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On Nonlinear Elliptic Problems with Discontinuities.

ANTONELLA FIACCA (*) - NIKOLAOS MATZAKOS (**)
NIKOLAOS S. PAPAGEORGIOU (***)

ABSTRACT - In this paper we examine nonlinear elliptic equations driven by the p-Laplacian and with a discontinuous forcing term. To develop an existence theory we pass to an elliptic inclusion by filling in the gaps at the discontinuity points of the forcing term. We prove three existence theorems. The first is a multiplicity result and proves the existence of two bounded solutions one strictly positive and the other strictly negative. The other two theorems deal with problems at resonance and prove the existence of solutions using Landesman-Lazer type conditions.

1. Introduction.

In this paper we study quasilinear problems with discontinuities. We prove three existence theorems. The first is a multiplicity result, which proves the existence of two bounded solutions, one strictly positive and the other strictly negative. The other two existence theorems concern a resonant eigenvalue problem and prove the existence of a solution using Landesman-Lazer type conditions.

Elliptic equations with discontinuities have been studied in the past,

(*) Indirizzo dell'A.: University of Perugia, Department of Mathematics, Via Vanvitelli 1, Perugia 06123, Italy.

(**) Indirizzo dell'A.: National Technical University, Department of Mathematics, Zografou Campus, Athens 157 80, Greece.

(***) Indirizzo dell'A.: National Technical University, Department of Mathematics, Zografou Campus, Athens 157 80, Greece.

E-mail: npapg@math.ntua.gr

After the completion of this work Prof. A.Fiacca died. So this paper is dedicated to the memory of Antonella, a dear friend, colleague and collaborator.

almost exclusively for semilinear problems. A representative sample of the techniques used to analyze the problem, can be found in the following works: Ambrosetti-Badiale [2] (they use Clarke's dual variational principle), Chang [9] (he uses nonsmooth critical point theory), Rauch [21] (his approach is based on truncation and penalization techniques) and finally Stuart [23] (he uses the method of upper and lower solutions). Recently there has been an increasing interest for the study of quasilinear elliptic problems involving the p-Laplacian differential operator. We refer to the works of Anane-Gossez [4], Arcoya-Calahorrano [6], Boccardo-Drabek-Giachetti-Kucera [7], Bougouima [8], Costa-Magalhaes [10] and El Hachimi-Gossez [11]. Of these works, only Arcoya-Calahorrano [6], and Carl-Dietrich [8] deal with problems with discontinuities and Arcoya-Calahorrano [6] assume that the right hand side function f has only a jump discontinuity at $x = 0$. The approach of Arcoya-Colahorrano [6] is variational, while Carl-Dietrich [8] combine variational techniques with the method of upper and lower solutions. The other works assume a continuous forcing term and use either a variational approach based on the smooth critical point theory (Anane-Gossez [4], Costa-Magalhaes [10] and El Hachimi-Gossez [11]) or degree theoretic methods (Boccardo-Drabek-Giachetti-Kucera [7]). Concerning the resonant eigenvalue problem studied in the second part of the paper using a Landesman-Lazer type condition, previous works in this direction deal with semilinear equations. We refer to the classical work of Landesman-Lazer [17] and the more recent ones by Landesman-Robinson-Rumbos [18] and Robinson-Landesman [22]. Here we examine a quasilinear resonant problem driven by the p-Laplacian and our approach uses degree theoretic methods. In the first part, in the analysis of the discontinuous quasilinear problem we employ the method of upper and lower solutions combined with techniques from the theory of nonlinear operators of monotone type.

2. Preliminaries.

Let X be a reflexive Banach space and X^* its topological dual. A map $A : D \subseteq X \to 2^{X^*}$ is said to be «*monotone*«, if for all $x^* \in A(x)$, $y^* \in A(y)$ we have $(x^* - y^*, x - y) \geq 0$ (here by $(\cdot, \cdot)$ we denote the duality brackets for the pair (X, X^*)). If $(x^* - y^*, x - y) = 0$, implies that $x = y$ we say that A is «*strictly monotone*». The map A is said to be «*maximal monotone*», if $(x^* - y^*, x - y) \geq 0$ for all $x \in D$ and all $x^* \in A(x)$, imply

$y \in D$ and $y^* \in A(y)$. It easy to see that this condition implies that the graph of A, $GrA = \{[x, x^*] \in X \times X^* : x^* \in A(x)\}$, is maximal with respect to inclusion among the graphs of all monotone maps. Using the above definition of maximality, we can check that the graph of a maximal monotone map A is sequentially closed in $X \times X_w^*$ and in $X_w \times X^*$ (here by X_w and X_w^* we denote the spaces X and X^* with their respective weak topologies). A map $A : X \to X^*$ which is single-valued and everywhere defined (i.e. $D = X$) is said to be «*demicontinuous*», if $x_n \to x$ in X, implies $A(x_n) \xrightarrow{w} A(x)$. A monotone, demicontinuous map $A : X \to X^*$ is maximal monotone. A map $A : D \subseteq X \to 2^{X^*}$ is said to be «*coercive*», if D is bounded or D is unbounded and $\inf \{\|x^*\| : x^* \in A(x)\} \to \infty$ as $\|x\| \to \infty$. A maximal monotone and coercive map, is surjective.

An operator $A : X \to 2^{X^*}$ is said to be «*pseudomonotone*» if

(a) for every $x \in X$, $A(x)$ is nonempty, weakly compact and convex in X^*;

(b) A as a set-valued map is upper semicontinuous from every finite dimensional subspace Z of X into X_w^* (i.e. for every $C \subseteq X^*$ nonempty and weakly closed, the set $A^{-1}(C) = \{x \in Z : A(x) \cap C \neq\}$ is closed in Z);

and

(c) if $x_n \xrightarrow{w} x$ in X, $x_n^* \in A(x_n)$, $n \geq 1$, and $\overline{\lim}(x_n^*, x_n - x) \leq 0$, then for every $y \in X$, we can find $x^*(y) \in A(x)$ such that $(x^*(y), x - y) \leq \leq \underline{\lim}(x_n^* . x_n - y)$.

If A is bounded (i.e. map bounded sets into bounded sets) and satisfies condition (c) above, then it satisfies condition (b) too. An operator $A : X \to 2^{X^*}$ is said to be «*generalized pseudomonotone*», if for $x_n^* \in \in A(x_n)$, $n \geq 1$, which satisfy $x_n \xrightarrow{w} x$ in X, $x_n^* \xrightarrow{w} x$ in X^* and $\overline{\lim}(x^*, x_n - - x) \leq 0$, we have $x^* \in A(x)$ and $(x^*, x_n) \to (x^*, x)$. Every maximal monotone operator is generalized pseudomonotone. Also a pseudomonotone operator is generalized pseudomonotone. The converse is true if A has nonempty, weakly compact and convex values and it is bounded. A pseudomonotone and coercive operator is surjective. For details on these and related issues we refer to the books of Hu-Papageorgiou [14] and Zeidler [25].

In our analysis we will need some facts about the spectrum of the negative p-Laplacian $-\Delta_p x = -\operatorname{div}(\|Dx\|^{p-2} Dx)$ with Dirichlet boundary conditions, i.e of $(\Delta_p, W_0^{1,p}(Z))$. More precisely let $Z \subseteq \mathbb{R}^N$ be a bounded domain with a Lipschitz boundary Γ and consider the following nonlinear eigenvalue problem

$$(1) \qquad \left. \begin{cases} -\operatorname{div}(\|Dx(z)\|^{p-2}Dx(z)) = \lambda\,|x(z)|^{p-2}x(z) \text{ a.e. on } Z \\ x_{|\Gamma} = 0 \end{cases} \right\}.$$

The least real number λ for which (1) has a nontrivial solution, is the first (principal) eignevalue of $(-\Delta_p, W_0^{1,p}(Z))$ and is denoted by λ_1. The first eigenvalue λ_1 is positive, isolated and simple (i.e. the associated eigenfunctions are constant multiples of each other). Furthermore we have a variational characterization of λ_1 via the Rayleigh quotient, i.e.

$$(2) \qquad \lambda_1 = \min\left[\frac{\|Dx\|_p^p}{x_p^p} : x \in W_0^{1,p}(Z),\, x \neq 0\right].$$

The minimum is realized at the normalized eigenfunction u_1. Note that if u_1 minimizes the Rayleigh quotient, then so does $|u_1|$ and so we infer that the first eigenfunction u_1 does not change sign on Z. In fact we can show that $u_1(z) \neq 0$ a.e. on Z and so we may assume that $u_1(z) > 0$ a.e. on Z. For details we refer to Anane [3] and Lindqvist [20].

The Liusternik-Schnirelmann theory gives, in addition to λ_1, a whole strictly increasing sequence of positive numbers $0 < \lambda_1 < \lambda_2 < \ldots < \lambda_n < < \ldots$ for which there exist nontrivial solutions of problem (1). In other words, the spectrum $\sigma(-\Delta_p)$ of the negative p-Laplacian on $W_0^{1,p}(Z)$ contains at least these points $\{\lambda_n\}_{n \geq 1}$. Nothing is know in general about the possible existence of other points in $\sigma(-\Delta_p) \subseteq [\lambda_1, \infty)$. Since $\lambda_1 > 0$ is isolated, we can define

$$\overline{\lambda}_2 = \inf[\lambda > 0 : \lambda \text{ is an eigevalue of } (-\Delta_p, W_0^{1,p}(Z)), \lambda \neq \lambda_1] > \lambda_1.$$

Recently Anane-Tsouli [5] proved that the second Liusternik-Schnirelmann eigenvalue λ_2 equals $\overline{\lambda}_2$.

3. Multiple bounded solutions.

In this section we prove the existence of two bounded solutions, one positive and the other negative for a quasilinear elliptic equation with a discontinuous right hand side. So let $Z \subseteq \mathbb{R}^N$ be a bounded domain with

C^2-boundary Γ. We examine the following Dirichlet problem:

$$(3) \qquad \left\{ \begin{array}{l} -\operatorname{div}(\|Dx(z)\|^{p-2} Dx(z)) = f(z, x(z)) \text{ a.e. on } Z \\ x_{|\Gamma} = 0,\ 2 \leqslant p < \infty \end{array} \right\}.$$

We do not assume that $f(z, \cdot)$ is continuous. So problem (3) need not have a solution. To develop an existence theory, we need to pass to a multivalued approximation of (3) by, roughly speaking, filling in the gaps at the discontinuity points of $f(z, \cdot)$ (see Chang [9], Rauch [21] and Stuart [23]. So we introduce following two functions:

$$f_1(z, x) = \underline{\lim}_{x' \to x} f(z, x') = \lim_{\varepsilon \downarrow 0} ess \inf_{|x'-x| < \varepsilon} f(z, x')$$

and

$$f_2(z, x) = \overline{\lim}_{x' \to x} f(z, x') = \lim_{\varepsilon \downarrow 0} ess \sup_{|x'-x| < \varepsilon} f(z, x').$$

Then instead of (3) we consider the following quasilinear elliptic inclusion:

$$(4) \qquad \left\{ \begin{array}{l} -\operatorname{div}(\|Dx(z)\|^{p-2} Dx(z)) \in \widehat{f}(z, x(z)) \text{ a.e. on } Z \\ x_{|\Gamma} = 0 \end{array} \right\},$$

where $\widehat{f}(z, x) = [f_1(z, x), f_2(z, x)] = \{y \in \mathbb{R} : f_1(z, x) \leqslant y \leqslant f_2(z, x)\}$. It is problem (4) that we will investigate.

We introduce the following hypotheses on the forcing term $f(z, x)$

H(f)$_1$: $f: Z \times \mathbb{R} \to \mathbb{R}$ is a measurable function such that

(i) f_1 and f_2 are N-measurable functions (i.e. if $x: Z \to \mathbb{R}$ is a measurable function, then $z \to f_1(z, x(z))$ and $z \to f_2(z, x(z))$ are both measurable);

(ii) there exist $a \in L^\infty(Z)$ and $c > 0$ such that for almost all $z \in Z$ and all $x \in \mathbb{R}$

$$|f(z, x)| \leqslant a(z) + c|x|^{p-1};$$

(iii) there exists $\theta \in L^\infty(Z)_+$ such that $\overline{\lim}_{|x| \to \infty} \frac{f(z, x)}{|x|^{p-2} x} \leqslant \theta(z)$ uniformly for almost all $z \in Z$ and $\theta(z) \leqslant \lambda_1$ a.e. on Z with strict inequality on a set positive Lebesgue measure;

(iv) $\underline{\lim}_{|x| \to 0} \frac{f(z, x)}{|x|^{p-2} x} > \lambda_1$ uniformly for almost all $z \in Z$.

By virtue of hypothesis $H(f)_1(iii)$, we see that given $\varepsilon > 0$, we can find $M = M(\varepsilon) > 0$ such that for almost all $z \in Z$ and all $x > M$ we have

$$f(z, x) \leqslant (\theta(z) + \varepsilon) |x|^{p-2} x ,$$

while for almost all $z \in Z$ and all $x < -M$, we have

$$f(z, x) \geqslant (\theta(z) + \varepsilon) |x|^{p-2} x .$$

Moreover, hypothesis $H(f)_1(ii)$ implies that for almost all $z \in Z$ and all $|x| \leqslant M$, we have $|f(z, x)| \leqslant a_M(z)$, with $a_M \in L^{\infty}(Z)_+$, $a_M \neq 0$. So finally we can write that for almost all $z \in Z$

$$f(z, x) \leqslant (\theta(z) + \varepsilon) |x|^{p-2} x + a_M(z) \text{ for all } x \geqslant 0$$

$$f(z, x) \geqslant (\theta(z) + \varepsilon) |x|^{p-2} x - a_M(z) \text{ for all } x \leqslant 0 .$$

We will start our investigations, by examining the following two auxiliary problems:

$$\text{(5)} \quad \left\{ \begin{array}{l} -\operatorname{div}(\|D\varphi(z)\|^{p-2} D\varphi(z)) = (\theta(z) + \varepsilon) |\varphi(z)|^{p-2} \varphi(z) + a_M(z) \text{ a.e. on } Z \\ \varphi_{|\Gamma} = 0 \end{array} \right\},$$

and

$$\text{(6)} \quad \left\{ \begin{array}{l} -\operatorname{div}(\|D\psi\|(z)^{p-2} D\psi(z)) = (\theta(z) + \varepsilon) |\psi(z)|^{p-2} \psi(z) - a_M(z) \text{ a.e. on } Z \\ \psi_{|\Gamma} = 0 \end{array} \right\}.$$

PROPOSITION 1. *If hypotheses $H(f)_1$ hold and $\varepsilon > 0$ is small, then problem* (5) *has a solution $\varphi \in C^1(\overline{Z})$ such that $\varphi(z) > 0$ for all $z \in Z$ and $\frac{\partial \varphi}{\partial n}(z) < 0$ for all $z \in \Gamma$ such that $\varphi(z) = 0$.*

PROOF. Let $A : W_0^{1,p}(Z) \to W^{-1,q}(Z) \left(\frac{1}{p} + \frac{1}{q} = 1 \right)$ be the nonlinear operator defined by

$$\langle A(x), y \rangle = \int_Z \|Dx(z)\|^{p-2} (Dx(z), D(y(z))_{\mathbb{R}^N} dz \text{ for all } x, y \in W_0^{1,p}(Z).$$

Here by $\langle \cdot, \cdot \rangle$ we denote the duality brackets for the pair $(W_0^{1,p}(Z), W^{-1,q}(Z))$. Also let $J_\vartheta \colon W_0^{1,p}(Z) \to L^q(Z) \subseteq W^{-1,q}(Z)$ be defined by

$$J_\vartheta(x)(\cdot) = (\theta(\cdot) + \varepsilon)\,|x(\cdot)|^{p-2}x(\cdot) + a_M(\cdot)\,.$$

Let $K = A - J_\theta\colon W_0^{1,p}(Z) \to W^{-1,q}(Z)$.

Claim 1: K is pseudomonotone.

Indeed let $x_n \xrightarrow{w} x$ in $W_0^{1,p}(Z)$ and assume that

(7) $$\overline{\lim}\langle A(x_n) - J_\theta(x_n),\, x_n - x\rangle \leqslant 0\,.$$

If by $(\cdot,\cdot)_{pq}$ we denote the duality brackets for the pair $(L^p(Z), L^q(Z))$, we see that $\langle J_\theta(x_n), x_n - x\rangle = (J_\theta(x_n), x_n - x)_{pq}$. From the compact embedding of $W_0^{1,p}(Z)$ into $L^p(Z)$, we have that $x_n \to x$ in $L^p(Z)$ and so $(J_\theta(x_n), x_n - x)_{pq} \to 0$ as $n \to \infty$. So from (7) we obtain that

$$\overline{\lim}\langle A(x_n),\, x_n - x\rangle \leqslant 0\,.$$

But it is easy to check that A is monotone, demicontinuous, hence maximal monotone and generalized pseudomonotone. Therefore

$$\langle A(x_n), x_n\rangle \to \langle A(x), x\rangle \quad \text{as } n \to \infty\,.$$

Hence K is generalized pseudomonotone and obviously bounded. So K is pseudomonotone.

Claim 2: There exists $\beta > 0$ such that $V(x) = \|Dx\|_p^p - \int_Z \vartheta(z)\,|x(z)|^p\,dz \geqslant \beta\|Dx\|_p^p$ for all $x \in W_0^{1,p}(Z)$.

Note that because $\vartheta(z) \leqslant \lambda_1$ a.e. on Z and (2), $V \geqslant 0$. Suppose that the claim was not true. Then we can find $\{x_n\}_{n\geqslant 1} \subseteq W_0^{1,p}(Z)$ with $\|Dx_n\|_p = 1$ such that $V(x_n) \downarrow 0$ as $n \to \infty$. From the weak lower semicontinuity of the norm in a Banach space, we have $\|Dx\|_p^p \leqslant \underline{\lim}\|Dx_n\|_p^p$. So

$$0 = \lim V(x_n) = \lim\left[\|Dx_n\|_p^p - \int_Z \vartheta(z)\,|x_n(z)|^p\,dz\right]$$

$$\geqslant \underline{\lim}\|Dx_n\|_p^p - \overline{\lim}\int_Z \vartheta(z)\,|x_n(z)|^p\,dz$$

$$\geqslant \|Dx\|_p^p - \int_z \vartheta(z)\,|x(z)|^p\,dz = V(x) \geqslant 0\,.$$

Therefore we obtain

$$\|Dx\|_p^p = \int_Z \vartheta(z)\,|x(z)|^p\,dz \leqslant \lambda_1 \|x\|_p^p \ \ \text{(hypothesis } H(f)_1(iii))$$

$$\text{(8)} \qquad \text{hence } \|Dx\|_p^p = \int_Z \vartheta(z)\,|x(z)|^p\,dz = \lambda_1 \|x\|_p^p \ \ \text{(see (2.3))}.$$

From the choice of the sequence $\{x_n\}_{n \geqslant 1} \subseteq W_0^{1,p}(Z)$, we have

$$V(x_n) = 1 - \int_Z \vartheta(z)\,|x_n(z)|^p\,dz \to 0 \ \text{ as } n \to \infty$$

hence $1 = \int_Z \vartheta(z)\,|x(z)|^p\,dz = \|Dx\|_p^p$ (see (8)).

If follows that $x \neq 0$. Then from (8), we see that $x = u_1$ and so $x(z) > 0$ for all $z \in Z$. So because of hypothesis $H(f)_1(iii)$ we have

$$\int_Z \vartheta(z)\,|x(z)|^p\,dz < \lambda_1 \|x\|_p^p$$

which contradicts (8). This proves the claim.

<u>Claim 3</u>: If $\varepsilon > 0$ small $K : W_0^{1,p}(Z) \to W^{-1,q}(Z)$ is coercive.

For every $x \in W_0^{1,p}$ we have

$$\langle K(x), x\rangle = \langle A(x) - J_\theta(x), x\rangle = \|Dx\|_p^p - \int_Z \vartheta(z)\,|x(z)|^p\,dz - \varepsilon\|x\|_p^p - \|a_M\|_\infty$$

$$\geqslant \beta\|Dx\|_p^p - \frac{\varepsilon}{\lambda_1}\|Dx\|_p^p - \|a_M\|_\infty \ \ \text{(from claim 2 and (2))}$$

$$= \left(\beta - \frac{\varepsilon}{\lambda_1}\right)\|Dx\|_p^p - \|a_M\|_\infty .$$

Let $\varepsilon > 0$ be such that $\varepsilon < \beta\lambda_1$. From the above inequality we infer that K is coercive.

Now recall that the pseudomonotone (claim 1), coercive (claim 2) operator K is surjective. So we can find $\phi \in W_0^{1,p}(Z)$ such that

$$A(\phi) = J_\vartheta(\phi) \ \text{ in } L^q(Z).$$

Let $\eta \in C_0^\infty(Z)$. We have

$$\langle A(\phi), \eta\rangle = (J_\vartheta(\phi), \eta)_{pq}$$
$$\Rightarrow \int_Z \|D\phi\|^{p-2}(D\phi, D\eta)_{\mathbb{R}^N}\, dz = (J_\vartheta(\phi), \eta)_{pq}.$$

Note that $-\operatorname{div}(\|D\varphi\|^{p-2}D\varphi) \in W^{-1,q}(Z)$ (Adams [1], theorem 3.10, p. 50 or Hu-Papageorgiou [15], theorem A.1.25, p. 866). So we obtain

$$\langle -\operatorname{div}(\|D\varphi\|^{p-2}D\varphi, \eta\rangle = (J_\vartheta(\varphi), \eta)_{pq} = \langle J_\vartheta(\varphi), \eta\rangle.$$

Since $C_0^\infty(Z)$ is dense in $W_0^{1,p}(Z)$ (the predual of $W^{-1,q}(Z)$), we conclude that

$$-\operatorname{div}(\|D\varphi\|^{p-2}D\varphi) = J_\vartheta(\varphi) \text{ in } L^q(Z)$$

hence

$$\left.\begin{cases} -\operatorname{div}(\|D\varphi(z)\|^{p-2}D\varphi(z) = (\vartheta(z)+\varepsilon)\,|\varphi(z)|^{p-2}\varphi(z) + a_M(z) \text{ a.e. on } Z \\ \varphi_{|\Gamma} = 0 \end{cases}\right\}. \tag{9}$$

From Ladyzenskaya-Uraltseva [16] (theorem 7.1, p. 286, see also Gilbarg-Trudinger [13], p. 277), we have that $\varphi \in L^\infty(Z)$. Then from theorem 1 of Lieberman [19], we have that $\varphi \in C^1(\overline{Z})$.

Next we will show that $\varphi(z) > 0$ for all $z \in Z$. Let $\varphi^-(z) = \max[-\varphi(z), 0]$. From Gilbarg-Trudinger [13], p. 146, we know that $\varphi^- \in W_0^{1,p}(Z)$

$$D\varphi^-(z) = \left.\begin{cases} -D\varphi(z) & \text{a.e. on } \{\varphi < 0\} \\ 0 & \text{a.e. on } \{\varphi \geqslant 0\} \end{cases}\right\}$$

and of course $\varphi^- \geqslant 0$. Using φ^- as our test function we have

$$\langle A(\varphi), \varphi^-\rangle = (J_\theta(\varphi), \varphi^-)_{pq}$$
$$\Rightarrow -\int_Z \|D\varphi^-\|^p\, dz + \int \vartheta(z)\,|\varphi^-|^p\, dz + \varepsilon\|\varphi^-\|_p^p = \int_Z a_M \varphi^-\, dz \geqslant 0$$

(since $a_M \geqslant 0$).

Using claim 2, we obtain

$$-\left(\beta - \frac{\varepsilon}{\lambda_1}\right) \|D\varphi^-\|_p^p \geqslant 0 .$$

Since from the choice of $\varepsilon > 0$, we have $0 < \beta - \frac{\varepsilon}{\lambda_1}$, it follows that $\varphi^- =$constant on $\overline{Z}$. From (9) we have

$$-\operatorname{div}(\|D\varphi(z)\|^{p-2} D\varphi(z)) + \|\vartheta + \varepsilon\|_\infty |\varphi(z)|^{p-2}\varphi(z) \geqslant a_M(z) \geqslant 0 \text{ a.e. on } Z .$$

Invoking theorem 5 of Vazquez [24], since $\varphi \neq 0$ (see (9) and recall the $a_M \neq 0$), we have that $\varphi(z) > 0$ for $z \in Z$ and $\frac{\partial \varphi}{\partial n}(z) < 0$ for all $z \in \Gamma$ such that $\varphi(z) = 0$. ■

In a similar fashion we can prove the following proposition.

PROPOSITION 2. *If hypotheses* $H(f)_1$ *hold and* $\varepsilon > 0$ *is small, then problem* (6) *has a solution* $\psi \in C^1(\overline{Z})$ *such that* $\psi(z) < 0$ *for all* $z \in Z$ *and* $\frac{\partial \psi}{\partial n}(z) < 0$ *for all* $z \in \Gamma$ *such that* $\psi(z) = 0$.

As we already mentioned in the introduction, our approach will also use the method of upper and lower solutions. For this reason we introduce the following notions (see Carl-Dietrich [8], p. 267):

DEFINITION. (a) A function $x \in W_0^{1,p}(Z)$ is an «upper solution» for problem (2) if

$$\int_Z \|Dx\|^{p-2}(Dx, Du)_{\mathbb{R}^N} dz \geqslant \int_Z f_2(z, x(z))\, u(z)\, dz \text{ for all } u \in W_0^{1,p}(Z)_+ ,$$

$$x_{|\Gamma} \geqslant 0 .$$

(b) A function $y \in W^{1,p}(Z)$ is «lower solution» for problem (2) if

$$\int_Z \|Dy\|^{p-2}(Dy, Du)_{\mathbb{R}^N} dz \leqslant \int_Z f_1(z, x(z))\, u(z)\, dz \text{ for all } u \in W_0^{1,p}(Z)_+ ,$$

$$y_{|\Gamma} \leqslant 0 .$$

Recall that for almost all $z \in Z$

$$f_2(z, x) \leqslant (\vartheta(z) + \varepsilon)\,|x|^{p-2} + a_M(z) \ \text{ for all } \ x \geqslant 0$$
$$\text{and } f_1(z, x) \geqslant (\vartheta(z) + \varepsilon)\,|x|^{p-2} - a_M(z) \ \text{ for all } \ x \leqslant 0\,.$$

So from propositions 1 and 2, we infer that φ and ψ are upper and lower solutions of (2) respectively.

Next by virtue of hypothesis $H(f)(iv)$, we can find $\delta > 0$ such that for almost all $z \in Z$ and all $0 < x \leqslant \delta$, we have

$$\lambda_1\,|x|^{p-2}x < f(z, x)$$
$$\text{hence } \lambda_1\,|x|^{p-2}x \leqslant f_1(z, x)\,. \tag{10}$$

Let u_1 be the principal eigenfunction corresponding to the eigenvalue $\lambda_! > 0$. Since by hypothesis the boundary Γ is a C^2-manifold, as before by virtue of theorem 1 of Lieberman [19], we have that $u_1 \in C^1(\overline{Z})$ and furthermore we can say that $u(z) > 0$ for all $z \in Z$ (see also Anane [4]). Let $0 < \xi_1 < 1$ be small enough so that $0 < \xi_1 u_1(z) \leqslant \delta$ for all $z \in Z$. Also from the comparison principle (see theorem 5 of Garcia Melian-Sabina de Lis [12]), we know that we can find $r_1 > 0$ such that $\xi_1 u_1(z) < r_1 \varphi(z)$ for all $z \in Z$. Then $\frac{\xi_1}{r_1} u_1(z) < \varphi(z)$ for all $z \in Z$. If we set $\xi = \frac{\xi_1}{r_1}$ and $u = \xi u_1$, we have that $0 < u(z) \leqslant \delta$ for all $z \in Z$ and so for all $v \in W_0^{1,p}(Z)_+$ we can write

$$\int_Z \|Du\|^{p-2}(Du, Dv)_{\mathbb{R}^N}\,dz = \lambda_1 \int_Z |u|^{p-2} uv dz$$
$$\leqslant \int_Z f_1(z, u(z))\,v(z)\,dz, \ u_{|\Gamma} = 0\,.$$

Hence by definition $u \in C^1(\overline{Z})$ is a lower solution for problem (2). Then we work with the ordered upper and lower solution pair $\{\varphi, u\}$ and obtain the following existence result.

PROPOSITION 3. *If hypotheses $H(f)$ hold, then problem (4) has a bounded solution $x \in W_0^{1,p}(Z)$ such that $x(z) > 0$ a.e. on Z.*

PROOF. Our proof is based on truncation and penalization techniques and on the theory of nonlinear operators of monotone type.

We introduce the trunction map $\tau : W_0^{1,p}(Z) \to W_0^{1,p}(Z)$ defined by

$$\tau(x)(z) = \begin{cases} \varphi(z) & \text{if } \varphi(z) \leqslant x(z) \\ x(z) & \text{if } u(z) \leqslant x(z) \leqslant \varphi(z) \\ u(z) & \text{if } x(z) \leqslant u(z) . \end{cases}$$

It is easy to check that $\tau(\cdot)$ is continuous.

Also we introduce the penalty function $\beta : Z \times \mathbb{R} \to \mathbb{R}$ defined by

$$\beta(z, x) = \begin{cases} (x - \varphi(z))^{p-1} & \text{if } \varphi(z) \leqslant x \\ 0 & \text{if } u(z) \leqslant x \leqslant \varphi(z) \\ -(u(z) - x)^{p-1} & \text{if } x \leqslant u(z) . \end{cases}$$

From this definition it is clear that $\beta(z, x)$ is Caratheodory function (i.e. $z \to \beta(z, x)$ is measurable and $x \to \beta(z, x)$ is continuous) and $|\beta|(z, x) \leqslant a_1(z) + c_1 |x|^{p-1}$ a.e. on Z with $a_1 \in L^q(Z)$, $c_1 > 0$. Moreover, we have

$$\int_Z \beta(z, x(z))\, x(z)\, dz \geqslant c_2 \|x\|_p^p - c_3$$

for some $c_2, c_3 > 0$ and for all $x \in L^p(Z)$.

We consider the following auxiliary problem

$$\text{(11)} \qquad \left\{ \begin{array}{l} -\operatorname{div}(\|Dx(z)\|^{p-2} Dx(z)) \in \widehat{f}(t, \tau(x)(z)) - \beta(z, x(z)) \text{ a.e. on } Z \\ x_{|\Gamma} = 0 \end{array} \right\} .$$

As before let $A : W_0^{1,p}(Z) \to W^{-1,q}(Z)$ be the operator defined by

$$\langle A(y), y \rangle = \int_Z \|Dx(z)\|^{p-2} (Dx(z), Dy(z))_{\mathbb{R}^N}\, dz .$$

We know the A is monotone, demicontinuous, hence maximal monotone. Also let $B : L^p(Z) \to L^q(Z)$ be the Nemitsky operator corresponding to the penalty function β, i.e. $B(x)(\cdot) = \beta(\cdot, x(\cdot))$. From Krasnoselskii's theorem, we know that B is continuous. Finally let $F : W_0^{1,p}(Z) \to 2^{L^q(Z)}$ be defined by $F(x) = \{h \in L^q(Z) : h(z) \in \widehat{f}(z, \tau(x)(z)) \text{ a.e. on } Z\}$. Note that by virtue of hypothesis $H(f)_1(i)$ $z \to \widehat{f}(z, \tau(x)(z))$ is a graph measurable multifunction and so by the Yankov-von Neumann-Aumann selection theorem (see Hu-Papageorgiou [14], theorem II. 2.14, p. 158) we see that F has nonempty values which are clearly weakly compact and convex. Moreover, F is bounded. Introduce the multifunction $R = A + B - F : W_0^{1,p}(Z) \to 2^{W^{-1,q}(Z)} \setminus \{\emptyset\}$ (recall that $W_0^{1,p} \subseteq L^p(Z)$ and $L^q(Z) \subseteq W^{-1,q}(Z)$).

Claim 1: R is pseudomonotone and coercive.

Clearly R is bounded. Thus in order to prove the pseudomonotonicity of R it suffices to show that R is generalized pseudomonotone. To this end let $x_n \xrightarrow{w} x$ in $W_0^{1,p}(Z)$, $v_m \xrightarrow{w} v$ in $W^{-1,q}(Z)$ $v_n \in R(x_n)$, $n \geq 1$, and assume that $\overline{\lim} \langle v_n, x_m - x \rangle \leq 0$. We have $v_n = A(x_n) + B(x_n) - w_n$, with $w_n \in F(x_n)$, $n \geq 1$.

Then

$$\langle v_n, x_n - x \rangle = \langle A(x_n) + B(x_n) - w_n, x_n - x \rangle$$
$$= \langle A(x_n), x_n - x \rangle + (B(x_n), x_n - x)_{pq} - (w_n, x_n - x)_{pq}.$$

Since $W_0^{1,p}(Z)$ is embedded compactly in $L^p(Z)$, we have $x_n \to x$ in $L^p(Z)$. So $(B(x_n), x_n - x)_{pq} \to 0$. Also since F is bounded, we have that $\{w_n\}_{n \geq 1} \subseteq L^q(Z)$ is bounded and so $(w_n, x_n - x_{pq}) \to 0$ as $n \to \infty$. Therefore finally we can say that

$$\overline{\lim} \langle A(x_n), x_n - x \rangle \leq 0 .$$

Because A is maximal monotone, it is generalized pseudomonotone and so

$$A(x_n) \xrightarrow{w} A(x) \text{ in } W_0^{1,p}(Z) \text{ and } \langle A(x_n), x_n \rangle \to \langle A(x), x \rangle$$
$$\Rightarrow \|Dx_n\|_p \to \|Dx\|_p .$$

Recall that $Dx_n \xrightarrow{w} Dx$ in $L^p(Z, \mathbb{R}^N)$ and because $L^p(Z, \mathbb{R}^N)$ is uniformly convex it has the Kadec-Klee property and so $Dx_n \to Dx$ in $L^p(Z, \mathbb{R}^N)$, i.e. $x_n \to x$ in $W_0^{1,p}(Z)$ (see Hu-Papageorgiou [14] Lemma I.1.7.4, p. 28). Hence we have $A(x_n) \xrightarrow{w} A(x)$ in $W^{-1,q}(Z)$ (demicontinuity of A), $B(x_n) \to B(x)$ in $L^q(Z)$ (continuity of B) and by passing to a subsequense if necessary, $w_n \xrightarrow{w} w$ in $L^q(Z)$, hence $w_n \to w$ in $W^{-1,q}(Z)$. Using proposition VII.3.9, p. 694, of Hu-Papageorgiou [14], we have that $w \in F(x)$. So $v_n \xrightarrow{w} v = A(x) + B(x) - w$ in $W^{-1,q}(Z)$ with $w \in F(x)$ and $\langle v_n, x_n \rangle \to \langle v, x \rangle$. This proves the generalized pseudomonotonicity of R, thus the pseudomonotonicity.

Also for every $x \in W_0^{1,p}(Z)$ and every $v \in R(x)$, we have

$$\begin{aligned}\langle v, x\rangle &= \langle A(x), x\rangle + (B(x), x)_{pq} - (w, x)_{pq} \quad (\text{for some } w \in F(x)) \\ &\geq \|Dx\|_p^p + c_2\|x\|_p^p - c_3 - \|w\|_q \|x\|_p \quad (\text{by Hölders inequality}).\end{aligned}$$

Recall that $|w(z)| \leq a(z) + c\|\varphi\|_\infty^{p-1}$ a.e. on Z with $a \in L^\infty(Z)$. So we obtain

$$\langle v, x\rangle \geq \|Dx\|_p^p + c_2\|x\|_p^p - c_4 \quad \text{for some } c_4 > 0 .$$

From this inequality it follows that R is coercive. This completes the proof of the claim.

We know that a pseudomonotone, coercive operator is surjective. So there exists $x \in W_0^{1,p}(Z)$ such that $0 \in R(x)$, hence $0 = A(x) + B(x) - w$ for some $w \in F(x)$. As in the proof of proposition 1, from the operator equation $A(x) = w - B(x)$ in $W^{-1,q}(Z)$, we obtain that $x \in W_0^{1,p}(Z)$ is a solution of (11).

Claim 2: $0 < u(z) \leq x(z) \leq \varphi(z)$ for almost all $z \in Z$.

Since $u \in W_0^{1,p}(Z)$ is a lower solution of (2), we have

$$(12) \quad \int_Z \|Du\|^{p-2}(Du, D\eta)_{\mathbb{R}^N}\, dz \leq \int_Z f_1(z, u(z))\, \eta dz \quad \text{for all } \eta \in W_0^{1,p}(Z)_+ .$$

Also $x \in W_0^{1,p}(Z)$ being a solution of (11), it satisfies

$$(13) \qquad \int_Z \|Dx\|^{p-2}(Dx, D\eta)_{\mathbb{R}^N}\, dz = \int_Z w\eta dz - \int_Z \beta(z, x(z))\, \eta dz$$

with $w \in L^q(Z), f_1(z, \tau(x)(z)) \leq w(z) \leq f_2(z, \tau(x)(z))$ a.e. on Z. In (12) and (??) use as test function $\eta = (u - x)_+ = \max[(u - x), 0] \in W_0^{1,p}(Z)_+$ and then subtract (12) from (13). We obtain

$$(14) \quad \begin{aligned}&\int_Z (\|Dx\|^{p-2}Dx - \|Du\|^{p-2}Du, D(u - x)_+)_{\mathbb{R}^N} \\ &\geq \int_Z (w(z) - f_1(z, u(z)))(u - x)_+\, dz - \int_Z \beta(z, x(z))(u - x)_+\, dz .\end{aligned}$$

We know that (see Gilbarg-Trudinger [13], p. 46)

$$D(u - x)_+(z) = \begin{cases} D(u - x)(z) & \text{a.e. on } \{x < u\} \\ 0 & \text{a.e. on } \{x \geq u\} .\end{cases}$$

So we obtain

$$\int_Z (\|Dx\|^{p-2}Dx - \|Du\|^{p-2}Du,\, D(u-x)_+)_{\mathbb{R}^N} = \int_{\{x<u\}} (\|Dx\|^{p-2}Dx - \|Du\|^{p-2}Du,\, Du - Dx)_{\mathbb{R}^N}\, dz \leqslant 0\,. \tag{15}$$

Also we have

$$\int_Z (w(z) - f_1(z, u(z)))(u-x)_+\, dz \geqslant 0 \quad \text{since } f_1(z, u(z)) \leqslant w(z) \text{ a.e. on } Z \tag{16}$$

and

$$\int_Z \beta(z,\, x(z))(u-x)_+\, dz = -\int_{\{x<u\}} (u-x)^{p-1}(u-x)\, dz\,. \tag{17}$$

Using (15), (16) and (17) in (14), we obtain

$$\int_Z (u-x)_+^p\, dz \leqslant 0$$

hence $u(z) \leqslant x(z)$ for almost all $z \in Z$.

With a similar argument we can show that $x(z) \leqslant \varphi(z)$ a.e. on Z. Thus finally we have that $u(z) \leqslant x(z) \leqslant \varphi(z)$ a.e. on Z and so $\tau(x) = x$, $\beta(z,\, x(z)) = 0$. From these facts it follows that $x \in W_0^{1,p}(Z)$ solves (2). ■

Using once again hypothesis $H(f)_1(iv)$, we can find $\delta > 0$ such that for almost all $z \in Z$ and all $-\delta \leqslant x < 0$, we have

$$\begin{gathered} \lambda_1 |x|^{p-2}x > f(z,\, x) \\ \text{hence } \lambda_1 |x|^{p-2}x \geqslant f_2(z,\, x)\,. \end{gathered} \tag{18}$$

As before let $0 < \xi_2 < 1$ small enough so that $0 < -\xi_2 u_1(z) \leqslant \delta$ for all $z \in Z$. From proposition 2 we have that $\psi \in C^1(\overline{Z})$, $\psi(z) < 0$ for all $z \in Z$. As before we can find $c_2 > 1$ such that $c_2\psi(z) < -\xi_2 u_1(z)$ for all $z \in Z$. Then $\psi(z) < -\frac{\xi_2}{r_2} u_1(z)$ for all $z \in Z$. If we set $\xi = \frac{\xi_2}{r_2} > 0$ and $\widehat{u} = -\xi u_1$, we see using (18) that $\widehat{u} \in C^1(\overline{Z})$ is an upper solution of (2). Working with the upper-lower solution pair $\{\widehat{u},\, \psi\}$ as in proof of proposition 3, we obtain:

PROPOSITION 4. *If hypotheses* $H(f)_1$ *hold, then problem* (4) *has a bounded solution* $y \in W_0^{1,p}(Z)$ *such that* $y(z) < 0$ *a.e. on* Z.

Combining propositions 3 and 4, we obtain the following multiplicity theorem for problem (4).

THEOREM 5. *If hypotheses* $H(f)$ *hold, then problem* (4) *has two bounded solutions* $x, y \in W_0^{1,p}(Z)$ *such that* $y(z) < 0 < x(z)$ *a.e. on* Z.

4. Quasilinear resonant problems.

As before let $Z \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundery Γ. In this section we study the following quasilinear resonant problem:

$$(19)\quad \left\{\begin{array}{l} -\operatorname{div}(\|Dx(z)\|^{p-2}Dx(z)) - \lambda_1 |x(z)|^{p-2}x(z) = f(z, x(z)) - h(z) \text{ a.e. on } Z \\ x|_\Gamma = 0 \end{array}\right\}.$$

As in the previous section, we do not assume that $f(z, \cdot)$ is continuous. So in order to have an existence theory, we pass to a multivalued version of (19), by introducing the multifunction $\widehat{f}(z, x) = [f_1(z, x), f_2(z, x)]$. So instead of (19), we consider the following elliptic inclusion:

$$(20)\quad \left\{\begin{array}{l} -\operatorname{div}(\|Dx(z)\|^{p-2}Dx(z)) - \lambda_1 |x(z)|^{p-2}x(z) \in \widehat{f}(z, x(z)) - h(z) \text{ a.e. on } Z \\ x|_\Gamma = 0 \end{array}\right\}.$$

Our hypotheses on the forcing term f are the following:

$\mathbf{H(f)_2}$: $f : Z \times \mathbb{R} \to \mathbb{R}$ is a measurable function such that

(i) f_1 and f_2 are N-measurable functions;

(ii) for almost all $z \in Z$ and all $x \in \mathbb{R}$, $|f(z, x)| \leqslant a(z)$ with $a \in \in L^q(Z)$.

Let $f_+(z) = \underline{\lim}_{x \to +\infty} f_1(z, x) = \lim_{M \to +\infty} \text{ess} \inf_{x > M} f_1(z, x)$ and $f_-(t) = = \overline{\lim}_{x \to -\infty} f_2(z, x) = \lim_{M \to +\infty} \text{ess} \sup_{x < -M} f_2(z, x)$. The Landesman-Lazer type condition that we will use is the following:

$\mathbf{H_0}$: $h \in L^q(Z)$ and $\int_Z f_+(z) u_1(z)\, dz < \int_Z h(z) u_1(z)\, dz < \int_Z f_-(z) u_1(z)\, dz$ where $u_1 \in C^1(\overline{Z})$ is the normalized principal eigenfunction of $(-\Delta_\varrho, W_0^{1,p}(Z))$.

LEMMA 6. *If $Z \subseteq \mathbb{R}^N$ is an open subset $F : Z \times \mathbb{R}^k \to \mathbb{R}^k$ is a multifunction which has nonempty, compact and convex values and*

(i) F is measurable, i.e. for all $y \in \mathbb{R}^k$, the $\mathbb{R}$-valued function $(z, x) \to d(y, F(z, x)) = \inf[\|y - v\| : v \in F(z, x)]$ is measurable;

(ii) for every $z \in Z$, $F(z, \cdot)$ is upper semicontinuous, i.e. for all $C \subseteq \mathbb{R}^k$ nonempty, closed $F_z^-(C) = \{x \in \mathbb{R}^k : F(z, x) \cap C \neq \emptyset\}$ is closed,

then for every $\varepsilon > 0$ there exists a Caratheodory function $g_\varepsilon : Z \times \mathbb{R}^k \to \mathbb{R}^k$ such that $g_\varepsilon(z, x) \in F(z, \overline{B}_\varepsilon(x)) + \varepsilon \overline{B}_1$ for all $(z, x) \in Z \times \mathbb{R}^k$ where $\overline{B}_\varepsilon(x) = \{y \in \mathbb{R}^k : \|y - x\| \leq \varepsilon\}$ and $\overline{B}_1 = \overline{B}_1(0)$.

PROOF. Let $S_\varepsilon(z) = \{\eta\} \in C(\mathbb{R}^k, \mathbb{R}^k) : \eta(x) \in F(z, \overline{B}_\varepsilon(x)) + \varepsilon \overline{B}_1$ for all $x \in \mathbb{R}^k$. From theorem I.4.41, p. 106 of Hu-Papageorgiou [14] we know that $S_\varepsilon(z) \neq$ for all $z \in Z$. Set $F_1(z, x) = F(z, \overline{B}_\varepsilon(x)) + \varepsilon \overline{B}_1$. By virtue of corollary I.2.20, p. 42, of Hu-Papageorgiou [14], we have that F_1 has nonempty, closed values. Moreover, we can easily verify that $x \to F_1(z, x)$ has closed graph and this by virtue of proposition I.2.23, p. 43, of Hu-Papageorgiou [14] implies that $x \to F_1(z, x)$ is upper semicontinuous.

For every $x \in \mathbb{R}^k$ we have

$$GrF_1(\cdot, x) = \{(z, y) \in Z \times \mathbb{R}^k : y \in F(z, \overline{B}_\varepsilon(x)) + \varepsilon \overline{B}_1\}.$$

Set $G = \{(z, y, u) \in Z \times \mathbb{R}^k \times \mathbb{R}^k : y \in F(z, x + u) + \varepsilon \overline{B}_1, u \in \overline{B}_\varepsilon(0)\}$. Because of hypothesis (i), we have that $G \in B(Z) \times B(\mathbb{R}^k) \times B(\mathbb{R}^k)$, with $B(Z)$ (resp $B(\mathbb{R}^k)$) being the Borel σ-field of Z (resp. of $\mathbb{R}^k$) (see proposition II.1.7, p. 142, of Hu-Papageorgiou [14]). Note that

$$GrF_1(\cdot, x) = proj_{Z \times B(\mathbb{R}^k)} G .$$

Moreover, from the Levin-Novikov projection theorem (see Hu-Papageorgiou [14], theorem II.1.21, p. 146), we have that

$$GrF_1(\cdot, x) = proj_{Z \times B(\mathbb{R}^k)} G \in B(Z) \times B(\mathbb{R}^k),$$

hence $z \to F_1(z, x)$ is a measurable multifunction (Hu-Papageorgiou [14], p. 150).

Let $\{x_m\}_{m \geq 1} \subseteq \mathbb{R}^k$ be a dense sequence and recall that because $F_1(z, \cdot)$ is upper semicontinuous, for every $v \in \mathbb{R}^k$, $x \to d(v, F_1(z, x))$ is lower a semicontinuous $\mathbb{R}_+$-valued function (see Hu-Papageorgiou [14],

p. 61). We have

$$GrS_\varepsilon = \{(z, \eta) \in Z \times C(\mathbb{R}^k, \mathbb{R}^k): \eta(x) \in F_1(z, x) \text{ for all } x \in \mathbb{R}^k\}$$

$$= \{(z, \eta) \in Z \times C(\mathbb{R}^k, \mathbb{R}^k): d(\eta(x), F_1(z, x)) = 0 \text{ for all } x \in \mathbb{R}^k\}$$

$$= \bigcap_{m \geq 1} \{(z, \eta) \in Z \times C(\mathbb{R}^k, \mathbb{R}^k): d(\eta(x_m), F_1(z, x_m) = 0)\},$$

hence $GrS_\varepsilon \in B(Z) \times B(C(\mathbb{R}^k, \mathbb{R}^k))$.

Thus we can apply the Yankov-von Neumann-Aumann selection theorem (see Hu-Papageorgiou [14], theorem II.2.14, p. 158) to obtain $\eta_\varepsilon: Z \to C(\mathbb{R}^k, \mathbb{R}^k)$ a measurable map such that $\eta_\varepsilon(x) \in S_\varepsilon(z)$ for all $z \in Z$. Then $g_\varepsilon(z, x) = \eta_\varepsilon(z)(z)$ is the desired Caratheodory selector. ■

Now we can state and prove our existence theorem for problem (20). Our method of proof is based on degree theoretic arguments.

THEOREM 7. *If hypotheses $H(f)_2$ and H_0 hold, then problem* (20) *has a solution $x \in W_0^{1,p}(Z)$.*

PROOF. As in previous proofs let $A: W_0^{1,p}(Z) \to W^{-1,q}(Z)$ be the nonlinear operator defined by

$$\langle A(x), y \rangle = \int_Z \|Dx\|^{p-2} (Dx, Dy)_{\mathbb{R}^N} dz .$$

The operator A is strictly monotone, demicontinuous, hence maximal monotone. It is also coercive, thus it is surjective. Therefore A is a bijection and so we can define $A^{-1}: W^{-1,q}(Z) \to W_0^{1,p}(Z)$. We claim that A^{-1} is continuous and bounded. To this end let $v_n \to v$ in $W^{-1,q}(Z)$ and set $x_n = A^{-1}(v_n)$, $n \geq 1$. We have $A(x_n) = v_n$ and so $\langle A(x_n), x_n \rangle = \|Dx_n\|_p^p = \langle v_n, x_n \rangle \leq \|v_n\|_* \|x_n\|$. From Poincare's inequality it follows that $\{x_n\}_{n \geq 1} \subseteq W_0^{1,p}(Z)$ is bounded. So by passing to a subsequence if necessary, we may assume that $x_n \xrightarrow{w} x$ in $W_0^{1,p}(Z)$. Note that $[x_n, v_n] \in GrA$, $n \geq 1$, and $[x_n, v_n] \to^{w \times s} [x, v]$ in $W_0^{1,p}(Z) \times W^{-1,q}(Z)$. Since the graph of the maximal monotone map A is sequentially closed in $W_0^{1,p}(Z)_w \times W^{-1,q}(Z)$, it follows that $[x, v] \in GrA$, i.e. $x = A^{-1}(v)$. Also $\langle A(x_n), x_n - x \rangle = \langle v_n, x_n - x \rangle \to 0$ and so by generalized pseudomonotonicity we have $\langle A(x_n), x_n \rangle \to \langle A(x), x \rangle$, i.e. $\|Dx_n\|_p \to \|Dx\|_p$. As before by the Kadec-Klee property we have $x_n \to x$ in $W_0^{1,p}(Z)$ and so A^{-1} is continuous and bounded.

From their definition, it is clear that $x \to f_1(z, x)$ is a lower semicontinuous function and $x \to f_2(z, x)$ is an upper semicontinuous functions. So from Hu-Papageorgiou [14], p. 37, we have that $x \to \widehat{f}(t, x)$ is an upper semicontinuous multifunction. Moreover, because of hypothesis $H(f)_2(i)$, $(z, x) \to \widehat{f}(z, x)$ is measurable. Using lemma 6, for every $0 < \varepsilon < 1$, we can find a Caratheodory function $g_\varepsilon \colon Z \times \mathbb{R} \to \mathbb{R}$ satisfying $g_\varepsilon(z, x) \in \widehat{f}(z, \overline{B}_\varepsilon) + \overline{B}_\varepsilon$ for all $(z, x) \in Z \times \mathbb{R}$, where $\overline{B}_\varepsilon = [-\varepsilon, \varepsilon]$. Let $\widehat{g}_\varepsilon \colon L^p(Z) \to L^q(Z)$ be the Nemitsky operator corresponding to the function $g_\varepsilon(z, x)$; i.e. $\widehat{g}_\varepsilon(x)(\cdot) = g_\varepsilon(\cdot, x(\cdot))$. We know that $\widehat{g}_\varepsilon$ is continuous and bounded (Krasnoselskii's theorem). Also let $J : L^p(Z) \to L^q(Z)$ be defined by $J(x)(\cdot) = |x(\cdot|)^{p-2}x(\cdot)$. Clearly J is continuous and bounded. Exploiting the compact embedding of $W_0^{1,p}(Z)$ into $L^p(Z)$, we infer that $A^{-1}(\lambda_1 J + \widehat{g}_\varepsilon - h) \colon L^p(Z) \to L^p(Z)$ is a compact map (i.e is continuous and maps bounded sets into relatively compact sets).

Consider the following parametric family of fixed point problems:

$$x = tA^{-1}(\lambda_1 J + \widehat{g}_\varepsilon - h)(x), \qquad 0 < t < 1 . \tag{21}$$

We will obtain an a priori bound in $L^p(Z)$, independent of t. Suppose that is not possible. Then we can find $x_n \in W_0^{1,p}(Z) \subseteq L^p(Z)$, $t_n \in (0, 1)$, $n \geq 1$, such that

$$A(x_n) = t_n^{p-1}(\lambda_1 J(x_n) + \widehat{g}_\varepsilon(x_n) - h) \tag{22}$$

(from th $(p-1)$0-homogeneity of A) and $\|x\|_p \to \infty$, $t_n \to t \in [0, 1]$.

Let $y_n = \frac{x_n}{\|x_n\|_p}$, $n \geq 1$. Divide equation (22) by $\|x_n\|_p^{p-1}$. We obtain

$$A(y_n) = t_n^{p-1}\lambda_1 J(y_n) + \frac{t_n^{p-1}}{\|x_n\|_p^{p-1}}\widehat{g}_\varepsilon(x_n) - \frac{t_n^{p-1}}{\|x_n\|_p^{p-1}} h .$$

Note that $\|g_\varepsilon(x_n)\|q \leq \|a\|_q + 1$ for all $n \geq 1$ and so $\left\{ \frac{t_n^{p-1}}{\|x_n\|_p^{p-1}} \right\}_{n \geq 1} \subseteq$ $\subseteq L^p(Z)$ is bounded. Also we have

$$\langle A(y_n), y_n \rangle = t_n^{p-1}\lambda_1 (J(y_n), y_n)_{pq} + \frac{t_n^{p-1}}{\|x_n\|_p^{p-1}}(\widehat{g}_\varepsilon(x_n), y_n)_{pq}$$

$$- \frac{t_n^{p-1}}{\|x_n\|_p^{p-1}}(h, y_n)_{pq}$$

$$\Rightarrow \|Dy_n\|_p^p \leq \lambda_1 \|y_n\|_p^p + \xi_1 \|y_n\|_p \text{ for some } \xi_1 > 0 ,$$

$$\Rightarrow \|Dy_n\|_p^p \leq \lambda_1 + \xi_1 \ (\text{since } \|y_n\|_p = 1) .$$

So by Poincare's inequality, it follows that $\{y_n\}_{n \geq 1} \subseteq W_0^{1,p}(Z)$ is bounded. So we may assume that

$$y_n \xrightarrow{w} y \text{ in } W_0^{1,p}(Z),\ y_n \to y \text{ in } L^p(Z),\ y_n(z) \to y(z) \text{ a.e. on } Z$$

and $|y_n(z)| \leq k(z)$ a.e. on Z with $k \in L^p(Z)$.

Note that $t_n^{p-1}\lambda_1 J(y_n) \to t^{p-1}\lambda_1 J(y)$ in $L^q(Z)$ and $\frac{t_n^{p-1}}{\|x_n\|_p^{p-1}} J(y_n)$, $\frac{t_n^{p-1}}{\|x_n\|_p^{p-1}} h \to 0$ in $L^q(Z)$.

As before exploiting the fact that A being maximal monotone has a graph which is sequentially closed in $W_0^{1,p}(Z)_w \times W^{-1,q}(Z)$, in the limit we obtain

$$A(y) = t^{p-1}\lambda_1 J(y),\ 0 \leq t \leq 1 \tag{23}$$

and so $\|Dy_n\|_p \to \|Dy\|_p$. As before from this convergence and since $Dy_n \xrightarrow{w} Dy$ in $L^p(Z, \mathbb{R}^N)$, we conclude that $y_n \to y$ in $W_0^{1,p}(Z)$, with $y \neq 0$, since $\|y\|_p = 1$. Moreover, from (23) and (2), we have that $t = 1$ and $y = \pm u_1$. Assume without any loss of generality that $y = u_1$ (the analysis of the other case is similar). Recall that $u_1(z) > 0$ for all $z \in Z$ and so $x_n(z) \to +\infty$ a.e. on Z. We have

$$\begin{aligned} &\langle A(y_n), y_n\rangle - t_n^{p-1}\lambda_1 (J(y_n), y_n)_{pq} = \frac{t_n^{p-1}}{\|x_n\|_p^{p-1}} (\widehat{g}_\varepsilon(x_n), y_n)_{pq} \\ &- \frac{t_n^{p-1}}{\|x_n\|_p^{p-1}} (h, y_n)_{pq} \\ \Rightarrow\ & \frac{t_n^{p-1}}{\|x_n\|_p^{p-1}} [(\widehat{g}_\varepsilon(x_n), y_n)_{pq} - (h, y_n)_{pq}] > 0 \\ & \text{(from (2) and since } 0 < t_n < 1 \text{ for all } n \geq 1) \\ \Rightarrow\ & (\widehat{g}_\varepsilon(x_n), y_n)_{pq} > (h, y_n)_{pq} \text{ for all } n \geq 1 . \end{aligned}$$

By construction, we have that for almost all $z \in Z$ and all $n \geq 1$

$$f_1(z, v_n(z)) - \varepsilon \leq \widehat{g}_\varepsilon(x_n)(z) \leq f_2(z, v_n(z)),\ + \varepsilon |v_n(z) - x_n(z)| < \varepsilon .$$

So $v_n(z) \to +\infty$ a.e. on Z. Hence in the limit we have $g_\varepsilon(z) \leq f_+(z) + \varepsilon$ a.e. on Z. Also $(\widehat{g}_\varepsilon(x_n), y_n)_{pq} \to (\widehat{g}_\varepsilon(x), u_1)_{pq}$ and $(h, y_n)_{pq} \to (h, u_1)_{pq}$.

Therefore we can write that

$$(h, u_1)_{pq} \leqslant (f_+ + \varepsilon, u_1)_{pq}$$
$$\Rightarrow \ 0 < \xi = \int_Z h(z)\, u_1(z)\, dz - \int_Z f_+(z)\, u_1(z)\, dz \leqslant \varepsilon \|u_1\|_1 \ \text{(hypothesis } H_0).$$

Choose $\varepsilon > 0$ so that $\varepsilon\|u_1\|_1 < \xi$. Then we have a contradiction. This means that the solution of (22) are bounded in $L^p(Z)$ and the bound is indepent of $0 < t < 1$. Invoking the Leray-Schauder alternative theorem, we obtain $x_\varepsilon \in W_0^{1,p}(Z)$ such that

$$A(x_\varepsilon) - \lambda_1 J(x_\varepsilon) = \widehat{g}_\varepsilon(x_\varepsilon) - h\,.$$

Next let $\varepsilon_m = \dfrac{1}{m}$ and $x_{\varepsilon_m} = x_m \in W_0^{1,p}(Z)$ be the coresponding solutions. We claim that $\{x_m\}_{m\geqslant} \subseteq W_0^{1,p}(Z)$ is bounded. Suppose this is not the case. This we may assume that $\|x_m\| \to +\infty$. Setting $\widehat{g}_{\varepsilon_m} = \widehat{g}_m$, we have

$$A(x_m) - \lambda_1 J(x_m) + \widehat{g}_m(x_m) - h$$
$$\Rightarrow \langle A(x_m), x_m\rangle = \lambda_1 (J(x_m), x_m)_{pq} + (\widehat{g}_m(x_m), x_m)_{pq} - (h, x_m)_{pq}$$
$$\Rightarrow \|Dx_m\|_p^p \leqslant \lambda_1 \|x_m\|_p^p + \|a_1\|_q \|x_m\|_p \ \text{(with } a_1(\cdot) = a(\cdot) + 1 + h(\cdot) \in L^q(Z))$$
$$\Rightarrow \|x_m\|_p \to +\infty \ \text{(by Poincare's inequality)}.$$

Let $y_m = \dfrac{x_m}{\|x_m\|_p}$, $m \geqslant 1$. Note that $\{y_m\}_{m\geqslant 1} \subseteq W_0^{1,p}(Z)$ is bounded and so as before by passing to a subsequence if necessary, we may assume that

$$y_m \xrightarrow{w} y \ \text{in } W_0^{1,p}(Z),\ y_m \to y \ \text{in } L^p(Z),\ y_m(z) \to y(z) \ \text{a.e. on } Z$$

and $|y_m(z)| \leqslant k_2(z)$ a.e. on Z, $k_2 \in L^p(Z)$.

We have

$$A(y_m) = \lambda_1 J(y_m) + \frac{1}{\|x_m\|_p^{p-1}} \widehat{g}_m(x_m) - \frac{1}{\|x_m\|_p^{p-1}} h$$
$$\Rightarrow A(y) = \lambda_1 J(y) \ \text{(as before since A is maximal monotone)} \tag{24}$$
$$\Rightarrow y = \pm u_1 \ \text{(since } \ y \neq 0 \ \text{ because } \|y\|_p = 1).$$

Again without any loss of generality we assume that $y = u_1$. This means that $x_m(z) \to +\infty$ a.e. on Z. From the construction of $g_m(z, x)$ we have

$$f_1(z, v_m(z)) - \frac{1}{m} \leqslant \hat{g}_m(x_m)(z) \leqslant f_2(z, v_m(z)) + \frac{1}{m}$$

with $v_m \in L^p(Z)$ such that $|v_m(z) - x_m(z)| < \frac{1}{m}$ a.e. on Z. Hence $v_m(z) \to +\infty$ a.e. on Z as $m \to \infty$.

Thus we have

$$\overline{\lim}\, \hat{g}_m(x_m)(z) \leqslant \overline{\lim}\, f_2(z, v_m(z)) \leqslant f_+(z) \text{ a.e. on } Z.$$

Also $\{\hat{g}_m(x_m)\}_{m \geqslant 1} \subseteq L^q(Z)$ is bounded and so may assume that $\hat{g}_m(x_m) \xrightarrow{w} g$ in $L^q(Z)$. We have

$$\begin{aligned} &(\hat{g}_m(x_m), y_m)_{pq} \geqslant (h, y_m)_{pq} \text{ (from (24) and (2))} \\ \Rightarrow\ &(g, u_1)_{pq} \geqslant (h, u_1)_{pq} \\ \Rightarrow\ &\int_Z f_+(z)\, u_1(z)\, dz \geqslant \int_Z h(z)\, u_1(z)\, dz\,, \end{aligned}$$

which contradicts hypothesis H_0. Therefore $\{x_n\}_{n \geqslant 1} \subseteq W_0^{1,p}(Z)$ is bounded and so we may assume that $x_m \to x$ in $W_0^{1,p}$, $x_m \to x$ in $L^p(Z)$, $x_m(z) \to x(z)$ a.e. on Z and $|x_m(z)| \leqslant k_3(z)$ a.e. on Z with $k_3 \in L^p(Z)$.

For every $m \geqslant 1$ we have

$$A(x_m) = \lambda_1 J(x_m) + \hat{g}_m(x_m) - h$$

and $J(x_m) \to J(x)$ in $L^q(Z)$, $\hat{g}_m(x_m) \xrightarrow{w} g$ in $L^q(Z)$, thus $\hat{g}_m(x_m) \to g$ in $W^{-1,q}(Z)$. As before exploiting the fact that GrA is sequentially closed in $W_0^{1,p}(Z)_w \times W^{-1,q}(Z)$, in the limit we obtain

$$A(x) = \lambda_1 J(x) + g - h\,. \tag{25}$$

Note that

$$f_1(z, v_m(z)) - \frac{1}{m} \leqslant \hat{g}_m(x_m)(z) \leqslant f_2(z, v_m(z)) + \frac{1}{m}$$

$$\text{with } |v_m(z) - x_m(z)| \leqslant \frac{1}{m}$$

a.e. on Z

Hence $v_m(z) \to x(z)$ a.e. on Z. Therefore in the limit as $m \to \infty$ we obtain

$$f_1(z, x(z)) \leqslant \varliminf_{m \to \infty} f_1(z, v_m(z)) \leqslant g(z) \leqslant \varlimsup f_2(z, v_m(z))$$
$$\leqslant f_2(z, x(z))$$

a.e. on Z.

$$\Rightarrow g(z) \in \hat{f}(z, x(z)) \text{ a.e. on } Z$$

Finally as in the proof of proposition 1, from (25) we conclude that $x \in W_0^{1,p}(Z)$ is a solution of (20). ■

We can have another existence theorem, using a different Landesman-Lazer type hypothesis.

H$_1$: $h \in L^q(Z)$ and $\int_Z f_-(z)\,dz < \int_Z h(z)\,dz < \int_Z f_+(z)\,dz$.

THEOREM 8. *If hypotheses $H(f)_2$ and H_1 hold, then problem* (20) *has a solution $x \in W_0^{1,p}(Z)$.*

PROOF. The proof follows the steps of that of theorem 7. So we only indicate where it differs. In this case we have (keeping the notation of the proof of theorem 7)

$$A(y_n) - t_n^{p-1} J(y_n) = \frac{t_n^{p-1}}{\|x_n\|_p^{p-1}} (\widehat{g_\varepsilon}(x_n) - h)$$
$$\Rightarrow \; -t_n^{p-1} (J(y_n), 1)_{pq} = \frac{t_n^{p-1}}{\|x_n\|_p^{p-1}} (\widehat{g_\varepsilon}(x_n) - h, 1)_{pq}$$

(1 is the constant equal to 1 function).

From the proof of theorem 7 we have that $-t_n^{p-1}(J(y_n), 1)_{pq} \to -(J(u_1), 1) < 0$. So for $n \geqslant 1$ large, we have

$$-t_n^{p-1}(J(y_n), 1)_{pq} < 0$$
$$\Rightarrow (\widehat{g}_\varepsilon(x_n), 1)_{pq} < (h, 1)_{pq}.$$

Passing to the limit and since $\varliminf \widehat{g}_\varepsilon(x_n)(z) \geqslant \varliminf f_1(z, v_n(z)) - \varepsilon \geqslant f_+(z) - \varepsilon$ a.e. on Z (since $x_n(z) \to \infty$ a.e. on Z and $|v_n(z) - x_n(z)| \leqslant \varepsilon$ a.e.

on Z), we obtain via Fatou's lemma

$$(f_+ - \varepsilon)_{pq} \leqslant (h, 1)_{pq}$$
$$\Rightarrow \int_Z (f_+(z) - h(z))\, dz = \xi \leqslant \varepsilon |Z| .$$

By hypothesis H_1, $\xi > 0$. So if we choose $\varepsilon > 0$ such that $\varepsilon|Z| < \xi$, we have a contradiction. The rest of the proof follows the steps of the proof of theorem 7 with some minor obvious modifications. ■

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