

Kaplansky Classes.

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1. Introduction.

In 1979 L. Salce introduced the notion of a cotorsion theory [15]. But only after P. Eklof and J. Triflaj [4] proved their completeness theorem did it become clear what an important role the notion would play in homological algebra. For example an easy application of their completeness result settled the flat cover conjecture.

More recently, M. Hovey [11] has shown that there is an intimate connection between Quillen's model of structure on abelian categories and complete cotorsion theories with respect to some proper class of short exact sequences. Hovey gave examples of such model structures using the cotorsion theories which appear in the context of the so called Gorenstein modules (these involve generalizations of notions introduced by Auslander [1] in connections with his study of G-dimensions). It is now well known that over Gorenstein rings, the classes of Gorenstein injective and projective modules form part of two corresponding complete cotorsion theories (see [9] for example).

Our aim in this paper is to study classes of modules which we define as Kaplansky classes and which behave well under Eklof and Triflaj's techniques. Establishing the fact that certain classes are Kaplansky classes allows us to prove general existence theorems above covers and envelopes in the Gorenstein setting. We also relate these classes to the notions of μ -dimension of modules (see [10]).

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Given \mathcal{L} a class of R -modules, we will denote by \mathcal{L}^\perp , respectively ${}^\perp\mathcal{L}$, the class of R -modules M such that $\text{Ext}_R^1(L, M) = 0$ respectively $\text{Ext}_R^1(M, L) = 0$, for every $L \in \mathcal{L}$. \mathcal{L}^\perp and ${}^\perp\mathcal{L}$ are called the orthogonal classes of \mathcal{L} . Now, using this notation, we say that a pair of classes of R -modules $(\mathcal{L}, \mathcal{C})$ is a cotorsion theory if and only if $\mathcal{L}^\perp = \mathcal{C}$ and ${}^\perp\mathcal{C} = \mathcal{L}$.

Let \mathcal{F} be a class of R -modules. A morphism $\varphi : F \rightarrow M$ where $F \in \mathcal{F}$ is called an \mathcal{F} -precover of M if for any morphism $\psi : F' \rightarrow M$ with $F' \in \mathcal{F}$ we get a commutative diagram

$$\begin{array}{ccc} F' & & \\ \downarrow & \searrow \psi & \\ F & \xrightarrow{\varphi} & M \end{array}$$

In case that $F' = F$ and $\psi = \varphi$ and we can only complete the diagram by automorphisms we say that $\varphi : F \rightarrow M$ is an \mathcal{F} -cover. Dually we have the definitions of an \mathcal{F} -preenvelope and \mathcal{F} -envelope. We say that an \mathcal{F} -precover $\varphi : F \rightarrow M$ is special if $\text{Ker}(\varphi) \in \mathcal{F}^\perp$. Dually we have the definitions of an \mathcal{F} -preenvelope, \mathcal{F} -envelope and special \mathcal{F} -preenvelope.

We note that \mathcal{F} -precovers need not be surjective. But if \mathcal{F} contains all the projective modules, then we easily see that such a precover is surjective. Likewise, \mathcal{F} -preenvelopes are necessary injective when \mathcal{F} contains all injective modules.

By a right \mathcal{F} -resolution of M we will mean a complex $0 \rightarrow M \rightarrow F^0 \rightarrow F^1 \rightarrow \dots$ with each $F^i \in \mathcal{F}$ and such that $\text{Hom}(-, F)$ makes the complex exact for any $F \in \mathcal{F}$. A finite complex $0 \rightarrow M \rightarrow F^0 \rightarrow \dots \rightarrow F^n$ with all $F^i \in \mathcal{F}$ and which is made exact by all $\text{Hom}(-, F)$ with $F \in \mathcal{F}$ is called a partial right resolution of length n . It is clear that right and left \mathcal{F} -resolutions may be constructed using \mathcal{F} -preenvelopes and \mathcal{F} -precovers respectively.

Finally we recall from [10] the definitions of $\lambda_{\mathcal{F}}$ and $\mu_{\mathcal{F}}$ -dimension of a module. We say that $\mu_{\mathcal{F}}(M) = -1$ if M does not have an \mathcal{F} -preenvelope. If $n \geq 0$, we write $\mu_{\mathcal{F}}(M) = n$ if the maximum length of a right partial \mathcal{F} -resolution of M is n . We will say that $\mu_{\mathcal{F}}(M) = \infty$ if there exists such a partial right \mathcal{F} -resolution of M for every $n \geq 0$.

Note that by the dual of [10, Corollary 2.6], if $\mu_{\mathcal{F}}(M) = \infty$ then M has a right \mathcal{F} -resolution. The $\lambda_{\mathcal{F}}$ -dimension is defined dually.

2. Kaplansky classes.

DEFINITION 2.1. *Let \mathcal{F} be a class of R -modules. Then \mathcal{F} is said to be a Kaplansky class if there exists a cardinal \mathcal{N} such that for every $M \in \mathcal{F}$ and for each $x \in M$, there exists a submodule F of M such that $x \in F \subseteq M$, $F, M/F \in \mathcal{F}$ and $\text{Card}(F) \leq \mathcal{N}$.*

REMARK 1. *The preceding definition is based on a result of Kaplansky [13] which says that if P is a projective R -module and $x \in P$ then there is a countably generated submodule S of P with $x \in S$ and with S and P/S projective (or equivalently, with S a summand of P). It is easy to argue that the class of injective modules and the class of flat modules are Kaplansky (see [9, Lemma 5.3.12] for a proof of the latter).*

The next results will be used to prove the existence of preenvelopes relative to Kaplansky classes. The two lemmas are due to F. Maeda [14] (also see C. Jensen [12, Lemma 1.4]).

LEMMA 2.1. *Let I be a right directed set. If $S \subseteq I$ is a set such that $\text{Card}(S) \leq \mathcal{N}$ for \mathcal{N} some infinite cardinal then there exists a right directed set J such that $S \subseteq J \subseteq I$ and with $\text{Card}(J) \leq \mathcal{N}$.*

PROOF. For each pair of elements $i, j \in S$ let $k \in I$ such that $i, j \leq k$. Now let $S_0 = S$ and S_1 the set formed by all the preceding k 's. Then $\text{Card}(S_1) \leq \text{Card}(S) + \text{Card}(S \times S) \leq \mathcal{N}$. Let now S_n be defined as before from S_{n-1} for $n \geq 2$. Then $J = \bigcup_{n \geq 0} S_n$ is the desired set. ■

LEMMA 2.2. *Let I be a set such that $\text{Card}(I) > \aleph_0$. Then there exists a well ordered chain*

$$J_0 \subseteq J_1 \subseteq J_2 \subseteq \dots \subseteq J_\omega \subseteq J_{\omega+1} \subseteq \dots \subseteq J_\alpha \subseteq \dots$$

with $\alpha < \lambda$ (λ an ordinal) such that $\bigcup_{\alpha < \lambda} J_\alpha = I$ and where each J_i is a right directed set such that $\text{Card}(J_i) < \text{Card}(I)$.

PROOF. We well order I so that $\alpha = \text{Ord}(I)$ is an initial ordinal, i. e., $\text{Card}(\beta) < \text{Card}(\alpha)$ for every $\beta < \alpha$.

Let now $(i_\beta)_{\beta < \alpha}$ be a family such that the i_β 's are the elements of I . We will construct

$$J_0 \subseteq J_1 \subseteq J_2 \subseteq \dots \subseteq J_\omega \subseteq J_{\omega+1} \subseteq \dots \subseteq J_\beta \subseteq \dots$$

such that each J_β is right directed and $i_\beta \in J_\beta$ for $\beta < \alpha$ and with $\text{Card}(J_\beta) < \text{Card}(I)$ also for each $\beta < \alpha$.

If $\text{Card}(I) = \aleph_0$ we may take every J_n finite, so let us assume that $\text{Card}(I) > \aleph_0$. Then we may get $J_0 \subseteq J_1 \subseteq \dots \subseteq J_n \subseteq \dots$ with each J_i finite and with $i_n \in J_n$. Then let $J' = \bigcup_{n=0}^{\infty} J_n$. By the preceding Lemma we find a right directed set J_ω such that $J_\omega = J' \cup \{i_\omega\}$ verifying that $\text{Card}(J_\omega) = \aleph_0$.

Now we proceed by transfinite induction. If we have J_β with $\text{Card}(J_\beta) \leq \text{Card}(\beta)$ (β infinite) then we get $J_\alpha \subseteq J_{\alpha+1}$ having the desired property. If $\lambda < \alpha$ is a limit ordinal and we have obtained J_β for $\beta < \alpha$ then we define $J' = \bigcup_{\beta < \lambda} J_\beta$ and we enlarge J' to J_λ such that $i_\lambda \in J_\lambda$ by the preceding Lemma and $\text{Card}(J_\lambda) = \text{Card}(J')$ and then the result follows. ■

PROPOSITION 2.3. *Let \mathcal{F} be a class of R -modules. If \mathcal{F} is closed under will ordered direct limits then it is closed under arbitrary direct limits.*

PROOF. Let $((M_i), (\varrho_{ij}))_{i \in I}$ be a directed system. We will make transfinite induction on $\text{Card}(I)$.

If $\text{Card}(I) = n < \aleph_0$ there is nothing to prove.

If $\text{Card}(I) = \aleph_0$ then there exists a cofinal set $J \subseteq I$ with $J = \{j_0, j_1, \dots\}$ where $j_0 < j_1 < \dots$. In this way $\varinjlim_{i \in I} M_i = \varinjlim_{i \in J} M_j \in \mathcal{F}$ by hypothesis.

We now assume that $\text{Card}(I) > \aleph_0$. In this case there exists

$$J_0 \subseteq J_1 \subseteq J_2 \subseteq \dots \subseteq J_\omega \subseteq J_{\omega+1} \subseteq \dots \subseteq J_\alpha \subseteq \dots$$

$\alpha < \lambda$ for λ an ordinal such that $\bigcup_{\alpha < \lambda} J_\alpha = I$ where each J_α is a right directed set and $\text{Card}(J_\alpha) < \text{Card}(I)$. Then $\varinjlim_{i \in I} M_i = \varinjlim_{\alpha < \lambda} \left(\varinjlim_{i \in J_\alpha} M_i \right)$. Since by induction hypothesis $\varinjlim_{i \in J_\alpha} M_i \in \mathcal{F}$, by the hypothesis in the statement we get that $\varinjlim_{i \in I} M_i \in \mathcal{F}$. ■

THEOREM 2.4. *Let \mathcal{F} be a class of R -modules closed under direct limits. Then the class \mathcal{C} of R -modules M such that $\mu_{\mathcal{F}}(M) = \infty$ is also closed under direct limits.*

PROOF. By the preceding we only have to show that the class of R -modules M such that $\mu_{\mathcal{F}}(M) = \infty$ is closed under well ordered direct limits. So let us assume that $((M_\alpha), (\varrho_{\beta\alpha}))_{\alpha < \lambda}$ is a directed system in \mathcal{C} . If $\lambda = n < \omega$ then $\varinjlim M_\alpha = M_{n-1}$ and so we have finished.

Then let $\lambda = \omega$ and let us start by showing that $\varinjlim M_n$ ($n < \omega$) is in \mathcal{C} . We may suppose that the \mathcal{F} -preenvelope of each M_n is an injection (if $\varphi : M \rightarrow F$ is an \mathcal{F} -preenvelope so is $\varphi(M) \rightarrow F$). If $M_0 \rightarrow F^0$ is an \mathcal{F} -preenvelope, we consider the pushout diagram of $M_0 \rightarrow F^0$ and $M_0 \rightarrow M_1$

$$\begin{array}{ccccccc} 0 & \longrightarrow & M_0 & \longrightarrow & F^0 & \longrightarrow & F^0/M_0 \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & M_1 & \longrightarrow & P & \longrightarrow & P/M_1 \longrightarrow 0 \end{array}$$

Since $P/M_1 \cong F^0/M_0$ and M_1 have an \mathcal{F} -preenvelope, say $P/M_1 \rightarrow F'$ and $M_1 \rightarrow F'$ and since there is a morphism $P \rightarrow F'$ we may construct an \mathcal{F} -preenvelope for P , $P \rightarrow F^1 = F \oplus F'$. Then $M_1 \rightarrow F^1$ is an \mathcal{F} -preenvelope.

Therefore we have the commutative diagram

$$\begin{array}{ccc} M_0 & \longrightarrow & F^0 \\ \downarrow & & \downarrow \\ M_1 & \longrightarrow & F^1 \end{array}$$

which has the property that if

$$\begin{array}{ccc} M_0 & \longrightarrow & F^0 \\ \downarrow & & \downarrow \\ M_1 & \longrightarrow & G \end{array}$$

is commutative with $G \in \mathcal{F}$ then there exists a morphism of diagrams

$$\begin{array}{ccc} M_0 & \longrightarrow & F^0 \\ \downarrow & & \downarrow \\ M_1 & \longrightarrow & F^1 \end{array} \longrightarrow \begin{array}{ccc} M_0 & \longrightarrow & F^0 \\ \downarrow & & \downarrow \\ M_1 & \longrightarrow & G \end{array}$$

which is the identity on M_0, M_1 and F^0 . If we continue with this process we get a commutative diagram

$$\begin{array}{ccc} M_0 & \longrightarrow & F^0 \\ \downarrow & & \downarrow \\ M_1 & \longrightarrow & F^1 \\ \downarrow & & \downarrow \\ M_2 & \longrightarrow & F^2 \\ \downarrow & & \downarrow \\ \vdots & & \vdots \end{array}$$

Now since each

$$\begin{array}{ccc} M_n & \longrightarrow & F^n \\ \downarrow & & \downarrow \\ M_{n+1} & \longrightarrow & F^{n+1} \end{array}$$

has the preceding property we easily see that $\varinjlim M_n \rightarrow \varinjlim F^n$ is an \mathcal{F} -preenvelope.

Then by the dual of [10, Corollary 2.6] ($\mu_{\mathcal{F}}(M) = \infty$) $C^n = \text{Coker}(M_n \rightarrow F^n)$ has an \mathcal{F} -preenvelope $\forall n \geq 0$, and so, if we consider the system $C^0 \rightarrow C^1 \rightarrow \dots$, by the reasoning above we get that

$$C = \varinjlim C^n = \text{Coker}(\varinjlim M^n \rightarrow \varinjlim F^n)$$

has an \mathcal{F} -preenvelope. Continuing in this manner we see that

$$\mu_{\mathcal{F}}\left(\varinjlim_{n < \omega} M_n\right) = \infty.$$

Now we reindex the modules $M_0, M_1, \dots, M_\omega, M_{\omega+1}, \dots$ such that $M_\omega = \varinjlim M_n$ and $M_{\omega+1}$ is the old M_ω and so forth. Therefore we may as-

sume that the system $(M_\alpha)_{\alpha < \lambda}$ is continuous, i. e., $M_\beta = \varinjlim_{\alpha < \omega} M_\alpha$ if β is a limit ordinal with $\beta < \lambda$. Then using transfinite induction it is clear that the argument generalizes and we get that $\varinjlim_{\alpha < \lambda} M_\alpha$ is in \mathcal{K} . ■

THEOREM 2.5. *Let R be a ring and \mathcal{F} a Kaplansky class closed under direct limits. The following assertions are equivalent:*

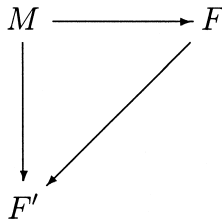
- i) $\mu_{\mathcal{F}}(M) = \infty$ for every finitely presented R -module M .*
- ii) $\mu_{\mathcal{F}}(M) = \infty$ for every R -module M .*
- iii) Every R -module has an \mathcal{F} -preenvelope.*
- iv) \mathcal{F} is closed under direct products.*

PROOF. *i) \Rightarrow ii)* is a consequence of Theorem 2.4.

ii) \Rightarrow iii) follows from the definition of $\mu_{\mathcal{F}}$ -dimension.

iii) \Rightarrow iv) Let $(F_i)_{i \in I}$ be any family of R -modules in \mathcal{F} and $\varphi : \prod_{i \in I} F_i \rightarrow F$ an \mathcal{F} -preenvelope. Then there is a morphism $f : F \rightarrow \prod_{i \in I} F_i$ such that $\varphi \circ f$ is the identity map and so $\prod_{i \in I} F_i \in \mathcal{F}$.

iv) \Rightarrow i) Let $M \in R\text{-Mod}$ finitely presented and let $F \in \mathcal{F}$. Since \mathcal{F} is a Kaplansky class, for every morphism $M \xrightarrow{f} F$ there exist $F' \in \mathcal{F}$ and a cardinal \mathcal{N} such that $\text{Card}(F') \leq \mathcal{N}$ and f factors $M \rightarrow F' \rightarrow F$. Then we say that any two morphisms $M \rightarrow F$ and $M \rightarrow F'$ with $F, F' \in \mathcal{F}$ and with $\text{Card}(F), \text{Card}(F') \leq \mathcal{N}$ are equivalent if and only if any diagram



can be completed by an isomorphism. Now if we take X a set of representatives of such $M \rightarrow F$, then $M \rightarrow \prod_X F$ is an \mathcal{F} -preenvelope. The result now follows in the usual way by considering the cokernel of this \mathcal{F} -preenvelope. ■

REMARK 2. *Given a class \mathcal{F} of R -modules we could say that a ring R is right \mathcal{F} -coherent if \mathcal{F} is closed under direct products. Then If \mathcal{F} is*

the class of injective modules then it is clear that every ring is \mathcal{F} -coherent. If \mathcal{F} is the class of flat modules then \mathcal{F} -coherency becomes usual coherency.

REMARK 3. Note that in Theorem 2.5 we don't need the hypothesis that the Kaplansky class is closed under direct limits to show the equivalence between ii), iii) and iv).

Now we give applications of Theorem 2.5 and prove the existence of Gorenstein injective preenvelopes.

PROPOSITION 2.6. If R is a left noetherian ring and \mathcal{F} is the class of Gorenstein injective left R -modules then \mathcal{F} is a Kaplansky class.

PROOF. Let $M \in \mathcal{F}$ and $x \in M$. Then there exist an exact sequence in $R\text{-Mod}$

$$S^* = \dots \rightarrow E^{-2} \xrightarrow{d^{-2}} E^{-1} \xrightarrow{d^{-1}} E^0 \xrightarrow{d^0} E^1 \xrightarrow{d^1} \dots$$

where every E^i is an injective R -module and such that $M = \text{Ker}(E^0 \rightarrow E^1)$ and remains exact whenever $\text{Hom}(E, -)$ is applied for any injective R -module E . We will construct a of a Gorenstein injective submodule S with $x \in S$ using a procedure with two steps: first we will construct an exact complex in $R\text{-Mod}$ of injective R -modules and from this we will find another complex in $R\text{-Mod}$ of injective R -modules such that $\text{Hom}(E, -)$ leaves this complex exact.

So if $x \in M$, since $E^{-1} \xrightarrow{f} M$ is surjective, there is $y \in E^{-1}$ such that $f(y) = x$. Then consider $\langle y \rangle \rightarrow F_0$ the inclusion and we get by Remark 1 a cardinal \mathcal{N}_0 and a submodule $S^{-1} \subseteq E^{-1}$ pure such that $\langle y \rangle \subseteq S^{-1}$ and $\text{Card}(S^{-1}) \leq \mathcal{N}_0$. Let now $f(S^{-1}) \subseteq M$ and observe that $f(S^{-1}) \subseteq E^0$. As before we get $S^0 \subseteq E^0$ pure and a cardinal \mathcal{N}_1 such that $\text{Card}(S^0) \leq \mathcal{N}_1$. Then consider the quotient $S^0/f(S^{-1})$ and get $S^1 \subseteq E^1$ and \mathcal{N}_2 such that $\text{Card}(S^1) \leq \mathcal{N}_2$.

Now we reverse the process in the opposite direction and consider $S^1 \cap E^0/M$. Then there exists a submodule D^0 of E^0 which applies in $S^1 \cap E^0/M$. We get again $S'^0 \subseteq E^0$ pure and \mathcal{N}_3 such that $D^0 \subseteq S'^0$ and $\text{Card}(S'^0) \leq \mathcal{N}_3$. Let $d^0(S'^0)$ and obtain $S'^1 \subseteq E^1$ pure and \mathcal{N}_4 such that $d^0(S'^0) \subseteq S'^1$ and $\text{Card}(S'^1) \leq \mathcal{N}_4$. Now let $S'^0 \cap M$ and since f is surjective there exists $D^{-1} \subseteq E^{-1}$ which applies in the preceding module. Let $f(D^{-1}) \subseteq M$ and $D^{-2} \subseteq E^{-2}$ which applies in

$f(D^{-1})$. We obtain $S'^{-2} \subseteq E^{-2}$ pure and \mathcal{N}_5 such that $D^{-2} \subseteq S'^{-2}$ and $\text{Card}(S'^{-2}) \leq \mathcal{N}_5$.

Again we start the construction going forward and we consider $d^{-2}(S'^{-2}) \subseteq E^{-1}$ and proceed as before, going n steps forward, going back $n + 1$ steps and $n + 2$ forward again.

Then we take the union of all the complexes constructed in the «zig-zag» process

$$S^* = \dots \rightarrow S^{-2} \rightarrow S^{-1} \rightarrow S^0 \rightarrow S^1 \rightarrow \dots$$

and we consider $S = \text{Ker}(S^0 \rightarrow S^1)$, which is a submodule of M which contains the element x and that by the construction, there exists a cardinal \mathcal{N} such that $\text{Card}(S) \leq \mathcal{N}$. The previous complex is exact by its construction and it is formed by injective modules since all of them are pure submodules of injective modules.

For the second step in the proof we make some preliminary remarks. Our aim is to construct $S \subseteq M$ with an exact sequence as before and such that it remains exact when $\text{Hom}(E, -)$ is applied for every E injective. But since R is left noetherian there is a set X of injective R -modules such that every injective R -modules is a direct sum of copies of modules in X . So let us take $I = \bigoplus_{E_x \in X} E_x$. Then if $\text{Hom}(I, -)$ makes the preceding complex exact, $\text{Hom}(E, -)$ will do the same for every injective E .

Now let us consider the complex

$$\dots \xrightarrow{d'^{-3}} \text{Hom}(I, S^{-2}) \xrightarrow{d'^{-2}} \text{Hom}(I, S^{-1}) \xrightarrow{d'^{-1}} \text{Hom}(I, S^0) \xrightarrow{d^0} \dots$$

This complex is a subcomplex of

$$\dots \xrightarrow{d^{-3}} \text{Hom}(I, E^{-2}) \xrightarrow{d^{-2}} \text{Hom}(I, E^{-1}) \xrightarrow{d^{-1}} \text{Hom}(I, E^0) \xrightarrow{d^0} \dots$$

which is exact by the above. Suppose then without lost of generality that $\text{Ker}(d'^{-1}) \neq \text{Im}(d'^{-2})$. Then there exist $S'^{-2} \subseteq E^{-2}$ pure and a cardinal \mathcal{N}_1 such that $S^{-2} \subseteq S'^{-2}$, $\text{Ker}(d'^{-1}) \subseteq \text{Im}(d'^{-2} |_{\text{Hom}(I, S'^{-2})})$ and $\text{Card}(S'^{-2}) \leq \mathcal{N}_1$. Consider now S' the image of S'^{-2} by the morphism $E^{-2} \rightarrow E^{-1}$ and let $S'^{-1} \subseteq E^{-1}$ pure and \mathcal{N}_2 such that $S' \subseteq S'^{-1}$, $\text{Im}(d'^{-2} |_{\text{Hom}(I, S'^{-2})}) \subseteq \text{Hom}(I, S'^{-1})$ and $\text{Card}(S'^{-1}) \leq \mathcal{N}_2$. Then let S' be the image of S'^{-1} by the morphism $E_0 \rightarrow M$. Now we enlarge S^0 to $S'^0 \subseteq F^0$ pure and we find \mathcal{N}_3 such that $\text{Im}(d^{-1} |_{\text{Hom}(I, S'^{-1})}) \subseteq \text{Hom}(I, S'^0)$ and $\text{Card}(S'^0) \leq \mathcal{N}_3$. Then we go back again and start another «zig-zag» process with $\text{Ker}(d^0 |_{\text{Hom}(I, S'^0)})$ and $\text{Im}(d^{-1} |_{\text{Hom}(I, S'^{-1})})$.

We consider the union of all the complexes we get in this last «zig-zag» process

$$T^* = \cdots \rightarrow T_1 \rightarrow T_0 \rightarrow T_{-1} \rightarrow T_{-2} \rightarrow \cdots$$

T^* has the property that when $\text{Hom}(E, -)$ is applied we get an exact complex for every injective R -module E , but it may happen that it is not exact. So we apply again the «zig-zag» process we used to get S^* and then we get an exact complex but that may not remain exact when $\text{Hom}_R(E, -)$ is applied. So again we use the same reasoning to get T^* and we obtain a new complex. The «limit» over these two procedures gives us a module S , a cardinal \mathcal{N} , and a complex S^* as we desired.

Finally M/S is also Gorenstein injective since the quotient complex E^*/S^* is exact and it remains exact when $\text{Hom}(E, -)$ is applied for any injective R -module E because E^* and S^* verify the two conditions. ■

Then as a direct consequence of Proposition 2.6 and Theorem 2.5 we get the following result.

COROLLARY 2.7. *If R is a left noetherian ring then every R -module has a Gorenstein injective preenvelope.*

REMARK 4. *We note that Gorenstein injective envelopes exist for every module over a Gorenstein ring (cf. [8]), but from the preceding Corollary we get that only the noetherian property is needed to get Gorenstein injective preenvelopes.*

The use of Kaplansky classes to get preenvelopes doesn't always require that the class be closed under direct products. If we drop this condition and we assume that the class is closed under direct limits then we get the following general result relating Kaplansky classes and envelopes.

The next result is a combination of [4, Theorem 10] and of [17, Theorem 2.2.2].

THEOREM 2.8. *Let \mathcal{F} be a Kaplansky class. If \mathcal{F} is closed under extensions and direct limits then every module has an \mathcal{F}^\perp -envelope.*

PROOF. Since \mathcal{F} is a Kaplansky class, if $F \in \mathcal{F}$ then F can be written as the direct union of a continuous chain of submodules $(F_\alpha)_{\alpha < \lambda}$ with λ an ordinal number such that $F_0 \in \mathcal{F}$, $F_{\alpha+1}/F_\alpha \in \mathcal{F}$ when $\alpha + 1 < \lambda$ with $\text{Card}(F_0), \text{Card}(F_{\alpha+1}/F_\alpha) \leq \mathcal{N}$ for some cardinal \mathcal{N} . Therefore if B is the direct sum of all representatives of \mathcal{F} such that their cardinals are less than or equal to \mathcal{N} , then $M \in \mathcal{F}^\perp$ if and only if $\text{Ext}_R^1(B, M) = 0$.

Now let N be any R -module. Now we use the procedure in [4, Theorem 10] to get an exact sequence $0 \rightarrow N \rightarrow A \rightarrow F \rightarrow 0$ such that $A \in \mathcal{F}^\perp$ and $F \in \mathcal{F}$ and the proof follows by [17, Theorem 2.2.2]. ■

REMARK 5. *As an application we have that if R is noetherian and \mathcal{F} is the class of injective R -modules, it follows immediatly by the preceding Theorem that every R -module has an \mathcal{F}^\perp -envelope. We note that when R is Gorenstein then \mathcal{F}^\perp becomes the class of Gorenstein injective modules.*

Now we give an application of Kaplansky classes to cotorsion theories. We recall that a cotorsion theory $(\mathcal{L}, \mathcal{C})$ is said to be complete (cf. [11]) if it has enough injectives and projectives, that is, for every R -module M there are exact sequences $0 \rightarrow M \rightarrow C \rightarrow L \rightarrow 0$ and $0 \rightarrow \overline{C} \rightarrow \overline{L} \rightarrow M \rightarrow 0$ respectively with $C, \overline{C} \in \mathcal{C}$ and $L, \overline{L} \in \mathcal{L}$. Then it is clear from the preceding that in this case every module has an \mathcal{L} -precover and a \mathcal{C} -preenvelope.

DEFINITION 2.2. *A cotorsion theory $(\mathcal{L}, \mathcal{C})$ is said to be perfect if every module has an \mathcal{L} -cover and a \mathcal{C} -envelope.*

By [9, Theorem 7.2.6] it easily proved that a cotorsion theory $(\mathcal{F}, \mathcal{C})$ where \mathcal{F} is closed under direct limits is perfect if and only if is complete. Now reasoning as in [9, Theorem 7.4.6] we get the following result.

THEOREM 2.9. *Let \mathcal{F} be a Kaplansky class. If \mathcal{F} contains the projective modules and it is closed under extensions and direct limits then $(\mathcal{F}, \mathcal{F}^\perp)$ is a perfect cotorsion theory in $R\text{-Mod}$.*

Now we give other applications to the so-called Gorenstein flat modules.

We recall from [7] that an R -module M is said to be Gorenstein flat if and only if there exists an exact sequence in $R\text{-Mod}$

$$\cdots \rightarrow F_1 \rightarrow F_0 \rightarrow F_{-1} \rightarrow F_{-2} \rightarrow \cdots$$

of flat R -modules such that $M = \text{Ker}(F_{-1} \rightarrow F_{-2})$ and which remains exact whenever $E \otimes_R -$ is applied for any injective right R -module E .

In the sequel we will denote by \mathcal{C} the smallest class of R -modules we obtain by the following recursive definition:

- i) $E(M) \in \mathcal{C}$ for all finitely generated modules M .
- ii) If $M \in \mathcal{C}$ and $T \subseteq M$ is a submodule, then $E(M/T) \in \mathcal{C}$.

This class verifies some useful and interesting properties. We easily see that all $E \in \mathcal{C}$ are injective. It is also shown that there exists a cardinal \mathcal{N} such that $\text{Card}(E) \leq \mathcal{N}$ for all $E \in \mathcal{C}$ and that every injective module E may be expressed as the direct limit of a directed system of submodules of E which are in \mathcal{C} .

Then we apply all these properties to proof the following result.

PROPOSITION 2.10. *Given a ring R , the class of Gorenstein flat modules is a Kaplansky class.*

PROOF. This follows by reasoning analogously to Proposition 2.6. For this we use that every injective module E may be expressed as the direct limit of a family of injective submodules which are in \mathcal{C} . On the other hand, as a consequence of the fact that the cardinal of every module in \mathcal{C} is bounded by a certain cardinal there exists a set of representatives of \mathcal{C} . Then we consider the module $I = \bigoplus E_i$ where E_i runs over the preceding set of representatives. Now if a sequence is such that $I \otimes_R -$ leaves it exact, by the commutativity of the tensor products with direct sums we get that $E_i \otimes_R -$ will also leave the sequence exact and from the commutativity of direct limits and the tensor products our sequence will remain exact under $E \otimes_R -$ for every injective module E . ■

If we denote by \mathcal{F} the class of Gorenstein flat modules, then Theorem 2.9 gives the following.

COROLLARY 2.11. *If R be a coherent ring then $(\mathcal{F}, \mathcal{F}^\perp)$ is a perfect cotorsion theory.*

Another application to Gorenstein flat modules is that the existence of Gorenstein flat preenvelopes for every module is equivalent to this class is closed under direct products. Examples of these rings are Gorenstein rings (cf. [7]) or even the more general class appearing in [3].

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