Examples of Birationality of Pluricanonical Maps.

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ABSTRACT - By generalizing an Enriques construction, in $\mathbb{P}^4$ we construct a double space $V$ of degree 12, whose branch locus has a 6-ple point of the type $z^6 + \cdots + x^{12} + \cdots + y^{12} = 0$. We demonstrate that a desingularization of $V$ has birational invariants $q_1 = q_2 = 0$, $p_g = P_1 = 3$, $P_2 = 7$, $P_3 = 13$, $P_4 = 22$, $P_5 = 34$, $P_6 = 51$. Moreover, we prove that the $m$-canonical transformation has fibers that are generically finite sets if and only if $m \geq 2$ and it is birational if and only if $m \geq 6$.

Introduction.

E. Bombieri [B] proved that the $m$-canonical transformation of any nonsingular surface of general type is birational if $m \geq 5$ and $m = 5$ is the minimum for the surfaces (minimal models) with $(K^2) = 1$ and $p_g = 2$.

F. Enriques constructed a surface with $(K^2) = 1$, $p_g = 2$ (see [E] § 14, pp. 303-304); this is a desingularization of a double plane with a branch curve of degree 10, having a singular [5,5] point on it.

At a seminar, E. Stagnaro suggested generalizing the Enriques double plane to a three-dimensional double space for constructing new examples of threefolds, whose $m$-canonical transformation becomes birational if $m$ is large enough.

This paper touches first on a demonstration of the fact that the $m$-canonical transformation of the Enriques example is birational if and only if $m \geq 5$, then such a situation is generalized, constructing a double

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space $V$. We thus have the birationality of the $m$-canonical transformation if and only if $m \geq 6$. A desingularization of $V$ has the birational invariants $q_1 = q_2 = 0$, $p_0 = P_1 = 3$, $P_2 = 7$, $P_3 = 13$, $P_4 = 22$, $P_5 = 34$, $P_6 = 51$.

We define double space of degree $2n$ the projective closure in $\mathbb{P}^4$ of the affine hypersurface given by $t^2 = f_{2n}(x, y, z)$, being $f_{2n}(x, y, z)$ a polynomial of degree $2n$; the surface of equation $f_{2n}(x, y, z) = 0$ is the branch locus of the double space.

We must bear in mind that a double plane with a branch curve of degree $10$ with a singular $[5, 5]$ point on it is affinely represented by an equation of the type $z^2 + y^5 + \cdots + x^{10} = 0$. In the following paragraphs, said situation will be generalized by constructing a double space affinely given by an equation of the type $t^2 + z^6 + \cdots + x^{12} + \cdots + y^{12} = 0$.

M. Chen [C] and S. Lee [L] proved that if the canonical divisor $K$ of a threefold is «nef» and $(K^2)$ is positive, then the $m$-canonical transformation is birational for $m \geq 6$. In the proposed example the said properties are not simultaneously satisfied, but the birationality of the $m$-canonical transformation holds true for $m \geq 6$.

In this paper we consider surfaces and threefolds on the field $\mathbb{C}$ of the complex numbers and we’ll write $\mathbb{P}^N$ instead of $\mathbb{P}^N_\mathbb{C}$.

1. Example of a double plane $S$ of degree $10$ in $\mathbb{P}^3$ whose $m$-canonical transformation is birational if and only if $m \geq 5$.

1.1. Description of $S$.

Let us choose a generic curve $C$ in the linear system of curves in $\mathbb{P}^2$ defined by

$$F_{10}(X_0, X_1, X_2) = aX_0^5X_2^5 + bX_0X_2^9 + cX_1^{10} + dX_2^{10}.$$ 

According to Bertini theorem, $C$ has its unique singularity at the point $A_0 = (1, 0, 0)$. To be more precise, $C$ has a $[5, 5]$ point at $A_0$, i.e. a 5-ple point with an infinitely near 5-ple point. By using the affine coordinates

$$x = \frac{X_1}{X_0}, \quad y = \frac{X_2}{X_0}, \quad z = \frac{X_3}{X_0}.$$
we obtain the polynomial 
\[ f_{10}(x, y) = ay^5 + by^9 + cx^{10} + dy^{10} \]
and hence the double plane of affine equation \( z^2 = f_{10}(x, y) \). Let \( S \) be its projective closure in \( \mathbb{P}^3 \):
\[
S : X_0^4 X_2^2 - aX_0^3 X_2^2 - bX_0 X_2^9 - cX_1^{10} - dX_2^{10} = 0.
\]

\( S \) is normal and its singularities are the points \( A_3 = (0, 0, 0, 1) \) and \( A_0 = (1, 0, 0, 0) \). To be more precise:

- \( S \) has an 8-ple point at \( A_3 \) and four double curves \( r_1, r_2, r_3, r_4 \) infinitely near in the next neighbourhoods;
- \( S \) has a double point at \( A_0 \) with a double curve \( r_5 \), a double point \( P \) and again two double curves \( r_6 \) and \( r_7 \) infinitely near, in the next neighbourhoods.

1.2. Birationality of the \( m \)-canonical transformation for \( m \geq 5 \).

We state the birationality of the \( m \)-canonical transformation, \( m \geq 5 \), using the theory of adjoints of Enriques. This theory has recently been revised by E. Stagnaro in [S2]. We keep the same nomenclature and notations as are used in said paper. In our examples all the singularities satisfy the hypothesis assumed in [S2].

The properties of a double plane are well known, but it may be useful to mention the ones that will be generalized to the hypersurface (double space) in \( \mathbb{P}^4 \) that we construct later on.

It is maybe less well known, however see [E], [S1], [S2] (a detailed calculation of the bicanonical adjoints is given in [S1]), that the \( m \)-canonical adjoints to a double plane of affine equation \( S : z^2 = f_{2n}(x, y) \), with a nonsingular branch curve \( f_{2n}(x, y) = 0 \), are:
\[
\phi_{m(n-3)}(x, y) + z\phi_{(m-1)n-3m}(x, y) = 0,
\]
where \( \phi_i(x, y) \) denotes a polynomial of degree \( i \) in \( x, y \).

In compliance with [S2], let us call the \( m \)-canonical adjoints defined by \( \phi_{m(n-3)}(x, y) = 0 \) as global and the \( m \)-canonical adjoints defined by \( z\phi_{(m-1)n-3m}(x, y) = 0 \) as non-global.

Let us emphasize the following facts.

1. The \( m \)-canonical transformation \( \psi_{[mK]} \) coincides (on an open set), up to isomorphisms, with the rational transformation \( \psi_{[m]|S} \) pro-
duced by the linear system of the $m$-canonical adjoints restricted to the double plane $S$ (see [S2], section 16).

2. If we want $\psi_{m|S}$ to be birational, it is necessary (but generally not sufficient) for at least one of the $m$-canonical adjoints to be of the kind $z\phi_{(m-1)-a} (x, y) = 0$. Conversely, the transformation is generically $2:1$, at most.

3. It is possible to prove (but we omit the demonstration) that in every $m$-canonical adjoint, $m \leq 4$, the $z$ coefficient vanishes as soon as the branch curve has a [5,5] point on it.

4. From 2 and 3 it follows for $m \leq 4$ that $\psi_{m|S}$, so $\varphi_{|mK|}$, cannot be birational. Moreover, one can prove directly that $\psi_{5|S}$ is birational and also that $\psi_{m|S}$ is birational for $m \geq 5$, because $p_g$ is positive.

The idea for generalizing all this to double spaces is to transfer the properties 1, 2, 3 and 4 to a suitable double space. As a result, in the case of our example at least, the birationality holds true if and only if $m \geq 6$.

2. Example of a double space $V$ of degree 12 in $\mathbb{P}^4$, whose $m$-canonical transformation is birational if and only if $m \geq 6$.

2.1. Description of $V$.

To extend the foregoing situation to $\mathbb{P}^4$, let $S$ be a generic surface in the linear system of surfaces in $\mathbb{P}^3$ defined by

$$F_{12}(X_0, X_1, X_2, X_3) = aX_0^6 X_3^6 + bX_0 X_3^{11} + cX_1^{12} + dX_2^{12} + eX_3^{12}.$$ 

According to Bertini theorem, $S$ has a unique singularity at the point $A_0 = (1, 0, 0, 0)$. To be more specific, $S$ has a 6-pie point at $A_0$ with an infinitely near 6-pie curve. By using the affine coordinates

$$x = \frac{X_1}{X_0}, \quad y = \frac{X_2}{X_0}, \quad z = \frac{X_3}{X_0}, \quad t = \frac{X_4}{X_0}$$

we obtain the polynomial

$$f_{12}(x, y, z) = az^6 + bz^{11} + cx^{12} + dy^{12} + ez^{12}$$

and hence the hypersurface of affine equation $t^2 = f_{12}(x, y, z)$. 

Let $V$ be its projective closure in $\mathbb{P}^4$:

$$V : X_0^{10} X_1^2 - aX_0^6 X_2^4 - bX_0 X_3^{11} - cX_1^{12} - dX_2^{12} - eX_3^{12} = 0.$$ 

We call $V$ a \textit{double space}, according to our definition.

$V$ is normal and only has singularities at $A_4 = (0, 0, 0, 0, 1)$ and at $A_0 = (1, 0, 0, 0, 0)$. To be more precise:

- $V$ has a 10-ple point at $A_4$ with 5 double surfaces $\alpha_1, \ldots, \alpha_5$ infinitely near, in the next neighbourhoods,
- $V$ has a double point at $A_0$ with 2 double surfaces $\alpha_6, \alpha_7$, 1 double curve $s$, and 2 double surfaces $\alpha_s, \alpha_3$ infinitely near, in the next neighbourhoods.

\section*{2.2. Computation of $p_g = P_1$ and $P_m$ of $V$.}

Now we compute the genus and plurigenera of $V$, i.e.

$$P_m = \dim \mathcal{H}^0(X, \mathcal{O}_X(mK_X)) = \dim |mK_X| + 1, \quad m \geq 1, \quad p_g = P_1,$$

where $X$ denotes a nonsingular model of $V$.

The path chosen for constructing $X$ consists in two sequences of relations owing to the singularities of $V$ at $A_4$ and $A_0$.

To solve the singularity at $A_4$ we have the following sequence of blow-ups:

$$V_6 \subset \mathbb{P}_6 \xrightarrow{\pi_6} \mathbb{P}_5 \xrightarrow{\pi_5} \mathbb{P}_4 \xrightarrow{\pi_4} \mathbb{P}_3 \xrightarrow{\pi_3} \mathbb{P}_2 \xrightarrow{\pi_2} \mathbb{P}_1 \xrightarrow{\pi_1} \mathbb{P}_0 = V$$

where $\pi_i$ denotes the blow-up of $\mathbb{P}_i$ at $A_4$ and $\pi_i$ ($2 \leq i \leq 6$) is the blow-up of $\mathbb{P}_{i-1}$ along $\alpha_{i-1}$. From (1) the relations follow:

$$\begin{aligned}
K_{\pi_i} &= \pi_i^*(K_{\pi_i}) + 3E_{A_i} \\
V_i &= \pi_i^*(V_{i-1}) - 10E_{A_i}
\end{aligned}$$

where $E_{A_i}, E_{\alpha_i}$ denotes the exceptional divisors of the blow-ups at $A_4$ and $\alpha_i$, and $V_i$ denotes the strict transformation of $V_{i-1}$.

To solve the singularity at $A_0$ we have the following sequence of blow-ups:

$$V_{12} \subset \mathbb{P}_{12} \xrightarrow{\pi_{12}} \mathbb{P}_{11} \xrightarrow{\pi_{11}} \mathbb{P}_{10} \xrightarrow{\pi_{10}} \mathbb{P}_9 \xrightarrow{\pi_9} \mathbb{P}_8 \xrightarrow{\pi_8} \mathbb{P}_7 \xrightarrow{\pi_7} \mathbb{P}_6 \xrightarrow{\pi_6} V_6$$

(in the following $V_{12}$ will be $X$), where $\pi_7$ is the blow-up of $\mathbb{P}_6$ at $A_0$, $\pi_8$ and $\pi_9$ are the blow-ups of $\mathbb{P}_7$ and $\mathbb{P}_8$ along $\alpha_6$ and $\alpha_7$, $\pi_{10}$ is the blow-up of $\mathbb{P}_9$ along $s$ and finally $\pi_{11}$ and $\pi_{12}$ are the blow-ups of $\mathbb{P}_{10}$ and $\mathbb{P}_{11}$ along
\( \alpha_s \) and \( \alpha_9 \). From (2) we can say that:

\[
\begin{align*}
K_{p_1} &= \pi^s_7(K_{p_1}) + 3E_{A_0} \\
V_7 &= \pi^s_7(V_6) - 2E_{A_s} \\
K_{p_2} &= \pi^s_8(K_{p_2}) + E_{a_8} \\
V_8 &= \pi^s_8(V_7) - 2E_{a_9}
\end{align*}
\]

where \( E_{A_0}, E_{a_8} \), and \( E_s \) denote the exceptional divisors of the blow-ups at \( A_0, \alpha_8 \) and \( s \).

Because \( X \) is nonsingular, we can apply the adjunction formula that states: if \( D \) is a divisor linearly equivalent to \( K_{p_{12}} + X \), i.e. \( D \equiv K_{p_{12}} + X \), and if \( D_{|X} \) is defined, then \( D_{|X} = K_X \), where \( K_X \) is a canonical divisor on \( X \).

Substituting from the above relations, we obtain

\[
K_{p_{12}} + X = \pi^s_{12}(\pi^s_{11}(\pi^s_{10}(\pi^s_9(\pi^s_8(\pi^s_7(\pi^s_6(\pi^s_5(\pi^s_4(\pi^s_3(\pi^s_2(\pi^s_1(\pi^s_0(K_{p_1} + V) - 7E_{A_1} - E_{a_1} - E_{a_2} - E_{a_3} - E_{a_4} - E_{a_5} - E_{a_6} - E_{a_7} - E_{a_8}) - E_{a_9} \).\]

We now have \( K_{p_{14}} \equiv -5H \) and \( V \equiv 12H \), where \( H \) is a hyperplane in \( \mathbb{P}^4 \). If \( \Phi_H \equiv 7H \) denotes a hypersurface of degree 7 in \( \mathbb{P}^4 \), we deduce from (3)

\[
K_{p_{14}} + X \equiv \pi^s_{14}(\pi^s_{13}(\pi^s_{12}(\pi^s_{11}(\pi^s_{10}(\pi^s_9(\pi^s_8(\pi^s_7(\pi^s_6(\pi^s_5(\pi^s_4(\pi^s_3(\pi^s_2(\pi^s_1(\pi^s_0(K_{p_1} + V - 7E_{A_1} - E_{a_1} - E_{a_2} - E_{a_3} - E_{a_4} - E_{a_5} - E_{a_6} - E_{a_7} - E_{a_8}) - E_{a_9} - E_{a_{10}} - E_{a_{11}} + E_{A_0} - E_{a_3} - E_{a_4} - E_{a_5} - E_{a_6} - E_{a_7} - E_{a_8} - E_{a_9} = D.\]

We see from the adjunction formula that, if \( D_{|X} \) is defined, then it is a canonical divisor \( K_X \) on \( X \), i.e. \( D_{|X} = K_X \).

If we multiply (4) by the integer \( m \geq 1 \), we obtain

\[
m(K_{p_{14}} + X) \equiv \pi^s_{14}(\pi^s_{13}(\pi^s_{12}(\pi^s_{11}(\pi^s_{10}(\pi^s_9(\pi^s_8(\pi^s_7(\pi^s_6(\pi^s_5(\pi^s_4(\pi^s_3(\pi^s_2(\pi^s_1(\pi^s_0(K_{p_1} + V - 7mE_{A_1} - mE_{a_1} - mE_{a_2} - mE_{a_3} - mE_{a_4} - mE_{a_5} - mE_{a_6} - mE_{a_7} - mE_{a_8} - mE_{a_9} = mD = D',
\]

where \( \Phi_{7m} \) is a hypersurface of degree 7m in \( \mathbb{P}^4 \).
As before we obtain $D'|_X = mK_X$.

Let $\sigma: X \rightarrow V$, where $\sigma = \pi_{12} \circ \ldots \circ \pi_2 \circ \pi_1$, be the desingularization of $V$ described.

Using the theory of adjoints and pluriadjoints, we can calculate $p_g = P_1$ and $P_m$; again we use the nomenclature and notations of [S 2]. $\Phi_{7m}, m \geq 1$, is an $m$-canonical adjoint to $V$ (with respect to $\sigma$) if $D'|_X$ is effective, i.e. $D'|_X \geq 0$ (see [S 2], section 2).

We see first how the presence of the singular point $A_4$ characterizes the canonical and $m$-canonical adjoints.

The condition $\pi^\sharp_7 (\Phi_7) - 7E_{A_4} \geq 0$ in (4), given by $A_4$, says that if $\Phi_7$ is a global canonical adjoint, then $A_4$ must be a 7-ple point for $\Phi_7$ itself, i.e. $\Phi_7$ is defined by a form $F_7$ in $X_0, X_1, X_2, X_3$. The further condition given by $A_4$

\[ \pi^\sharp_6 (\pi^\sharp_5 (\pi^\sharp_4 (\pi^\sharp_3 (\pi^\sharp_2 (\pi^\sharp_1 F_7)) - 7E_{A_4}) - E_{a_1}) - E_{a_2}) - E_{a_3}) \geq 0 \]

(see (4)), implies that it is

\[ F_7(X_0, X_1, X_2, X_3, X_4) = X^5_0 F_2(X_0, X_1, X_2, X_3). \]

The condition

\[ [\pi^\sharp_6 (\pi^\sharp_5 (\pi^\sharp_4 (\pi^\sharp_3 (\pi^\sharp_2 (\pi^\sharp_1 F_7)) - 7mE_{A_4}) - mE_{a_1}) - mE_{a_2}) - mE_{a_3}] \]

imposed by $A_4$ on the $m$-canonical adjoints (see (5)) implies that

\[ F_{7m}(X_0, X_1, X_2, X_3, X_4) = X^{5m}_0 [X^5_0 X^4_1 F_{2m-\delta}(X_0, X_1, X_2, X_3) + F_{2m}(X_0, X_1, X_2, X_3)]. \]

So we have a situation much the same as the double plane. To be more precise, the $m$-canonical adjoints to a double space of affine equation $t^2 = f_{2m}(x, y, z)$, with a nonsingular branch locus $f_{2m}(x, y, z) = 0$, are:

\[ \phi_{m(n-\delta)}(x, y, z) + t\phi_{(m-1)n-\delta}(x, y, z) = 0 \]

where $\phi_i(x, y, z)$ denotes a polynomial of degree $i$ in $x, y, z$.

Here again, let us call the $m$-canonical adjoints given by $\phi_{m(n-\delta)}(x, y, z) = 0$ global and those given by $t\phi_{(m-1)n-\delta}(x, y, z) = 0$ non-global.

Now let us examine the point $A_0$, which is a singular point for the double space because there is a 6-ple point on its branch locus.
From (4) it must be that
\[ F_7(X_0, X_1, X_2, X_3, X_4) = X_0^3 X_3 (a_1 X_1 + a_2 X_2 + a_3 X_3). \]

Let \( W_i \) be the vector space of the forms defining global canonical adjoints and \( \omega_i \) be the vector space of the forms defining canonical adjoints. Since \( W_i = \omega_i \) and \( p_g = \dim |K_X| + 1 \) (see [S2], section 3), it follows that \( p_g = 3 \).

We can move on now to consider the point \( A_0 \) for calculating the \( m \)-canonical adjoints \((m > 1)\). The conditions imposed by \( A_0 \) produce different results, depending on the value of \( m \).

For \( m < 6 \) the vector spaces of the forms defining global \( m \)-canonical adjoints, \( W_{7,m} \), and those of the forms defining \( m \)-canonical adjoints, \( \omega_{7,m} \), coincide; but the equality does not hold true for \( m = 6 \). Indeed, being an \( m \)-canonical adjoint implies that
\[ \Phi_{7m} : \phi_{m(6-4)}(x, y, z) + t \phi_{(m-1)6-4m}(x, y, z) = 0 \]
must satisfy the condition (see (5)):
\[ [\pi_{12}(\pi_{10}(\pi \Phi_{7m}) + mE_{A_0} - mE_{A_0}) - mE_{A_0}) - mE_{A_0})]|x \geq 0. \]

Now, if \( m < 6 \), the degree of the «t» coefficient is too low and it satisfies the condition (6) if and only if \( \phi_{(m-1)6-4m}(x, y, z) \) vanishes. So, for \( m < 6 \), \( \Phi_{7m} \) is an \( m \)-canonical adjoint if and only if it is defined by a form
\[ F_{7m}(X_0, X_1, X_2, X_3, X_4) = X_0^5 X_3^m F_m(X_0, X_1, X_2, X_3), \]
i.e. if and only if \( \Phi_{7m} \) is really a global \( m \)-canonical adjoint.

To be more precise, we have
\[ \omega_{74} = W_{74} = \{ X_0^{10} X_3^5 (b_1 X_0 X_1 + b_2 X_1 X_2 + b_3 X_1 X_2 + b_4 X_1 X_3 + + b_5 X_2^2 + b_6 X_2 X_3 + b_7 X_3^2), b_i \in \mathbb{C} \}; \]
\[ \omega_{721} = W_{721} = \{ X_0^{15} X_3^5 (b_1 X_0 X_1 X_2 + b_2 X_0 X_2 X_3 + \cdots + b_5 X_2 X_3 + b_6 X_3^2), b_i \in \mathbb{C} \}; \]
\[ \omega_{728} = W_{728} = \{ X_0^{20} X_3^5 (b_1 X_0 X_1 X_2 X_3 + b_2 X_0 X_2 X_3 + \cdots + b_5 X_2 X_3 + b_6 X_3^2), b_i \in \mathbb{C} \}; \]
\[ \omega_{735} = W_{735} = \{ X_0^{25} X_3^5 (b_1 X_0 X_1 X_2 X_3 + b_2 X_0 X_2 X_3 + \cdots + b_5 X_2 X_3 + b_6 X_3^2), b_i \in \mathbb{C} \}. \]
If $m=6$, the degree of the $a_t$ coefficient is $(m-1)6-4m=6$. This is the minimum that can satisfy condition (6) and we have the first non-global $m$-canonical adjoint which is affinely given by $tz^6=0$. To be more specific, $\Phi_{7m}$ is an $m$-canonical adjoint ($m=6$) if and only if it is defined by a form

$$F_{42}(X_0, X_1, X_2, X_3, X_4) = X_0^{20}[X_0^6 F_6(X_0, X_1, X_2, X_3) + X_0^3 X_3^6 X_4]$$

and, in affine coordinates, it has the equation

$$\phi_{42}(x, y, z, t) = z^6 \phi_6(x, y, z) + tz^6 = 0.$$

In a detailed expression we obtain

$$\mathcal{W}_{42} = \{X_0^{20} X_3^6 (aX_0^5 X_4 + b_1 X_0^3 X_3^5 + b_2 X_0^2 X_1^3 X_3^2 + \cdots + b_{20} X_2 X_3^5 + b_{10} X_2^5), a, b_i \in \mathbb{C}\}.$$

So we have a non-global 6-canonical adjoint defined by the form $X_0^{20} X_3^6 X_4$.

In particular, the plurigenera $P_i = \dim |iK_X| + 1$, $i \geq 1$ (see [S2]), are $p_7 = P_1 = 3$, $P_2 = 7$, $P_3 = 13$, $P_4 = 22$, $P_5 = 34$, $P_6 = 51$.

2.3. The $m$-canonical transformations $\varphi_{|mK_X|}$, $1 \leq m \leq 5$.

In this paragraph, we prove that $\varphi_{|mK_X|}$ is a generically 2:1 map for $2 \leq m \leq 5$.

Let us consider the following triangle

$$\begin{array}{c}
X \xrightarrow{\varphi_{|mK_X|}} P_{m-1} = P_{\dim |mK_X|} \\
\sigma_{|X|} \downarrow \quad \Psi_{m|V|} \\
V
\end{array}$$

where $\sigma_{|X|$ is the desingularization of $V$ and $\Psi_{m|V|$ is the rational transformation, restricted to $V$, defined by the linear system of bicanonical adjoints to $V$. The foregoing diagram is commutative because the divisors of $|mK_X|$ are of the kind

$$[\pi_2 \cdots (\pi_1^\tau (\Phi_{7m}) - 7mE_{A_1}) \cdots - mE_{A_n}) - mE_{A_n}]_X.$$

To prove that $\varphi_{|mK_X|}$ is generically 2:1, it suffices to consider such a transformation on an open set of $X$. $\sigma$ is a sequence of blow-ups and so it is an isomorphism outside the exceptional divisors of the single blow-ups; so, on an open set of $X$, $\sigma_{|X}$ is an isomorphism. As a result, to say that $\varphi_{|mK_X|}$ is generically 2:1 means that $\Psi_{m|V|$ generically 2:1.
Now let us demonstrate that $\psi_{2|V}$ is generically $2:1$.

Bearing in mind that $\mathcal{W}_{14} \subset W_{14}' = \{ X_0^{10} X_4^2 (b_1 X_0 X_1 + b_2 X_1^2 + b_3 X_2 + b_4 X_4 X_5 + b_5 X_2^2 + b_6 X_2 X_3 + b_7 X_3^2), b_i \in \mathbb{C} \}$, we shall have

$$V \subset \mathbb{P}^4 \xrightarrow{\psi_2} \mathbb{P}^6$$

$$(X_0, X_1, X_2, X_3, X_4) \mapsto (Y_0, \ldots, Y_6)$$

defined by

$$\begin{align*}
Y_0 &= (X_0^{10} X_4^2) X_0 X_3 \\
Y_1 &= (X_0^{10} X_4^2) X_1^2 \\
Y_2 &= (X_0^{10} X_4^2) X_1 X_2 \\
Y_3 &= (X_0^{10} X_4^2) X_4 X_3 \\
Y_4 &= (X_0^{10} X_4^2) X_2^2 \\
Y_5 &= (X_0^{10} X_4^2) X_2 X_3 \\
Y_6 &= (X_0^{10} X_4^2) X_3^2.
\end{align*}$$

Let $U = \mathbb{P}^4 - \{X_0 = X_1 = X_3 = 0\}$ be the affine open set chosen in $\mathbb{P}^4$, with the coordinates $x = \frac{X_0}{X_1}, \ y = \frac{X_2}{X_1}, \ z = \frac{X_3}{X_1}, \ t = \frac{X_4}{X_1}$.

Let $T = \mathbb{P}^6 - \{Y_1 = Y_3 = 0\}$ be the affine open set in $\mathbb{P}^6$ with the coordinates

$$y_1 = \frac{Y_0}{Y_1}, \ y_2 = \frac{Y_2}{Y_1}, \ldots, \ y_6 = \frac{Y_6}{Y_1}.$$ 

We shall thus have

$$\psi_{2|U} : U \rightarrow T$$

$$(x, y, z, t) \mapsto (y_1, \ldots, y_6) : \left\{ \begin{array}{lcl}
y_1 &=& xz \\
y_2 &=& y \\
y_3 &=& z \\
y_4 &=& y^2 \\
y_5 &=& yz \\
y_6 &=& z^2.\end{array} \right.$$
Let $\mathcal{P} = (\mathfrak{y}_1, ..., \mathfrak{y}_6)$ be a generic point of $\text{Im} \psi_{1|U}$; the fiber on $\mathcal{P}$ is

$$\psi_{1|U}^{-1}(\mathcal{P}) = \begin{cases} (x, y, z, t): & \begin{cases} xz = \mathfrak{y}_1 \\ y = \mathfrak{y}_2 \\ z = \mathfrak{y}_3 \\ y^2 = \mathfrak{y}_4 \\ yz = \mathfrak{y}_5 \\ z^2 = \mathfrak{y}_6 \end{cases} = (x, y, z, t): \begin{cases} xz = \mathfrak{y}_1 \\ y = \mathfrak{y}_2 \\ z = \mathfrak{y}_3 \end{cases} \end{cases}.$$ 

The fiber on $\mathcal{P}$ intersects $V_U = V \cap U$ at two points; indeed,

$$V_U \cap \psi_{1|U}^{-1}(\mathcal{P}) = \begin{cases} x^2 = \mathfrak{y}_1 \\ y = \mathfrak{y}_2 \\ z = \mathfrak{y}_3 \end{cases} = \begin{cases} \left( \begin{array}{c} \mathfrak{y}_1 \\ \mathfrak{y}_2 \\ \mathfrak{y}_3 \end{array} \right)^{10} t^2 = a \mathfrak{y}_1^6 + b \mathfrak{y}_1 \mathfrak{y}_2 + c + d \mathfrak{y}_2^2 + e \mathfrak{y}_3^2 \\ y = \mathfrak{y}_2 \\ z = \mathfrak{y}_3 \\ x = \mathfrak{y}_1 \end{cases}.$$ 

This means that $\psi_{1|U}: V \to \mathbb{P}^6$, so $\varphi_{|2K_X}: X \to \mathbb{P}^6$, is generically 2:1. In particular, we find that $V$ is of general type (Kodaira dimension 3). It follows that $\varphi_{|mK_X}$, $m > 2$, is also generically $n : 1$, with $n \leq 2$.

Let us consider an effective canonical divisor $K$, which exists because $p_g$ is positive; putting $nK + |2K_X| = \{ nK + D, D \in |2K_X| \}$ for $n = 1$, 2, ... ($nK$ fixed part of the linear system), we consider the linear systems

$$K + |2K_X| \subset 3K_X, 2K + |2K_X| \subset 4K_X, \ldots, (m - 2)K + |2K_X| \subset mK_X, \ldots.$$ 

All these linear systems $K + |2K_X|$, $2K + |2K_X|$, ... give rise to rational transformations which are generically $n : 1$, $n \leq 2$, and so are the transformations $\varphi_{|mK_X}$, $m \geq 2$.

If $2 \leq m \leq 5$, the absence of any non-global $m$-canonical adjoint implies that $n = 2$, which is the statement.

**Remark 1.** We said previously that the canonical transformation $\varphi_{|K_X}$ coincides, up to isomorphisms, with $\psi_{1|U}$ on an open set. We
can now note that $\psi_1|V$ is generically the projection map of $V$ from the straight line $X_1 = X_2 = X_3 = 0$ on a plane.

2.4. The 6-canonical transformation $\varphi|_{6K_X}$.

Our aim is to prove that $\varphi|_{6K_X}$ is birational. Unlike the foregoing cases, this will be based on the existence of the non-global 6-canonical adjoint defined by the form $G_7 = X_0^3 X_1^3 X_4$.

As we did previously, we choose a canonical effective divisor $R$ (e.g. let $R$ be given by $L = X_0^5 X_3 X_1$) and we construct the linear system $4R + \langle 2K_X \rangle \subset \langle 6K_X \rangle$. The linear system $4R + \langle 2K_X \rangle \subset \langle 6K_X \rangle$ defines a rational transformation which coincides with $\varphi|_{2K_X}$ on an open set, so it defines a generically 2:1 transformation. Now let's consider the non-global 6-canonical adjoint given by $G_7$ and let $\bar{D}$ be the divisor on $X$ defined by it. Note that $\bar{D} \equiv 6K_X$. Let $\Sigma$ be the linear system

$$\{L^4(\lambda_0 F_0 + \cdots + \lambda_6 F_6) + \lambda_7 G_7 = 0, \; \lambda_i \in \mathbb{C}\},$$

with $F_0 = (X_0^{10} X_1^5) X_0 X_2$, $F_1 = (X_0^{10} X_1^7) X_1^3$, $F_2 = (X_0^{10} X_1^5) X_1 X_2$, $F_3 = (X_0^{10} X_1^7) X_2$, $F_4 = (X_0^{10} X_1^5) X_1 X_2 X_3$, $F_5 = (X_0^{10} X_1^7) X_2 X_3$, $F_6 = (X_0^{10} X_1^5) X_3^2$.

Note that $F_0, \ldots, F_6$ span $W_{14}$ and $L^4 F_0, \ldots, L^4 F_6, G_7$ span a vector subspace of $\mathbb{W}_{14}$. We obtain $4R + \langle 2K_X \rangle \subset \Sigma \subset \langle 6K_X \rangle$. The linear system $\Sigma$ defines a rational transformation

$$V \subset \mathbb{P}^4 \longrightarrow \mathbb{P}^7,$$

$$(X_0, X_1, X_2, X_3, X_4) \mapsto (Y_0, \ldots, Y_7)$$

given by:

$$\begin{align*}
Y_0 &= (X_0^5 X_3 X_1)^4 (X_0^{10} X_1^7) X_0 X_3 \\
Y_1 &= (X_0^5 X_3 X_1)^4 (X_0^{10} X_1^7) X_1^3 \\
Y_2 &= (X_0^5 X_3 X_1)^4 (X_0^{10} X_1^7) X_1 X_2 \\
Y_3 &= (X_0^5 X_3 X_1)^4 (X_0^{10} X_1^7) X_1 X_3 \\
Y_4 &= (X_0^5 X_5 X_1)^4 (X_0^{10} X_1^7) X_2^2 \\
Y_5 &= (X_0^5 X_5 X_1)^4 (X_0^{10} X_1^7) X_2 X_3 \\
Y_6 &= (X_0^5 X_5 X_1)^4 (X_0^{10} X_1^7) X_3^2 \\
Y_7 &= (X_0^5 X_5 X_1)^4 (X_0^{10} X_1^7) X_4.
\end{align*}$$

Let us now consider the open affine set $U = \mathbb{P}^4 - \{X_0 = X_1 = X_3 = 0\}$ in $\mathbb{P}^4$ with the coordinates

$$x = \frac{X_0}{X_1}, \quad y = \frac{X_2}{X_1}, \quad z = \frac{X_3}{X_1}, \quad t = \frac{X_4}{X_1}.$$
and the open affine set $T = \mathbb{P}^7 - \{ Y_1 = Y_2 = 0 \}$ in $\mathbb{P}^7$ with the coordinates

$$y_1 = \frac{Y_0}{Y_1}, \ldots, y_7 = \frac{Y_7}{Y_1}.$$ 

We obtain:

$$\psi |_U : U \to T \quad \begin{cases} y_1 = xz \\ y_2 = y \\ y_3 = z \\ y_4 = y^2 \\ y_5 = yz \\ y_6 = z^2 \\ y_7 = x^5 t \end{cases}.$$

$\psi |_U$ is 1:1. Indeed let $P_1(x_1, y_1, z_1, t_1)$ and $P_2(x_2, y_2, z_2, t_2)$ be two points on $U$ such that $\psi |_U(P_1) = \psi |_U(P_2)$, i.e.

$$x_1 z_1 = x_2 z_2, \quad y_1 = y_2, \quad z_1 = z_2, \quad x_1^5 t_1 = x_2^5 t_2.$$

From $y_1 = y_2$ and $z_1 = z_2$, it follows that $x_1 = x_2$ and finally that $t_1 = t_2$.

This proves that $\psi$, so $\psi |_{6K_X}$ is birational.

The birationality of $\psi |_{6K_X}$, $m > 6$, follows from this last fact. Indeed, let us consider an effective canonical divisor $K$ and let us construct the linear systems $K + [6K_X] \cap [7K_X]$, $2K + [6K_X] \cap [8K_X]$, $\ldots$ All these linear systems give rise to rational transformations which are generically 1:1. So all the transformations $\psi |_{6K_X}$, $m \geq 6$, are birational.

**Remark 2.** Note that if we «delete» $y_7 = x^5 t$ in the expression of $\psi |_U : U \to T$, we obtain the $\psi_{2|U}$ of section 2.3. So we have obtained all the informations we need on the pluricanonical transformations only considering the linear system of bicanonical adjoints to $V$ and the non-global 6-canonical adjoint given by $X_0^3 X_1^6 X_4$.

### 2.5. Irregularities of $V$.

We have to show that the following two relations hold true:

$$q_1(X) = \dim \mathbb{C} \cdot H^1(X, \mathcal{O}_X) = 0, \quad q_2(X) = \dim \mathbb{C} \cdot H^2(X, \mathcal{O}_X) = 0.$$ 

To do this, we use the arguments of [S1], section 4. We consider the surface of degree 12 $S = \sigma^{-1}(H \cap V)$, where $H$ is the generic hyperplane in $\mathbb{P}^4$. Since $A_0$ and $A_4$ are isolated singular points on $V$, then $H \cap V$, and so $S$, is nonsingular. Thus it is well known (and easy to see, cf. for instance
formula (36), that \( q(S) = 0 \). We deduce from remark 8 that
\[
q_1(X) = q(S) = 0.
\]

In addition from formula (36), we have
\[
q_2(X) = p_g(X) + p_g(S) - \dim W_8,
\]
where \( W_8 \) is the vector space of the forms defining global adjoints to \( V \) in \( \mathbb{P}^4 \) of degree 8. Thus
\[
q_2(X) = 3 + 165 - 168 = 0.
\]
This proves the statement.

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Manoscritto pervenuto in redazione il 20 febbraio 2001