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Abstract - The main result of this paper is a global existence theorem in suitable Sobolev spaces for 2D incompressible MHD system in the half-plane. The existence result derives by the existence of a global classical solution in Hölder spaces, by proving some a-priori estimates in Sobolev spaces and, finally, by applying the Banach-Caccioppoli fixed point theorem. Hence, the uniqueness of the solution follows.

1. Introduction.

Let $\Omega$ be the half-plane $\mathbb{R}^2_+ := \{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0\}$, and let $\Gamma$ be the boundary of $\Omega$. In $Q_T := \Omega \times (0, T)$, with $T > 0$, we consider the equations of magneto-hydrodynamics for 2D incompressible ideal fluid

$$
(1) \quad u_t + (u \cdot \nabla) u + \nabla p + \frac{1}{2} \nabla |B|^2 - (B \cdot \nabla) B = 0 \quad \text{in } Q_T,
$$

$$
(2) \quad B_t + (u \cdot \nabla) B - (B \cdot \nabla) u - \mu \Delta B = 0 \quad \text{in } Q_T,
$$

$$
(3) \quad \text{div } u = 0 \quad \text{in } Q_T,
$$

$$
(4) \quad \text{div } B = 0 \quad \text{in } Q_T,
$$

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Here $u = u(x, t) = (u^1(x, t), u^2(x, t)), B = B(x, t) = (B^1(x, t), B^2(x, t))$ and $\pi = \pi(x, t)$ denote the unknown velocity field, the magnetic field and the pressure of the fluid respectively. The functions $u_0 = (u_0^1(x), u_0^2(x))$ and $B_0 = (B_0^1(x), B_0^2(x))$ denote the given initial data, $\nu$ the unit outward normal on $\Gamma$ and $\mu$ a real positive constant. Moreover, we use the notation

$$f_i = \frac{\partial f}{\partial t}, \quad \partial_i = \frac{\partial}{\partial x_i}, \quad \nabla = (\partial_1, \partial_2), \quad u \cdot \nabla = u^1 \partial_1 + u^2 \partial_2,$$

$$\nabla^2 = \frac{\partial^2}{\partial_i \partial_j}, \quad \Delta = \nabla^2_{11} + \nabla^2_{22}.$$

In case the magnetic field $B$ is identically equal to zero, i.e. in the case of Euler equations, such a problem for global classical solutions was studied by many authors, starting from Lichtenstein [10] and Wolibner [15]. The existence of global solutions in Hölder spaces in bounded domains has been proven by Kato [6]. This result was extended to the exterior domain case by Kikuchi [8]. On the other hand, the existence of a classical solution for MHD system was shown by Kozono [9] and by Casella, Secchi and Trebeschi [5] in the bounded and unbounded case, respectively.

Existence results in Sobolev spaces were proved by several authors. For the Euler equation we refer to Temam [14], Kato and Lai [7] and Beirão Da Veiga [3], [4]. Existence and uniqueness results in $W^{k}$-spaces for the equations of magneto-hydrodynamics, when $\mu = 0$, have been proved by Alexseev [1]. Moreover, in this case, Secchi [12] and Schmidt [11] proved not only existence and uniqueness results, but also the continuous dependence on the data. In this paper we prove a global existence result in suitable Sobolev spaces for MHD system in the half-plane case. To prove this result, firstly, we show a local existence theorem in Sobolev spaces. Then we derive some a-priori estimates, global in time, which come from the all-time existence of classical solution of system (1)-
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(9) in Hölder-spaces. We underline that energy-method works well, since the classical solution \((u, B)\) is such that 
\[ \|u(t)\|_{L^p(\Omega)}, \|B(t)\|_{L^p(\Omega)}, \|\nabla u(t)\|_{L^p(\Omega)}, \|\nabla B(t)\|_{L^p(\Omega)} \] and 
\[ \|B_1(t)\|_{L^p(\Omega)} \] are uniformly bounded in time on the whole interval \([0, T]\).

We observe that the main result obtained in the present paper is a necessary first step in the analysis of slightly compressible MHD fluids, which will be the object of a forthcoming work.

The plan of the paper is the following. In next section we fix some notations and we introduce some preliminary results and the main theorem. In Section 3 we show some a-priori estimates, and finally in Section 4 we prove the main result.

\section*{2. – Notations and results.}

For a scalar-valued function \(\phi\), we set
\[ \text{Rot} \phi = (\partial_2 \phi, -\partial_1 \phi), \]
for a vector-valued function \(u = (u_1, u_2)\), we use the notation
\[ \text{rot} u = \partial_1 u^2 - \partial_2 u^1 \quad \text{and} \quad \text{div} u = \nabla \cdot u = \partial_1 u^1 + \partial_2 u^2. \]

We denote the norm of \(L^p(\Omega)\), \(1 \leq p \leq \infty\), by \(\|\cdot\|_{L^p} \). \(H^m(\Omega)\) denotes the usual Sobolev space of order \(m \geq 1\), and \(\|\cdot\|_{H^m}\) denotes its norm. For simplicity we use the abbreviated notation \(L^p, H^m\). We also use the same symbol for spaces of scalar and vector-valued functions.

Moreover, if \(X\) is a normed space, then \(L^p(0, T; X)\), with \(1 \leq p < +\infty\), denotes the set of all measurable functions \(u(t)\) with values in \(X\) such that:
\[ \|u\|_{L^p(0, T; X)} := \left( \int_0^T \|u(t)\|^p_X \, dt \right)^{1/p} < +\infty, \]
where \(\|\cdot\|_X\) is the norm in \(X\).

Given \(T > 0\) arbitrary, the set of all essentially bounded (with respect to the norm of \(X\)) measurable functions of \(t\) with values in \(X\) is denoted by \(L^\infty(0, T; X)\). We equip this space with the usual norm
\[ \|f\|_{L^\infty(0, T; X)} = \sup_{t \in [0, T]} \|f(t)\|_X. \]

In particular, the norm of \(L^p(0, T; L^p)\), \(1 \leq p < +\infty\), is denoted by \(\|\cdot\|_{L^p(0, T; L^p)}\).
Let $C^m([0,T];X)$ denote the set of all $X$-valued $m$-times continuously differentiable functions of $t$, for $0 \leq t \leq T$.

We define $X^m(T) := \bigcap_{k=0}^{m-1} C^k([0,T];H^{m-k})$ equipped with the usual norm

$$
\|u\|_{X^m} := \sup_{t \in [0,T]} \sum_{k=0}^{m-1} \|\partial_t^k u(t)\|_{H^{m-k}}.
$$

We denote by $\mathcal{B}(\overline{Q})$ (resp. $\mathcal{B}(\overline{Q}_T)$) the Banach space of all real valued continuous and bounded functions on $\overline{Q}$ (resp. $\overline{Q}_T$), with the usual norm.

For $0 < \alpha < 1$, $C^\alpha(\overline{Q})$ denotes the usual space of functions in $\mathcal{B}(\overline{Q})$, uniformly Hölder continuous on $\overline{Q}$ with exponent $\alpha$; the norm of $C^\alpha(\overline{Q})$ is $\|\cdot\|_{L^\infty} + [\cdot]_\alpha$, where

$$
[\phi]_\alpha := \sup_{x \neq y, x, y \in \overline{Q}} \frac{|\phi(x) - \phi(y)|}{|x - y|^\alpha}.
$$

For $0 < \alpha < 1$ and integer $k$, $C^{k+\alpha}(\overline{Q})$ denotes the space of functions $\phi$ with $D^\beta \phi \in \mathcal{B}(\overline{Q})$ for $|\beta| \leq k$, and $D^\gamma \phi \in C^\alpha(\overline{Q})$ for $|\gamma| = k$. The norm is

$$
[\phi]_{k+\alpha} = \max_{|\beta| \leq k} \|D^\beta \phi\|_{L^\infty} + \max_{|\gamma| = k} [D^\gamma \phi]_\alpha.
$$

With $C^{k,j}(\overline{Q}_T)$ for integers $k$, $j \geq 0$ we mean the set of all functions $\phi$ for which every $\partial_{t,j}^k \phi$ exists and is continuous on $\overline{Q}_T$, for $0 \leq q \leq k$, $0 \leq r \leq j$, $C^{k+\alpha,j+\beta}(\overline{Q}_T)$, for integers $k$, $j \geq 0$ and $0 \leq \alpha$, $\beta < 1$ is the subset of $C^{k,j}(\overline{Q}_T)$, consisting of Hölder continuous functions with exponents $\alpha$ in $x$ and $\beta$ in $t$.

For every function $\phi \in C^{k+\alpha,j+\beta}(\overline{Q}_T)$, we consider the following seminorm:

$$
[\phi]_{k+\alpha,j+\beta} := \sup_{x \neq y, t \in [0,T]} \frac{|\phi(x,t) - \phi(y,t)|}{|x - y|^\alpha} + \sup_{t \neq s, x \in \overline{Q}} \frac{|\phi(x,t) - \phi(x,s)|}{|t - s|^\beta},
$$

and the norm

$$
[\phi]_{k+\alpha,j+\beta} := \max_{|\beta| \leq k} \|\partial_{t,j}^k \phi\|_{L^\infty} + \max_{|\gamma| = k} [\partial_{t,j}^k \phi]_{\alpha,\beta}.
$$

We shall denote by $C$ and by $C_i$, $i \in \mathbb{N}$, some real positive constants which may be different in each occurrence, and by $C_i(t)$ a real function in $L^\infty(0,T)$ depending on $\|u(t)\|_{L^\infty}$, $\|B(t)\|_{L^\infty}$, $\|\nabla u(t)\|_{L^\infty}$, $\|\nabla B(t)\|_{L^\infty}$, $\|B_i(t)\|_{L^\infty}$ and some their suitable powers.
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We now set $Z := \text{rot } u$, $\xi := \text{rot } B$, $Z_0 := \text{rot } u_0$ and $\xi_0 := \text{rot } B_0$. By applying rot to both sides of equations (1) and (2), we get

\begin{align*}
Z_t + u \cdot \nabla Z - B \cdot \nabla \xi &= 0, \\
\xi_t + u \cdot \nabla \xi - B \cdot \nabla Z - 2 \partial_1 u^1 \mathbb{D} B + 2 \partial_2 B^2 \mathbb{D} u - \mu A \xi &= 0,
\end{align*}

where $\mathbb{D} u = \partial_1 u^2 + \partial_2 u^1$, and $\mathbb{D} B = \partial_1 B^2 + \partial_2 B^1$. Finally, let $F$, $\phi$, $\tilde{\phi}$, $\psi$ be defined as

\begin{align*}
F &= -u \cdot \nabla \xi + B \cdot \nabla Z - 2 \partial_1 u^1 \mathbb{D} B - 2 \partial_2 B^2 \mathbb{D} u, \\
\phi(s) &= \sum_{k=0}^{3} \| \partial^k \xi(s) \|_{H^1}, \\
\tilde{\phi}(s) &= \sum_{k=0}^{3} \| \partial^k F(s) \|_{H^{1-k}}, \\
\psi(s) &= \sum_{k=0}^{3} \| \partial^k \xi(s) \|_{L^2}.
\end{align*}

We now recall a result (see [5]) which will be fundamental to prove that, under suitable assumptions on initial data, the classical solution of problem (1)-(9) belongs to suitable Sobolev spaces.

**Theorem 2.1.** Let $T > 0$ be arbitrary. Let $u_0 \in C^{1+\theta}(\Omega)$, $\text{rot } u_0 \in L^1(\Omega)$, $B_0 \in C^{2+\theta}(\Omega) \cap H^1(\Omega)$ for some $0 < \theta < 1$, such that $\text{div } u_0 = \text{div } B_0 = 0$ in $\Omega$ and $u_0 \cdot v = B_0 \cdot v = 0$ on $\Gamma$. Then there exists a positive constant $C_\theta$ such that, if $\|B_0\|_{H^1} + |B_0|_{L^2} \leq C_\theta$, then there exists a solution $(u, B, \pi) \in C^{1+\theta}(Q_T) \times C^{2+\theta}(Q_T) \times C^{1,0}(Q_T)$ of system (1)-(9). Such a solution is unique up to an arbitrary function of $t$ which may be added to $\pi$.

**Remark 2.2.** In [5] Kozono’s result obtained in [9] and Kikuchi’s result, see [8], are extended to the exterior domain case and to the half-plane case, and to the MHD equations, respectively. In [5] the existence of the global classical solution for MHD system in Hölder spaces is proved by applying the Schauder fixed point theorem. The authors followed the idea of Kato [6], Kikuchi [8], Kozono [9]. The crucial step is the definition of a map, defined on a suitable class, the same already considered by Kozono in [9], which satisfies the conditions of the Schauder fixed point theorem. The uniqueness of the solutions of the studied problem is obtained by following standard techniques, see Temam [13].

The main result, we are going to prove, is:
THEOREM 2.3. Let $T > 0$ be arbitrary. Let the couple $(u_0, B_0) \in H^k \cap L^1$, \(\text{rot} u_0 \in L^1\), $\text{div} u_0 = \text{div} B_0 = 0$ in $\Omega$ and $u_0 \cdot v = B_0 \cdot v = 0$ on $\Gamma$. Assume also that, for some $0 < \theta < 1$,

$$\|B_0\|_{H^1} + \|B_0\|_{L^\infty} \leq C_*,$$

where $C_*$ is the constant obtained in Theorem 2.1.

Then problem (1)-(9) has a unique solution $(u, B, \pi)$ such that

$$u \in X^5(T), \quad B \in X^5(T) \cap L^2(0, T; H^k), \quad \nabla \pi \in X^4(T).$$

3. Some a-priori estimates.

We devote this section to prove

**Lemma 3.1.** The following energy-type estimate

$$\frac{1}{2} \frac{d}{dt} (\phi(t) + \|Z(t)\|_{H^1}^2) + C_1 \psi(t) \leq C_\mu(t)(\phi(t) + \|Z(t)\|_{H^1}^2)$$

holds in $[0, T].$

Note that Theorem 2.1 ensures us that the time functions $\|u(t)\|_{L^\infty(\Omega)}$, $\|B(t)\|_{L^\infty(\Omega)}$, $\|\nabla u(t)\|_{L^\infty(\Omega)}$, $\|\nabla B(t)\|_{L^\infty(\Omega)}$, and $\|B_t(t)\|_{L^\infty(\Omega)}$ are uniformly bounded in time on the whole interval $[0, T]$. Consequently, the real function $C_\mu(t)$ (appearing in (12) and in some preliminary lemmata given below) belongs to $L^\infty(0, T)$. We shall prove (12) for regular solutions.

**Lemma 3.2.** The couple $(u, B)$ satisfies the following energy-type estimate

$$\|u\|_{L^\infty(\Omega)} + \|B\|_{L^\infty(\Omega)} + 2\mu \|\nabla Z\|_{L^2(Q_T)} \leq \|u_0\|_{L^2} + \|B_0\|_{L^2}.$$  

**Proof.** We multiply equations (1) and (2) by $u$ and $B$ respectively. By standard calculations and by summing the resulting expressions, we get easily the thesis. □

**Lemma 3.3.** The following inequality holds

$$\|Z\|_{L^\infty(\Omega)} + \|\xi\|_{L^\infty(\Omega)} + \frac{\mu}{2} \|\nabla \xi\|_{L^2(Q_T)} \leq C(\|Z_0\|_{L^2} + \|\xi_0\|_{L^2}).$$
PROOF. We multiply equations (10) and (11) by \( Z \) and \( \xi \) respectively. Since
\[
\frac{2}{\|\nabla u(t)\|_{L^2}} \leq C \|Z(t)\|_{L^2} \quad \text{and} \quad \frac{1}{\|\nabla B(t)\|_{L^2}} \leq C \|\xi(t)\|_{H^2} \|\nabla \xi(t)\|_{H^2},
\]
we easily obtain that
\[
\frac{2}{\|\nabla u(t)\|_{L^2}} \leq C \|Z(t)\|_{L^2} \quad \text{and} \quad \frac{1}{\|\nabla B(t)\|_{L^2}} \leq C \|\xi(t)\|_{H^2} \|\nabla \xi(t)\|_{H^2}.
\]
By collecting (14) and (15), we get
\[
\frac{1}{2} \frac{d}{dt} \left( \|Z(t)\|_{L^2}^2 + \|\xi(t)\|_{H^2}^2 \right) + \frac{\mu}{2} \|\nabla \xi(t)\|_{H^2}^2 \leq C \|\xi(t)\|_{H^2} \|Z(t)\|_{L^2} + \|\xi(t)\|_{H^2}^2.
\]
The thesis follows by using Lemma 3.2 and the Gronwall lemma.

The next step is to estimate the \( L^\infty(0; T; L^2) \)-norm of \( \partial^\alpha Z \), where \( \alpha \) is a multi-index such that \( 1 \leq |\alpha| \leq 4 \). We get the following result.

**Lemma 3.4.** Let \( \varepsilon > 0 \). Then the following inequality
\[
\frac{1}{2} \frac{d}{dt} \|\partial^\alpha Z(t)\|_{L^2}^2 \leq C_\alpha(t) \|\partial^\alpha Z(t)\|_{L^2}^2 + \varepsilon \|\xi(t)\|_{H^2}^2,
\]
holds in \([0, T]\), where \( C_\alpha \) depends also on \( \varepsilon \).

**Proof.** By applying \( \partial^\alpha \) to both sides of equation (10), we get
\[
\partial^\alpha Z_t + (u \cdot \nabla) \partial^\alpha Z = -[\partial^\alpha, u \cdot \nabla] Z + \partial^\alpha((B \cdot \nabla) \xi),
\]
where \([\cdot, \cdot]\) denotes the commutator operator. We now multiply (16) by \( \partial^\alpha Z \) and we estimate term by term. We use the Hölder and Young inequalities and some suitable interpolation inequalities (obtained by the
well-known Gagliardo-Nirenberg one). More precisely,

\[ D^2 u \leq C \| \nabla u \| \ \| Z \| \]

\[ D^2 B \leq C \| \nabla B \| \ \| Z \| \]

\[ D^3 u \leq C \| \nabla u \| \ \| Z \| \]

\[ D^3 B \leq C \| \nabla B \| \ \| Z \| \]

\[ D^4 u \leq C \| \nabla u \| \ \| Z \| \]

\[ D^4 B \leq C \| \nabla B \| \ \| Z \| \]

By using (17)-(23), we easily obtain the thesis. ■

**Lemma 3.5.** The following estimate

\[
\frac{1}{2} \frac{d}{dt} \phi(t) + C_1 \psi(t) \leq C_2 (3(t) + \phi(t))
\]

holds in \([0, T]\).

**Proof.** We write (11) in the form \( \partial_t \xi - \mu A \xi = F \). For each integer \( k = 1, \ldots, 4 \), we take \((k-1)\) time derivatives and we obtain the following problems

\[
\begin{cases}
\partial_t^k \xi - \mu A \partial_t^{k-1} \xi = \partial_t^{k-1} F & \text{in } \Omega, \\
\partial_t^{k-1} \xi = 0 & \text{on } \partial \Omega.
\end{cases}
\]

For each fixed \( k \), we multiply the first equation of (24) by \( \partial_t^{k-1} \xi \) and by \( -\partial_t^{k-1} \xi \). By using the Hölder inequality one has

\[
\frac{1}{2} \frac{d}{dt} \| \partial_t^{k-1} \xi(t) \|_{L^2}^2 + \frac{\mu}{2} (\| \nabla \partial_t^{k-1} \xi(t) \|_{L^2}^2 + \| A \partial_t^{k-1} \xi(t) \|_{L^2}^2) \leq C (\| \partial_t^{k-1} F(t) \|_{L^2}^2 + \| \partial_t^{k-1} \xi(t) \|_{L^2}^2).
\]

We now write (24) in the form of the elliptic problem

\[
\begin{cases}
-\mu A \partial_t^{k-1} \xi = - \partial_t^k \xi + \partial_t^{k-1} F & \text{in } \Omega, \\
\partial_t^{k-1} \xi = 0 & \text{on } \partial \Omega.
\end{cases}
\]

By well-known results on the regularity of the solutions of problems
We now sum (25) for \( k = 1, \ldots, 4 \), and we add to both sides of the resulting expression the term
\[
\frac{\mu}{2} \left( \| \partial_t^k \xi(t) \|_{H^2}^2 + \sum_{k=0}^2 \| \partial_t^k \xi(t) \|_{H^{k-1}}^2 \right).
\]
By observing that \( \| \xi_{ttt}(t) \|_{H^2} \) is equivalent to \( \| \xi_{ttt}(t) \|_{H^2} + \| \Delta \xi_{ttt} \|_{L^2} \), we get
\[
\frac{1}{2} \frac{d}{dt} \phi(t) + C_1 \psi(t) \leq C_2 \left( \mathcal{R}(t) + \| \partial_t^3 \xi(t) \|_{L^2}^2 + \sum_{k=0}^2 \| \partial_t^k \xi(t) \|_{H^{k-1}}^2 \right).
\]
We use inequality (27) firstly for \( k = 1, 2, 3 \) and \( m = 4 - k \), and again in the cases \( k = 1, 2 \) and \( m = 1 \). By summing the resulting expressions we obtain the thesis.

We now estimate each term appearing in \( \mathcal{R}(t) \). The result we are going to show is

**Lemma 3.6.** The following inequality
\[
\mathcal{R}(t) \leq C_x(t) (\phi(t) + \| Z(t) \|_{H^1})
\]
holds in \([0, T]\).

**Proof.** We split the proof of the previous statement in several steps. As first step we write explicity \( \| F(t) \|_{H^2} \). By using the Hölder and Gagliardo-Nirenberg inequalities, we easily obtain
\[
\| F(t) \|_{H^2} \leq C_x(t) (\| \xi(t) \|_{H^1} + \| Z(t) \|_{H^1}),
\]
By using (27) in the following cases \((k, m) = (1, 2), (k, m) = (2, 0), \) and finally \((k, m) = (1, 0)\), one has
\[
\| \xi(t) \|_{H^1} \leq C \left( \sum_{k=0}^2 \| \partial_t^k \xi(t) \|_{L^2}^2 + \| F(t) \|_{H^2}^2 \right).
\]
By straightfull calculations, we get
\[
\| F(t) \|_{H^2} \leq C_x(t) (\| \xi(t) \|_{H^1} + \| \xi(t) \|_{H^1} + \| Z(t) \|_{H^1}),
\]
\[
\| F(t) \|_{H^2} \leq C_x(t) (\| \xi(t) \|_{H^1} + \| \xi(t) \|_{H^1} + \| Z(t) \|_{H^1}).
\]
Hence
\[ \|F(t)\|_{H^1} \leq C_x(t)(\phi(t) + \|Z(t)\|_{H^1}). \]

In order to estimate \( \|F(t)\|_{H^2}, \|F_u(t)\|_{H^1}, \) and \( \|F_{uu}(t)\|_{L^2}, \) we follow the same lines as in the previous step, and we consider the following interpolation inequalities

\[ \|D^2 B\|_{L^2} \leq C\|\nabla B\|_{L^4} \|\xi\|_{L^4}^4, \]
\[ \|D^2 u\|_{L^8} \leq C\|\nabla u\|_{L^4} \|Z\|_{L^4}^4, \]
\[ \|D^3 B\|_{L^2} \leq C\|B\|_{L^{\frac{36}{16}}} \|\xi\|_{L^{\frac{36}{16}}}^{\frac{36}{16}}, \]
\[ \|D^3 u\|_{L^8} \leq C\|u\|_{L^{\frac{36}{16}}} \|Z\|_{L^{\frac{36}{16}}}^{\frac{36}{16}}, \]
\[ \|D^4 B\|_{L^2} \leq C\|\nabla B\|_{L^4} \|\xi\|_{L^4}^{\frac{3}{2}}, \]
\[ \|D^4 B\|_{L^{\frac{36}{16}}} \leq C\|\nabla B\|_{L^4} \|\xi\|_{L^{\frac{36}{16}}}^{\frac{36}{16}}, \]
\[ \|D^5 B\|_{L^{\frac{36}{16}}} \leq C\|\nabla B\|_{L^4} \|\xi\|_{L^{\frac{36}{16}}}^{\frac{36}{16}}, \]
\[ \|D^5 B\|_{L^{\frac{36}{16}}} \leq C\|B\|_{L^{\frac{36}{16}}} \|\xi\|_{L^{\frac{36}{16}}}^{\frac{36}{16}}. \]

By virtue of the Hölder inequality, of (17)-(23) and of (28)-(35) we get

\[ \|F_u(t)\|_{H^1} \leq C_x(t)(\phi(t) + \|Z(t)\|_{H^1}) + \|D^2 B\|_{L^2} \|\xi\|_{L^4}^4. \]
\[ \|F_{uu}(t)\|_{L^2} \leq C_x(t)(\phi(t) + \|Z(t)\|_{H^1}) + \|D^3 B\|_{L^8} \|Z\|_{L^8}^4. \]

By (35), by recalling that \( B \in L^\infty(\Omega), \) and by using again (27), we get

\[ \|F_{uu}(t)\|_{H^1} \leq C_x(t)(\phi(t) + \|Z(t)\|_{H^1} + \|\xi(t)\|_{H^1}) \leq \]
\[ \leq C_x(t)(\phi(t) + \|Z(t)\|_{H^1} + \|\xi(t)\|_{H^1}^2 + \|\xi_{uu}(t)\|_{H^1}^2 + \|F_u(t)\|_{H^1}) \leq \]
\[ \leq C_x(t)(\phi(t) + \|Z(t)\|_{H^1}). \]

Hence, the claim follows. \[ \blacksquare \]

By collecting Lemmata 3.3-3.6 we obtain inequality (12).
4. – Proof of Theorem 2.3.

The first topic which we treat is a local existence result in Sobolev spaces for system (1)-(9). Obtained that the classical solution of (1)-(9) belongs locally to $H^3(\Omega)$, from the a-priori estimates (12) and (13) we can extend such a solution on the whole time interval $[0, T]$.

**Proof.** In order to show the local existence of a solution in Sobolev spaces, we apply the Banach-Caccioppoli theorem. In particular, let $0 < \hat{t} \leq T$ be sufficiently small and let

$$ S := \{(u, B) \in L^\infty((0, \hat{t}; H^3(\Omega)) : \|(u, B)\|_{L^\infty(0, \hat{t}; H^3)} \leq 2A \}, $$

where $A$ is a real positive constant such that $A > C_0(\|u_0\|_{H^3} + \|B_0\|_{H^3})$ for a suitable constant $C_0$, which will be fixed later.

Given the couple $(u, B)$ in $S$ and satisfying (3)-(9), let

$$ A : S \to A(S) $$

be the map defined by

$$ U := (u, B) \to \bar{U} := (\bar{u}, \bar{B}), $$

where $\bar{U} := (\bar{u}, \bar{B})$ is the solution of the following linear system

\begin{align*}
\bar{u}_t + (u \cdot \nabla) \bar{u} - (B \cdot \nabla) \bar{B} &= \frac{1}{2} \nabla |B|^2 \quad \text{in } Q_T, \\
\bar{B}_t + (u \cdot \nabla) \bar{B} - (B \cdot \nabla) \bar{u} - \mu A \bar{B} &= 0 \quad \text{in } Q_T, \\
\text{div } \bar{u} &= 0 \quad \text{in } Q_T, \\
\text{div } \bar{B} &= 0 \quad \text{in } Q_T, \\
\bar{u} \cdot v &= 0 \quad \text{on } \Gamma \times (0, T), \\
\bar{B} \cdot v &= 0 \quad \text{on } \Gamma \times (0, T), \\
\text{rot } \bar{B} &= 0 \quad \text{on } \Gamma \times (0, T), \\
\bar{u}(x, 0) &= u_0(x) \quad \text{in } \Omega, \\
\bar{B}(x, 0) &= B_0(x) \quad \text{in } \Omega.
\end{align*}
We now show that the map $L$ satisfies all the assumptions of the Banach-Caccioppoli theorem. We now apply to both sides of equation (36)-(37) $\text{rot}$ and $\overline{a}$, where $a$ is a multi-index with $N^a \leq 4$. We multiply the resulting expressions by the test functions $\overline{a} \overline{Z}$ and $\overline{a} \overline{B}$, where $\overline{Z} := \text{rot} \overline{u}$ and $\overline{B} := \text{rot} \overline{B}$. By suitable integrations by parts, and by using the Hölder and Young inequalities, an application of the Gronwall lemma yields that

$$
\max_{t \in [0, \hat{t}]} \| \overline{U}(t) \|_{H^1} \leq \left( C_0 \|U(0)\|_{H^1} + C \int_0^\hat{t} \|U(t)\|_{H^4} \, dt \right) e^{C(1 + \|U(0)\|_{H^4}^2)}
$$

where $\|U(0)\|_{H^1} = \|u_0\|_{H^1} + \|B_0\|_{H^1}$. Since $U$ belongs to $S$, we get

$$
\max_{t \in [0, \hat{t}]} \| \overline{U}(t) \|_{H^1} \leq (A + C \hat{t} (2A) \|U(0)\|_{H^4}^2) e^{C(1 + \|U(0)\|_{H^4}^2)}.
$$

Consequently, if $\hat{t}$ is small enough, $L$ maps the set $S$ into itself. We now show that $A$ is a contraction with respect to $L^\infty(0, \hat{t}; L^2)$-norm. Let $U_1 := (\overline{u}_1, \overline{B}_1)$ and $U_2 := (\overline{u}_2, \overline{B}_2)$ be solutions of system (36)-(44). We now consider the difference between equations (36), written for $i = 1, 2$, and equations (37), again written for $i = 1, 2$. We use as test functions $\overline{u}_1 - \overline{u}_2$ and $\overline{B}_1 - \overline{B}_2$ respectively. By standard arguments, we get

$$
\| \overline{U}_1 - \overline{U}_2 \|_{L^2(0, \hat{t}; L^2)} \leq C \hat{t} e^{A \hat{t}} \|U_1 - U_2\|_{L^2(0, \hat{t}; L^2)}.
$$

If $\hat{t}$ is sufficiently small, $A$ is a contraction and the unique fixed point of the map $A$ is a solution of system (1)-(9). The thesis follows by the uniqueness of the classical solution and by using (12) and (13). ■

REFERENCES


A global existence result in Sobolev spaces etc.


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