Universally Koszul Algebras Defined by Monomials.

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Let $K$ be a field and let $R = \bigoplus_{i \in \mathbb{N}} R_i$ be a homogeneous $K$-algebra, that is, an algebra of the form $K[x_1, \ldots, x_n]/I$ where $I$ is a homogeneous ideal. The minimal $R$-free resolution of a graded $R$-module $M$ is said to be linear if the matrices that represent the maps of the resolution have entries of degree 1. Recall that $R$ is said to be Koszul if $K$ has a linear $R$-free resolution. More generally, one says that $R$ is universally Koszul (uk for short) if all the ideals of $R$ generated by elements of degree 1 have a linear $R$-free resolution. For an updated survey on Koszul algebras we refer the reader to the recent paper of Fröberg [F]. For generalities on uk algebras we refer the reader to [C].

Our goal is to classify the uk algebras defined by monomials. We recall first a few facts. Given two homogeneous $K$-algebras $R = K[x_1, \ldots, x_n]/I$ and $S = K[y_1, \ldots, y_m]/J$ the fiber product $R \otimes S$ of $R$ and $S$ is $K[x_1, \ldots, x_n, y_1, \ldots, y_m]/H$ where $H = I + J + (x_i y_j : i = 1, \ldots, n \text{ and } j = 1, \ldots, m)$. One has [Lemma 1.6, C]:

Lemma 1. (a) A polynomial extension $R[x]$ of an algebra $R$ is uk if and only if $R$ is uk.

(b) The fiber product $R \otimes S$ of algebras $R$ and $S$ is uk if and only if $R$ and $S$ are both uk.

(c) If $R$ is uk and $I$ is an ideal of $R$ generated by elements of degree 1 then $R/I$ is uk.

Lemma 1.5 and Proposition 2.2 in [C] give two sufficient conditions for an algebra to be uk:

Lemma 2. Let $R$ be a homogeneous algebra.

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(a) Assume that for every non-zero element \( z \) of degree 1 in \( R \) one has that the algebra \( R/(z) \) is uk and that the ideal \( 0: z \) is generated by elements of degree 1. Then \( R \) is uk.

(b) Assume that for every non-zero element \( z \) of degree 1 in \( R \) one has that \( x_1 + \ldots + x_n \) is a non-zero element of degree 1 in \( R \) and let \( V \) be the degree 1 part of \( 0: z \). Note that if \( a \neq 0 \) then \( V \) contains \( x_1, \ldots, x_n \) and if \( a = 0 \) then \( V \) contains \( x_1 + \ldots + x_n \). This is enough to conclude that \( R_1V = R_2 \).

Then \( R \) is uk.

Let \( J \) be an ideal generated by monomials of degree 1 in a set of variables \( X \). The restriction of \( I \) to a subset \( Y \subset X \) is the ideal \( J \) generated by those monomial generators of \( I \) which involve only elements of the set \( Y \). Let \( I \) be an ideal generated by monomials of degree 2 in a set of variables \( X \). The restriction of \( I \) to a subset \( Y \subset X \) is the ideal \( J \) generated by those monomial generators of \( I \) which involve only elements of the set \( Y \).

If \( R = K[X]/I \) and \( H \) is the ideal of \( R \) generated by the elements in the set \( X \), then \( R/H = K[Y]/J \). Hence, by Lemma 1(c), if \( I \) defines a uk algebra, then its restrictions define uk algebras as well.

Given an integer \( n \geq 0 \) let us denote by \( H(n) \) the algebra \( K[x_1, \ldots, x_n]/I \) where \( I = (x_1, \ldots, x_{n-1})^2 + (x_n^2) \). Note that \( H(0) \) is simply \( K \) and \( H(1) = K[x_1]/(x_1)^2 \). One has:

**Lemma 3.** (a) \( H(n) \) is uk for all \( n \).

(b) Let \( K \) be a field of characteristic 2 and \( R = K[x_1, \ldots, x_n]/I \) be such that \( x_i^2 \in I \) for all \( i \). Let \( x \) be an indeterminate. Then \( R \) is uk if and only if \( R[x]/(x^2) \) is uk.

**Proof.** To prove (a) we apply the criterion (b) of Lemma 2 to \( R = H(n) \). To this end, let \( z = a_1 x_1 + \ldots + a_{n-1} x_{n-1} + a_n x_n \) be a non-zero element of degree 1 in \( R \) and let \( V \) be the degree 1 part of \( 0: z \). Note that if \( a_n = 0 \) then \( V \) contains \( x_1, \ldots, x_n \) and if \( a_n \neq 0 \) then \( V \) contains \( a_n x_n \). This is enough to conclude that \( R_1V = R_2 \).

To prove (b) set \( S = R[x]/(x^2) \). Since \( S/(x) = R \), the «if» part follows from Lemma 1(c). For the other implication we apply the criterion (a) of Lemma 2 and argue by induction on \( n \). Let \( z \) be an element of degree 1 is \( S \), say \( z = L_1 + ax \) with \( L_1 \) an element of degree 1 in \( R \) and \( a \in K \). We have to show that \( S/(z) \) is uk and that \( 0: z \) is generated by elements of degree 1. We discuss first the case \( a = 0 \). Then \( S/(z) = R/(L_1)[x]/(x^2) \) and by induction we know that this ring is uk. Furthermore \( 0: z = 0 + (0:xL_1)S \) and \( 0:xL_1 \) is generated in degree 1 since \( R \) is uk. Now assume \( a \neq 0 \). We may assume \( a = 1 \). Then \( S/(z) = R/(L_1^2) = R \) since, by assumption, \( L_1^2 = 0 \) in \( R \) and hence \( S/(z) \) is uk. We show now that \( 0: z = (z) \). As \( z^2 = 0 \) in \( S \) the inclusion \( \supset \) holds. For the other inclusion let \( f \in S \) be an
element in $0 : z$. Clearly $f$ can be written (in a unique way) as $f = h + xy$ with $h, g \in R$. Since $fz = 0$ we have that $h + gL_1 = 0$. That is, $h = gL_1$ and then $f = gz$. 

We have also

**Lemma 4.** Let $I$ be one of the following ideals:

1. $(xy, z^2)$,
2. $(x^2, xy, z^2)$,
3. a monomial ideal whose squarefree generators are $xy, zt$,
4. a monomial ideal whose squarefree generators are $xy, yz, zt$.

Then the algebra $R$ defined by $I$ is not uk. Furthermore the same conclusion holds if the characteristic of the base field is $\neq 2$ and $I$ is equal to

5. $(x^2, y^2, z^2)$.

**Proof.** In the cases (1) and (2) we claim that (the class of) $xz$ is a minimal generator of $0 : (y + z)$ in $R$. This implies that $R$ is not uk. That $xz \in 0 : (y + z)$ is clear. It is easy to see that there are no elements of degree 1 in $0 : (y + z)$. Hence $xz$ is a minimal generator of $0 : (y + z)$.

In the cases (3) and (4) we claim that $xt$ is a minimal generator of $0 : (y + z)$ in $R$. That $xt \in 0 : (y + z)$ is clear. To prove that $xt$ is a minimal generator we may assume that $I$ is the largest possible, i.e. $I = (xy, yz, zt, x^2, y^2, z^2, t^2)$. It is easy to see that the space of the elements of degree 1 in $0 : (y + z)$ is generated by $y$ and $z$. As $xt$ is not in the ideal generated by $y, z$ in $R$ we may conclude that $xt$ is a minimal generator of $0 : (y + z)$.

Finally (5) has been observed in [Example 1.10, C].

We are in the position to state our result. For a base field of characteristic $\neq 2$ we have:

**Theorem 5.** Let $R$ be an algebra defined over a field $K$ of characteristic $\neq 2$ by an ideal $I$ generated by monomials of degree 2 in a set of variables $X$. The following are equivalent:

1. $R$ is uk,
2. $R$ is obtained from the algebras $H(n)$ by iterated polynomial extensions and fiber products.
3. The restriction of $I$ to any subset of variables of $X$ does not give an ideal of type (1)-(5) of the list of Lemma 4.
In characteristic 2 we have:

**THEOREM 6.** Let $R$ be an algebra defined over a field $K$ of characteristic 2 by an ideal $I$ generated by monomials of degree 2 in a set of variables $X$. The following are equivalent:

1. $R$ is $uk$,
2. $R$ is obtained from the field $K$ by iterated polynomial extensions, fiber products and extension of the type of Lemma 3(b).
3. The restriction of $I$ to any subset of variables of $X$ does not give an ideal of type (1) – (4) of the list of Lemma 4.

**PROOF OF THEOREMS 5 and 6.** (2) $\Rightarrow$ (1) follows from Lemma 1 and 2. (1) $\Rightarrow$ (3) follows from Lemma 4. We prove (3) $\Rightarrow$ (2) by induction on the cardinality $\#X$ of $X$. If $\#X = 1$ then the assertion is clearly true. So assume that $\#X > 1$. Let $V$ be a subset of $X$ such that for all pairs $x, y \in V$ with $x \neq y$ one has $xy \not\in I$ and assume that $V$ is maximal with respect to this property. Say $V = \{v_1, v_2, \ldots, v_k\}$. Set $W = X \setminus V$ and $G_j = \{x \in W: xy \in I\}$. By the definition of $V$ we have that $W = \bigcup_{i=1}^{k} G_i$. We claim that:

(a) For $i = 1, \ldots, k$, $x \in G_i$ and $y \in W \setminus G_i$ then $xy \in I$,
(b) For $1 \leq i < j \leq k$ then either $G_i \subseteq G_j$ or $G_j \subseteq G_i$.

To prove (a), let $j$ be such that $y \in G_j$. Since $i \neq j$ we have that $I$ contains $xv_i$ and $yv_j$ and by construction does not contain $v_i, v_j$ and $v_i y$. Hence $I$ must contain also $xy$ otherwise the square free part of its restriction to $\{x, y, v_i, v_j\}$ would be either $xv_i, yv_j$ or $xv_i, yv_j, v_i y$, a contradiction since these are ideals of type (3) and (4) in the list of Lemma 4. To prove (b), assume by contradiction that there exist $x \in G_i \setminus G_j$ and $y \in G_j \setminus G_i$ and argue as in case (a).

After renumbering if needed, by (b) we may assume that

$$G_1 \subseteq G_2 \subseteq \ldots \subseteq G_k = W.$$ 

If $G_1 \neq \emptyset$, then by (a) and definition of the $G_i$ we have for each $x \in G_1$ and $y \in X \setminus G_1$ then $xy \in I$. Then $R$ is the fiber product of the algebra $R_1$ defined by the restriction of $I$ to $G_1$ with the algebra $R_2$ defined by the restriction of $I$ to $X \setminus G_1$. As $R_1$ and $R_2$ clearly satisfy condition (3) of the theorem, we may assume by induction that they also satisfy (2). As $R = R_1 \circ R_2$, also $R$ satisfies (2) and we are done.
If instead $G_1 = \emptyset$ and $v_1^2 \notin I$ then $R$ is a polynomial extension, and again we are done by induction.

So we are left with the case in which $G_1 = \emptyset$ and $v_1^2 \in I$. Let $h$ be the largest index such $G_h = \emptyset$. We may also assume that $v_i^2 \in I$ for $i = 1, \ldots, h$ otherwise we conclude as above.

If $W = \emptyset$ (equivalently, $h = k$) then $R$ is equal to $K[v_1, \ldots, v_k]/(v_1^2, \ldots, v_k^2)$. This ring is obtained by iterated extensions of the type of Lemma 3(b) if the characteristic of $K$ is 2. If, instead, the characteristic of $K$ is $\neq 2$, then $k = 2$ (otherwise a restriction would be of type (5)) and $R$ is $H(2)$.

Therefore we may assume $W \neq \emptyset$ (equivalently $h < k$). Let $i > h$ and let $x \in G_i$. Then $I$ contains $v_i^2$, $xv_i$, and does not contain $v_i$, $xv_i$. It follows that $v_i^2$ and $x^2$ must be in $I$ otherwise $I$ would have a restriction of type (1) or (2). In particular $x^2 \in I$ for all $x \in X$. But then, if the characteristic of $K$ is 2, $R$ is an extension of the type of Lemma 3(b) (with $x = v_i$). By induction, this concludes the proof in the characteristic 2 case. Assume that the characteristic of $K$ is not 2. Since $x^2 \in I$ for all $x \in X$ and $W$ is not empty, then $h = 1$ and $k = 2$, otherwise there would be a restriction of type (5). Now let $x, y \in W$. Since we know that $x^2, y^2, v_1^2 \in I$ and $xv_1, yv_1 \notin I$ it follows that $xy \in I$. Summing up, $R$ is (isomorphic to) $H(n)$.

**REFERENCES**
