
EMANUELA CASELLA (*) - PAOLA TREBESCHI (*)

ABSTRACT - The main result of this paper is a global existence theorem in suitable Sobolev spaces for 2D incompressible MHD system in the half-plane. The existence result derives by the existence of a global classical solution in Hölder spaces, by proving some a-priori estimates in Sobolev spaces and, finally, by applying the Banach-Caccioppoli fixed point theorem. Hence, the uniqueness of the solution follows.

1. Introduction.

Let \( \Omega \) be the half-plane \( \mathbb{R}^2_+ := \{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0\} \), and let \( \Gamma \) be the boundary of \( \Omega \). In \( Q_T := \Omega \times (0, T) \), with \( T > 0 \), we consider the equations of magneto-hydrodynamics for 2D incompressible ideal fluid

\[
\begin{align*}
  u_t + (u \cdot \nabla) u + \nabla p + \frac{1}{2} \nabla |B|^2 - (B \cdot \nabla) B &= 0 \quad \text{in } Q_T, \\
  B_t + (u \cdot \nabla) B - (B \cdot \nabla) u - \mu \Delta B &= 0 \quad \text{in } Q_T, \\
  \text{div } u &= 0 \quad \text{in } Q_T, \\
  \text{div } B &= 0 \quad \text{in } Q_T,
\end{align*}
\]

(*) Indirizzo degli AA.: Dip. di Matematica, Università di Brescia, Facoltà di Ingegneria, Via Valotti 9, 25133 Brescia.
Here $u = u(x, t) = (u^1(x, t), u^2(x, t))$, $B = B(x, t) = (B^1(x, t), B^2(x, t))$ and $\pi = \pi(x, t)$ denote the unknown velocity field, the magnetic field and the pressure of the fluid respectively. The functions $u_0 = (u^1_0(x), u^2_0(x))$ and $B_0 = (B^1_0(x), B^2_0(x))$ denote the given initial data, $n$ the unit outward normal on $G$ and $\mu$ a real positive constant. Moreover, we use the notation

$$f_t = \frac{\partial f}{\partial t}, \quad \partial_i = \frac{\partial}{\partial x_i}, \quad \nabla = (\partial_1, \partial_2), \quad u \cdot \nabla = u^1 \partial_1 + u^2 \partial_2,$$

$$\partial^2_{ij} = \frac{\partial^2}{\partial_i \partial_j}, \quad \Delta = \partial^2_{11} + \partial^2_{22}.$$
(9) in Hölder-spaces. We underline that energy-method works well, since the classical solution \((u, B)\) is such that \(\|u(t)\|_{L^\infty(\Omega)}, \|B(t)\|_{L^\infty(\Omega)}, \|\nabla u(t)\|_{L^\infty(\Omega)}, \|\nabla B(t)\|_{L^\infty(\Omega)}\) and \(\|B_t(t)\|_{L^\infty(\Omega)}\) are uniformly bounded in time on the whole interval \([0, T]\).

We observe that the main result obtained in the present paper is a necessary first step in the analysis of slightly compressible MHD fluids, which will be the object of a forthcoming work.

The plan of the paper is the following. In next section we fix some notations and we introduce some preliminary results and the main theorem. In Section 3 we show some a-priori estimates, and finally in Section 4 we prove the main result.

---

2. – Notations and results.

For a scalar-valued function \(\phi\), we set

\[
\text{Rot} \phi = (\partial_2 \phi, -\partial_1 \phi),
\]

for a vector-valued function \(u = (u^1, u^2)\), we use the notation

\[
\text{rot} u = \partial_1 u^2 - \partial_2 u^1 \quad \text{and} \quad \text{div} u = \nabla \cdot u = \partial_1 u^1 + \partial_2 u^2.
\]

We denote the norm of \(L^p(\Omega)\), \(1 \leq p \leq \infty\), by \(\|\cdot\|_{L^p}\). \(H^m(\Omega)\) denotes the usual Sobolev space of order \(m \geq 1\), and \(\|\cdot\|_{H^m}\) denotes its norm. For simplicity we use the abbreviated notation \(L^p, H^m\). We also use the same symbol for spaces of scalar and vector-valued functions.

Moreover, if \(X\) is a normed space, then \(L^p(0, T; X)\), with \(1 \leq p < +\infty\), denotes the set of all measurable functions \(u(t)\) with values in \(X\) such that:

\[
\|u\|_{L^p(0, T; X)} := \left( \int_0^T \|u(t)\|_X^p \, dt \right)^{1/p} < +\infty,
\]

where \(\|\cdot\|_X\) is the norm in \(X\).

Given \(T > 0\) arbitrary, the set of all essentially bounded (with respect to the norm of \(X\)) measurable functions of \(t\) with values in \(X\) is denoted by \(L^\infty(0, T; X)\). We equip this space with the usual norm

\[
\|f\|_{L^\infty(0, T; X)} = \sup_{t \in [0, T]} \|f(t)\|_X.
\]

In particular, the norm of \(L^p(0, T; L^p)\), \(1 \leq p < +\infty\), is denoted by \(\|\cdot\|_{L^p(0, T; L^p)}\).
Let \( C^m([0, T]; X) \) denote the set of all \( X \)-valued \( m \)-times continuously differentiable functions of \( t \), for \( 0 \leq t \leq T \).

We define \( X^m(T) := \bigcap_{k=0}^{m-1} C^k([0, T]; H^{m-k}) \) equipped with the usual norm

\[
\|u\|_{X^m} := \sup_{[0,T]} \sum_{k=0}^{m-1} \|D^k u(t)\|_{H^{m-k}}.
\]

We denote by \( \beta(\Omega) \) (resp. \( \beta(\mathcal{Q}_T) \)) the Banach space of all real valued continuous and bounded functions on \( \Omega \) (resp. \( \mathcal{Q}_T \)), with the usual norm.

For \( 0 < \alpha < 1 \), \( C^\alpha(\Omega) \) denotes the usual space of functions in \( \beta(\Omega) \), uniformly Hölder continuous on \( \Omega \) with exponent \( \alpha \); the norm of \( C^\alpha(\Omega) \) is \( \|\cdot\|_{L^\alpha} \) where

\[
[\phi]_{\alpha} := \sup_{x \neq y, x, y \in \Omega} \frac{|\phi(x) - \phi(y)|}{|x - y|^\alpha}.
\]

For \( 0 < \alpha < 1 \) and integer \( k \), \( C^{k+a}(\Omega) \) denotes the space of functions \( \phi \) with \( D^\beta \phi \in \beta(\Omega) \) for \( |\beta| \leq k \), and \( D^\gamma \phi \in C^\alpha(\Omega) \) for \( |\gamma| = k \). The norm is

\[
[\phi]_{k+a} = \max_{|\beta| \leq k} \|D^\beta \phi\|_{L^\alpha} + \max_{|\gamma| = k} \|D^\gamma \phi\|_{L^\alpha}.
\]

With \( C^{k,j}(\mathcal{Q}_T) \) for integers \( k, j \geq 0 \) we mean the set of all functions \( \phi \) for which every \( \partial_x^k \partial_t^j \phi \) exists and is continuous on \( \mathcal{Q}_T \), for \( 0 \leq q \leq k \), \( 0 \leq r \leq j \), \( C^{k+a,j+\beta}(\mathcal{Q}_T) \), for integers \( k, j \geq 0 \) and \( 0 \leq \alpha, \beta < 1 \) is the subset of \( C^{k,j}(\mathcal{Q}_T) \), consisting of Hölder continuous functions with exponents \( \alpha \) in \( x \) and \( \beta \) in \( t \).

For every function \( \phi \in C^{k+a,j+\beta}(\mathcal{Q}_T) \), we consider the following seminorm:

\[
[\phi]_{k+a,j+\beta} := \sup_{x \neq y, t \in [0, T]} \frac{|\phi(x, t) - \phi(y, t)|}{|x - y|^\alpha} + \sup_{t \neq s, x \in \Omega} \frac{|\phi(x, t) - \phi(x, s)|}{|t - s|^\beta},
\]

and the norm

\[
[\phi]_{k+a,j+\beta} := \max_{|\beta| \leq k, |\gamma| \leq j} \|\partial_x^\beta \partial_t^\gamma \phi\|_{L^\alpha} + \max_{|\beta| \leq k} \|\partial_x^\beta \partial_t^\gamma \phi\|_{L^\alpha}.
\]

We shall denote by \( C \) and by \( C_i, i \in \mathbb{N} \), some real positive constants which may be different in each occurrence, and by \( C_{\nu}(t) \) a real function in \( L^\infty(0, T) \) depending on \( \|u(t)\|_{L^\infty}, \|B(t)\|_{L^\infty}, \|\nabla u(t)\|_{L^\infty}, \|\nabla B(t)\|_{L^\infty}, \|B_i(t)\|_{L^\infty} \) and some their suitable powers.
We now set $Z := \text{rot } u$, $\xi := \text{rot } B$, $Z_0 := \text{rot } u_0$ and $\xi_0 := \text{rot } B_0$. By applying rot to both sides of equations (1) and (2), we get

$$Z_t + u \cdot \nabla Z - B \cdot \nabla \xi = 0,$$

$$\xi_t + u \cdot \nabla \xi - B \cdot \nabla Z + 2 \partial_1 u^1 \nabla B + 2 \partial_2 B^2 \nabla u - \mu \Delta \xi = 0,$$

where $\nabla u = \partial_1 u^2 + \partial_2 u^1$, and $\nabla B = \partial_1 B^2 + \partial_2 B^1$. Finally, let $F$, $\phi$, $\psi$ be defined as

$$F = -u \cdot \nabla \xi + B \cdot \nabla Z - 2 \partial_1 u^1 \nabla B - 2 \partial_2 B^2 \nabla u,$$

$$\phi(s) := \sum_{k=0}^{3} \| \partial_t^k \xi(s) \|_{L^1},$$

$$\psi(s) := \sum_{k=0}^{3} \| \partial_t^k F(s) \|_{H^{1-k}}.$$
THEOREM 2.3. Let $T > 0$ be arbitrary. Let the couple $(u_0, B_0) \in H^2 \cap L^1$, rot $u_0 \in L^1$, div $u_0 = \text{div} B_0 = 0$ in $\Omega$ and $u_0 \cdot \nu = B_0 \cdot \nu = 0$ on $\Gamma$. Assume also that, for some $0 < \theta < 1$, 
\[
\|B_0\|_{L^1} + |B_0|_{L^1} + \theta \leq C_\theta,
\]
where $C_\theta$ is the constant obtained in Theorem 2.1.

Then problem (1)-(9) has a unique solution $(u, B, \pi)$ such that

$u \in X^5(T), \quad B \in X^5(T) \cap L^2(0, T; H^1), \quad \nabla \pi \in X^4(T)$.

3. Some a-priori estimates.

We devote this section to prove

**Lemma 3.1.** The following energy-type estimate

(12) \[
\frac{1}{2} \frac{d}{dt} (\phi(t) + \|Z(t)\|^2_{H^1}) + C_1 \psi(t) \leq C_\theta (t) (\phi(t) + \|Z(t)\|^2_{H^1})
\]
holds in $[0, T]$.

Note that Theorem 2.1 ensures us that the time functions $\|u(t)\|_{L^\infty(\Omega)}$, $\|B(t)\|_{L^\infty(\Omega)}$, $\|\nabla u(t)\|_{L^\infty(\Omega)}$, $\|\nabla B(t)\|_{L^\infty(\Omega)}$, and $\|B_t(t)\|_{L^\infty(\Omega)}$ are uniformly bounded in time on the whole interval $[0, T]$. Consequently, the real function $C_\theta(t)$ (appearing in (12) and in some preliminary lemmata given below) belongs to $L^\infty(0, T)$. We shall prove (12) for regular solutions.

**Lemma 3.2.** The couple $(u, B)$ satisfies the following energy-type estimate

(13) \[
\|u\|^2_{L^\infty(\Omega)} + \|B\|^2_{L^\infty(\Omega)} + 2 \mu \|\nabla Z\|^2_{L^2(Q_T)} \leq \|u_0\|^2_{L^2} + \|B_0\|^2_{L^2}.
\]

**Proof.** We multiply equations (1) and (2) by $u$ and $B$ respectively. By standard calculations and by summing the resulting expressions, we get easily the thesis. $\blacksquare$

**Lemma 3.3.** The following inequality holds

\[
\|Z\|^2_{L^\infty(\Omega)} + \|Z\|^2_{L^\infty(\Omega)} + \frac{1}{2} \|\nabla Z\|^2_{L^2(Q_T)} \leq C(\|Z_0\|^2_{L^2} + \|Z_0\|^2_{L^2}).
\]
PROOF. We multiply equations (10) and (11) by $Z$ and $\xi$ respectively. Since

$$2 \int (B \cdot \nabla) \xi \cdot Z \, dx = \int (B \cdot \nabla) Z \cdot \xi \, dx,$$

by summing the resulting expressions, we obtain

$$\frac{1}{2} \frac{d}{dt} \left( \|Z(t)\|_{L^2}^2 + \|\xi(t)\|_{L^2}^2 \right) + 2 \int (\partial_t^1 u_1 DB + \partial_2 B_2 Du) \xi \, dx +$$

$$+ \mu \|\nabla \xi(t)\|_{L^2}^2 = 0.$$

Since $\|\nabla u(t)\|_{L^2} \leq C \|Z(t)\|_{L^2}$ and $\|\nabla B(t)\|_{L^2} \leq C \|\xi(t)\|_{H^{1/2}} \|\nabla \xi(t)\|_{H^{1/2}}$, we easily obtain that

$$(15) \quad 2 \int (\partial_t^1 u_1 DB + \partial_2 B_2 Du) \xi \, dx \leq \frac{\mu}{2} \|\nabla \xi(t)\|_{L^2}^2 + C \|Z(t)\|_{L^2}^2 \|\xi(t)\|_{H^{1/2}}^2.$$

By collecting (14) and (15), we get

$$\frac{1}{2} \frac{d}{dt} \left( \|Z(t)\|_{L^2}^2 + \|\xi(t)\|_{L^2}^2 \right) + \frac{\mu}{2} \|\nabla \xi(t)\|_{L^2}^2 \leq C \|\xi(t)\|_{H^{1/2}} \|Z(t)\|_{L^2}^2 + \|\xi(t)\|_{H^{1/2}}^2.$$

The thesis follows by using Lemma 3.2 and the Gronwall lemma.

The next step is to estimate the $L^\infty(0; T; L^2)$-norm of $\partial^\alpha Z$, where $\alpha$ is a multi-index such that $1 \leq |\alpha| \leq 4$. We get the following result.

**Lemma 3.4.** Let $\epsilon > 0$. Then the following inequality

$$\frac{1}{2} \frac{d}{dt} \|\partial^\alpha Z(t)\|_{L^2}^2 \leq C_\alpha(t) \|\partial^\alpha Z(t)\|_{L^2}^2 + \epsilon \|\xi(t)\|_{H^1}^2,$$

holds in $[0, T]$, where $C_\alpha$ depends also on $\epsilon$.

**Proof.** By applying $\partial^\alpha$ to both sides of equation (10), we get

$$\partial^\alpha Z_t + (u \cdot \nabla) \partial^\alpha Z = -[\partial^\alpha, u \cdot \nabla] Z + \partial^\alpha((B \cdot \nabla) \xi),$$

where $[\cdot, \cdot]$ denotes the commutator operator. We now multiply (16) by $\partial^\alpha Z$ and we estimate term by term. We use the Hölder and Young inequalities and some suitable interpolation inequalities (obtained by the
well-known Gagliardo-Nirenberg one). More precisely,

\[ \|D^2 u\|_{L^2} \leq C\|\nabla u\| L^2 \| Z\| L^2, \]

(17)

\[ \|D^2 B\|_{L^2} \leq C\|\nabla B\| L^2 \| \xi\| L^2, \]

(18)

\[ \|D^3 u\|_{L^2} \leq C\|\nabla u\| L^2 \| Z\| L^2, \]

(19)

\[ \|D^3 B\|_{L^2} \leq C\|\nabla B\| L^2 \| \xi\| L^2, \]

(20)

\[ \|D^4 u\|_{L^2} \leq C\|\nabla u\| L^2 \| Z\| L^2, \]

(21)

\[ \|D^4 B\|_{L^2} \leq C\|\nabla B\| L^2 \| \xi\| L^2, \]

(22)

\[ \|D^2 B_t\|_{L^2} \leq C\|B_t\| L^2 \| \xi\| L^2. \]

(23)

By using (17)-(23), we easily obtain the thesis. □

Lemma 3.5. The following estimate

\[ \frac{1}{2} \frac{d}{dt} \phi(t) + C_1 \psi(t) \leq C_2 (\bar{\partial}(t) + \phi(t)) \]

holds in \([0, T]\).

Proof. We write (11) in the form \( \partial_t \xi - \mu A \xi = F \). For each integer \( k = 1, \ldots, 4 \), we take \( (k-1) \) time derivatives and we obtain the following problems

(24)

\[
\begin{align*}
\partial_t^k \xi - \mu A \partial_t^{k-1} \xi &= \partial_t^{k-1} F \text{ in } \Omega, \\
\partial_t^{k-1} \xi &= 0 \text{ on } \partial \Omega. 
\end{align*}
\]

For each fixed \( k \), we multiply the first equation of (24) by \( \partial_t^{k-1} \xi \) and by \( -A \partial_t^{k-1} \xi \). By using the Hölder inequality one has

(25)

\[
\frac{1}{2} \frac{d}{dt} \| \partial_t^{k-1} \xi(t) \|_{H^1}^2 + \frac{\mu}{2} (\| \nabla \partial_t^{k-1} \xi(t) \|_{L^2}^2 + \| A \partial_t^{k-1} \xi(t) \|_{L^2}^2) \leq C(\| \partial_t^{k-1} F(t) \|_{L^2}^2 + \| \partial_t^{k-1} \xi(t) \|_{L^2}^2). 
\]

We now write (24) in the form of the elliptic problem

(26)

\[
\begin{align*}
- \mu A \partial_t^{k-1} \xi &= - \partial_t^{k} \xi + \partial_t^{k-1} F \text{ in } \Omega, \\
\partial_t^{k-1} \xi &= 0 \text{ on } \partial \Omega. 
\end{align*}
\]

By well-known results on the regularity of the solutions of problems
(26), we get
\[ \| \partial_t \xi(t) \|_{H^{m-1}} \leq C(\| \partial_t \xi(t) \|_{H^m} + \| \partial_t \xi(t) \|_{H^m} + \| \partial_t F(t) \|_{H^m}). \]

We now sum (25) for \( k = 1, \ldots, 4 \), and we add to both sides of the resulting expression the term
\[ \frac{\mu}{2} \left( \| \partial_t \xi(t) \|_{L^2}^2 + \sum_{k=0}^{2} \| \partial_t \xi(t) \|_{H^{m-1}}^2 \right). \]

By observing that \( \sum_{k=0}^{2} \| \partial_t \xi(t) \|_{H^{m-1}}^2 \) is equivalent to \( \sum_{k=0}^{2} \| \partial_t \xi(t) \|_{L^2}^2 \), we get
\[ \frac{1}{2} \frac{d}{dt} \phi(t) + C_1 \psi(t) \leq C_2 \left( \| \partial_t \xi(t) \|_{L^2}^2 + \sum_{k=0}^{2} \| \partial_t \xi(t) \|_{H^{m-1}}^2 \right). \]

We use inequality (27) firstly for \( k = 1, 2, 3 \) and \( m = 4 - k \), and again in the cases \( k = 1, 2 \) and \( m = 1 \). By summing the resulting expressions we obtain the thesis.

We now estimate each term appearing in \( \mathcal{R}(t) \). The result we are going to show is

**Lemma 3.6.** The following inequality
\[ \mathcal{R}(t) \leq C_\infty(t)(\phi(t) + \| Z(t) \|_{H^1}). \]

holds in \([0, T]\).

**Proof.** We split the proof of the previous statement in several steps. As first step we write explicity \( \| F(t) \|_{H^1} \). By using the Hölder and Gagliardo-Nirenberg inequalities, we easily obtain
\[ \| F(t) \|_{H^1} \leq C_\infty(t)(\| \xi(t) \|_{H^1} + \| Z(t) \|_{H^1}). \]

By using (27) in the following cases \( (k, m) = (1, 2), (k, m) = (2, 0) \), and finally \( (k, m) = (1, 0) \), one has
\[ \| \xi(t) \|_{H^1} \leq C \left( \sum_{k=0}^{2} \| \partial_t \xi(t) \|_{L^2}^2 + \| F(t) \|_{H^1}^2 \right). \]

By straightforward calculations, we get
\[ \| F(t) \|_{H^1} \leq C_\infty(t)(\| \xi(t) \|_{H^1} + \| Z(t) \|_{H^1}), \]
\[ \| F(t) \|_{H^1} \leq C_\infty(t)(\| \xi(t) \|_{H^1} + \| Z(t) \|_{H^1}). \]
Hence

\[ \|F(t)\|_{L^2} \leq C_\infty(t)(\phi(t) + \|Z(t)\|_{L^4}). \]

In order to estimate \( \|F'(t)\|_{L^2} \), \( \|F''(t)\|_{L^1} \), and \( \|F'''(t)\|_{L^2} \), we follow the same lines as in the previous step, and we consider the following interpolation inequalities

\begin{align*}
\|D^2 B\|_{L^8} &\leq C\|\nabla B\|^{1/4}_2 \|\xi\|^{1/4}_2, \\
\|D^2 u\|_{L^8} &\leq C\|\nabla u\|^{1/4}_2 \|Z\|^{1/4}_2, \\
\|D^3 B\|_{L^8} &\leq C\|B\|^{1/2}_2 \|\xi\|^{1/2}_2, \\
\|D^3 u\|_{L^8} &\leq C\|u\|^{1/2}_2 \|Z\|^{1/2}_2, \\
\|D^4 B\|_{L^8} &\leq C\|\nabla B\|^{1/4}_2 \|\xi\|^{1/4}_2, \\
\|D^4 u\|_{L^8} &\leq C\|\nabla u\|^{1/4}_2 \|\xi\|^{1/4}_2.
\end{align*}

By virtue of the Hölder inequality, of (17)-(23) and of (28)-(35) we get

\begin{align*}
\|F'\|_{L^2} &\leq C_\infty(t)(\phi(t) + \|Z(t)\|_{L^4}), \\
\|F''\|_{L^1} &\leq C_\infty(t)(\phi(t) + \|Z(t)\|_{L^4}) + \|D^3 B_1 DZ\|_{L^2}. \\

\text{By (35), by recalling that } B_1 \in L^\infty(\Omega), \text{ and by using again (27), we get}
\end{align*}

\begin{align*}
\|F'\|_{L^2} &\leq C_\infty(t)(\phi(t) + \|Z(t)\|_{L^4} + \|\xi_1(t)\|_{L^4}) \\
&\leq C_\infty(t)(\phi(t) + \|Z(t)\|_{L^4} + \|\xi_1(t)\|_{L^4} + \|\xi_2(t)\|_{L^4} + \|F_1(t)\|_{L^4}) \\
&\leq C_\infty(t)(\phi(t) + \|Z(t)\|_{L^4}).
\end{align*}

Hence, the claim follows. 

By collecting Lemmata 3.3-3.6 we obtain inequality (12).
4. Proof of Theorem 2.3.

The first topic which we treat is a local existence result in Sobolev spaces for system (1)-(9). Obtained that the classical solution of (1)-(9) belongs locally to $H^5(\Omega)$, from the a-priori estimates (12) and (13) we can extend such a solution on the whole time interval $[0, T]$.

PROOF. In order to show the local existence of a solution in Sobolev spaces, we apply the Banach-Caccioppoli theorem. In particular, let $0 < \delta T$ be sufficiently small and let

$$S := \{(u, B) \in L^\infty(0, \delta T; H^5(\Omega)) : \|(u, B)\|_{L^\infty(0, \delta T; H^5)} \leq 2A\},$$

where $A$ is a real positive constant such that $A > C_0(\|u_0\|_{H^5} + \|B_0\|_{H^5})$ for a suitable constant $C_0$, which will be fixed later.

Given the couple $(u, B)$ in $S$ and satisfying (3)-(9), let

$$A : S \rightarrow A(S)$$

be the map defined by

$$U := (u, B) \mapsto \bar{U} := (\bar{u}, \bar{B}),$$

where $\bar{U} := (\bar{u}, \bar{B})$ is the solution of the following linear system

\begin{align*}
\bar{u}_t + (u \cdot \nabla) \bar{u} - (B \cdot \nabla) \bar{B} &= \frac{1}{2} \nabla |B|^2 \quad \text{in } QT, \\
\bar{B}_t + (u \cdot \nabla) \bar{B} - (B \cdot \nabla) \bar{u} - \mu A \bar{B} &= 0 \quad \text{in } QT, \\
\text{div } \bar{u} &= 0 \quad \text{in } QT, \\
\text{div } \bar{B} &= 0 \quad \text{in } QT, \\
\bar{u} \cdot \nu &= 0 \quad \text{on } \Gamma \times (0, T), \\
\bar{B} \cdot \nu &= 0 \quad \text{on } \Gamma \times (0, T), \\
\text{rot } \bar{B} &= 0 \quad \text{on } \Gamma \times (0, T), \\
\bar{u}(x, 0) &= u_0(x) \quad \text{in } \Omega, \\
\bar{B}(x, 0) &= B_0(x) \quad \text{in } \Omega.
\end{align*}
We now show that the map $A$ satisfies all the assumptions of the Banach-Caccioppoli theorem. We now apply to both sides of equation (36)-(37) $\text{rot}$ and $\bar{a}$, where $\bar{a}$ is a multi-index with $N_{\bar{a}} \leq N$. By suitable integrations by parts, and by using the Hölder and Young inequalities, an application of the Gronwall lemma yields that

$$
\max_{t \in [0, \tilde{t}]} \| \bar{U}(t) \|_{H^1} \leq C_{0} \| U(0) \|_{H^1} + C \int_{0}^{\tilde{t}} \| U(t) \|_{H^1} \, dt \, e^{c(1 + \| U(t) \|_{H^1}^2) \, dt},
$$

where $\| U(0) \|_{H^1} = \| u_0 \|_{H^1} + \| B_0 \|_{H^1}$. Since $U$ belongs to $S$, we get

$$
\max_{t \in [0, \tilde{t}]} \| \bar{U}(t) \|_{H^1} \leq (A + C\tilde{t}(2A)^4) \, e^{C(1 + 2A)^2}.
$$

Consequently, if $\tilde{t}$ is small enough, $A$ maps the set $S$ into itself. We now show that $A$ is a contraction with respect to $L^\infty(0, \tilde{t}; L^2)$-norm. Let $\bar{U}_1 := (\bar{u}_1, \bar{B}_1)$ and $\bar{U}_2 := (\bar{u}_2, \bar{B}_2)$ be solutions of system (36)-(44). We now consider the difference between equations (36), written for $i = 1, 2$, and equations (37), again written for $i = 1, 2$. We use as test functions $\bar{u}_1 - \bar{u}_2$ and $\bar{B}_1 - \bar{B}_2$ respectively. By standard arguments, we get

$$
\| \bar{U}_1 - \bar{U}_2 \|_{L^\infty(0, \tilde{t}; L^2)} \leq C\tilde{t} e^{4A\tilde{t}} \| U_1 - U_2 \|_{L^\infty(0, \tilde{t}; L^2)}.
$$

If $\tilde{t}$ is sufficiently small, $A$ is a contraction and the unique fixed point of the map $A$ is a solution of system (1)-(9). The thesis follows by the uniqueness of the classical solution and by using (12) and (13).

REFERENCES


A global existence result in Sobolev spaces etc. 91


Manoscritto pervenuto in redazione il 17 aprile 2002.