{p\in\Pi \;\middle|\; \bigoplus_{n\in\omega}\mathbb{Z}(p^n)\notin\mathfrak{C}\right\}$$

and let $S=\mathbb{Q}^{(M)}=\left\{\frac{m}{n}: m,\, n\in\mathbb{Z},\, n \text{ is a product of powers of primes from } M\right\}$. Moreover, let $[Q]$ be the maximal element in $\mathcal{M}_{\mathfrak{C}}$ and put $R=\left\langle\frac{1}{p}: p\in Q\right\rangle$. Note that the type of R is independent of the representative chosen for $[Q]$. By the proof of Theorem 4.1 (see [SW]) all we have to show is that for $C=\bigoplus_{p\in M}\mathbb{Q}^{(p)}\oplus\bigoplus_{P\in\mathcal{P}}R_P$ we have $\mathcal{T}\mathcal{C}(C)=\mathcal{T}\mathcal{C}(S\oplus R)$ where $\mathbb{Q}^{(p)}$ and R_P are defined as in the proof of Theorem 4.1 (see [SW]). It is easy to see that $\mathcal{T}\mathcal{C}(S)=\mathcal{T}\mathcal{C}\left(\bigoplus_{p\in M}\mathbb{Q}^{(p)}\right)$ using condition (vi). Moreover, since $Q \in \mathcal{P}$ we clearly have $\mathcal{TC}\left(\bigoplus_{P \in \mathcal{P}} R_P\right) \subseteq \mathcal{TC}(R)$. Conversely, if a torsion group T satisfies $\operatorname{Ext}(R, T) = 0$, then $T_p = 0$ for almost all primes $p \in Q$. But $[Q]$ is uniquely maximal in $\mathcal{M}_{\mathfrak{C}}$ and hence any set $P \in \mathcal{P}$ is almost contained in Q and therefore also $T_p = 0$ for almost all primes $p \in P$. Thus $\operatorname{Ext}(R_P, T) = 0$ which shows $\mathcal{TC}(R) \subseteq \mathcal{TC}\left(\bigoplus_{P \in \mathcal{P}} R_P\right)$. ■

By Theorem 2.4 any finite rank torsion-free group must satisfy the conditions (i) to (vii) of Theorem 4.6. It is surprising that condition (vi) is easily checked for finite rank torsion-free groups (in fact for countable groups G condition (vi) is equivalent to saying that $\operatorname{OT}(G)$ is ∞-closed) but condition (vii) is less obvious and hard to verify.

Note that both conditions (vi) and (vii) may fail in the countable rank case even for completely decomposable groups. (see Example 3.9 and Example 3.10).

Finally we state some properties of the set $\mathcal{M}_{\mathfrak{C}}$.

Lemma 4.7. *Let G be a torsion-free group. Then*

(i) *If $[P] \in \mathcal{M}_{\mathcal{TC}(G)}$ is uniquely maximal, then $[P]$ satisfies: If for some set of primes X, $X \cap P$ is finite, then $\bigoplus_{p \in X} \mathbb{Z}(p) \in \mathcal{TC}(G)$;*

(ii) *If $\mathcal{M}_{\mathcal{TC}(G)}$ is inductive, then there exists a unique maximal element $[P]$ in $\mathcal{M}_{\mathcal{TC}(G)}$;*

(iii) *every chain in $\mathcal{M}_{\mathcal{TC}(G)}$ is countable.*

Proof. Let $T_X = \bigoplus_{p \in X} \mathbb{Z}(p)$ and assume $T_X \notin \mathcal{TC}(G)$, then there exists by [SW, Proposition 3.5] an infinite subset Y of X such that $\bigoplus_{p \in Y} \mathbb{Z}(p) \notin \mathcal{TC}(G)$ and for every infinite $W \subseteq Y$ we have $\bigoplus_{p \in W} \mathbb{Z}(p) \notin \mathcal{TC}(G)$. But by the unique maximality of $[P]$ we obtain that Y is almost contained in P, hence $X \cap P$ is infinite - a contradiction. Thus (i) holds.

The proof of (ii) is trivial: for if $[P]$ and $[Q]$ are distinct maximal elements, then $[P \cup Q]$ is again an element of $\mathcal{M}_{\mathcal{TC}(G)}$ contradicting the maximality of $[P]$ and $[Q]$.

Finally, if we have a chain $\{[X_i] : i \in I\}$ in $\mathcal{M}_{\mathcal{TC}(G)}$, then I must be countable since by definition X_i is almost contained in X_{i+1} for every i and the set Π of primes is countable. ■

It is not clear for which torsion-free groups G the set $\mathcal{M}_{\mathcal{T}\mathcal{C}(G)}$ is inductive but it seems unlikely that there is a «more natural» characterization of the groups in $\mathcal{R}$.

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