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A Sufficient Condition for the Convexity of the Area of an Isoptic Curve of an Oval.

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ABSTRACT - The problem of convexity of the area function of an α-isoptic of the convex curve C is considered.

1. Introduction.

An α-isoptic C_α of a plane, closed, convex curve C is a set of those points in the plane from which the curve C is seen under a fixed angle $\pi - \alpha$. In [1] the authors discussed the problem of convexity of α-isoptic curves. We will study the problem of convexity of the area function.

Let $p(t)$, $t \in [0, 2\pi]$ be a support function of the curve C. It is known [1], [2] that the area $A(\alpha)$ of α-isoptic of C is given by formula

$$(1.1)\quad A(\alpha) = \int_0^{2\pi} \frac{1}{\sin^2 \alpha} [p^2(t) - p(t+\alpha)(p(t)\cos\alpha + \dot{p}(t)\sin\alpha)]\, dt\,.$$

Let $p(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt)$ be the Fourier series of the support function. We will assume that $p(t) \in C^2[0, 2\pi]$ and $p(t), \dot{p}(t) \in L^2[0, 2\pi]$. Moreover the curvature radius $R(t) = p(t) + \ddot{p}(t) > 0$ and the Fourier series of $R(t) = a_0 + \sum_{n=1}^{\infty} (1 - n^2)(a_n \cos nt + b_n \sin nt)$ is convergent.

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Let $f(x) \sim \sum_{\nu=-\infty}^{\infty} c_\nu e^{i\nu x}$, $g(x) \sim \sum_{\nu=-\infty}^{\infty} d_\nu e^{i\nu x} \in L^2$ where $c_{-\nu} = \overline{c_\nu}$ are complex numbers then we have ([5], p. 37, theorem 1.12)

THEOREM 1.1. *Suppose that f and g are in L^2 and have coefficients c_ν and d_ν respectively. Then*

$$\frac{1}{2\pi} \int_0^{2\pi} f(x-t)\, g(t)\, dt = \sum_{\nu=-\infty}^{\infty} c_\nu d_\nu e^{i\nu x} \tag{1.2}$$

for all x, and the series on the right converges absolutely and uniformly. In particular

$$\frac{1}{2\pi} \int_0^{2\pi} f(x+t)\, \overline{f(t)}\, dt = \sum_{\nu=-\infty}^{\infty} |c_\nu|^2 e^{i\nu x}, \tag{1.3}$$

$$\frac{1}{2\pi} \int_0^{2\pi} f(t)\, g(t)\, dt = \sum_{\nu=-\infty}^{\infty} c_\nu d_{-\nu}, \tag{1.4}$$

$$\frac{1}{2\pi} \int_0^{2\pi} |f(t)|^2\, dt = \sum_{\nu=-\infty}^{\infty} |c_\nu|^2. \tag{1.5}$$

If $f(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt)$ is a real valued function and $g = f'$ then we have for all $x \in \mathbb{R}$

$$\int_0^{2\pi} f^2(t)\, dt = 2\pi a_0^2 + \pi \sum_{n=1}^{\infty} (a_n^2 + b_n^2), \tag{1.6}$$

$$\int_0^{2\pi} f(t+x)\, f(t)\, dt = 2\pi a_0^2 + \pi \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \cos nx, \tag{1.7}$$

$$\int_0^{2\pi} f(t+x)\, f'(t)\, dt = \pi \sum_{n=1}^{\infty} n(a_n^2 + b_n^2) \sin nx \tag{1.8}$$

which, with formula (1.1), give the following formula for the area of the α-isoptic

(1.9) $A(\alpha) =$

$$= \frac{1}{\sin^2 \alpha} \Big[2\pi a_0^2 + \pi \sum_{n=1}^{\infty} (a_n^2 + b_n^2) - \cos\alpha \Big(2\pi a_0^2 + \pi \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \cos n\alpha \Big) -$$

$$-\sin\alpha\cdot\pi\sum_{n=1}^{\infty} n(a_n^2+b_n^2)\sin n\alpha\Big] =$$

$$=\frac{\pi}{\sin^2\alpha}\left[2a_0^2(1-\cos\alpha)+\sum_{n=1}^{\infty}(a_n^2+b_n^2)(1-\cos n\alpha\cos\alpha-n\sin\alpha\sin n\alpha)\right]=$$

$$=\frac{\pi}{\sin^2\alpha}\left[2a_0^2(1-\cos\alpha)+\sum_{n=0}^{\infty}(a_{n+1}^2+b_{n+1}^2)\left(1+\frac{n}{2}\cos(n+2)\alpha-\frac{n+2}{2}\cos n\alpha\right)\right].$$

2. The main result.

THEOREM 2.1. *Let* $p(t)=a_0+\sum_{n=1}^{\infty}(a_n\cos nt+b_n\sin nt)$ *be the Fourier series of the support function of* C. *If Fourier coefficients of* $p(t)$ *are such that the inequality holds*

$$a_0^2 \geqslant \sum_{n=0}^{\infty}(n(n+2))^2(a_{n+1}^2+b_{n+1}^2) \tag{2.1}$$

then $A(\alpha)$ *is a convex function.*

PROOF. We will show that $A''(t)\geqslant 0$. From the above considerations we get

$$\frac{A'(\alpha)}{\pi}=2a_0^2\,\frac{(1-\cos\alpha)^2}{\sin^3\alpha}+$$

$$+\sum_{n=0}^{\infty}(a_{n+1}^2+b_{n+1}^2)\frac{1}{\sin^3\alpha}\left[\frac{n(n+2)}{2}\sin\alpha\,(\sin n\alpha-\sin(n+2)\alpha)-\right.$$

$$\left.-2\cos\alpha\left(1+\frac{n}{2}\cos(n+2)\alpha-\frac{n+2}{2}\cos n\alpha\right)\right]$$

and by some calculations

$$A''(\alpha)=\frac{\pi}{\sin^4\alpha}\Big\{2a_0^2(1-\cos\alpha)^2(2-\cos\alpha)+ \tag{2.2}$$

$$+\sum_{n=0}^{\infty}(a_{n+1}^2+b_{n+1}^2)[n^3\sin^3\alpha\sin(n+1)\alpha+3n^2\sin^2\alpha\cos n\alpha-$$

$$-6n\sin\alpha\cos\alpha\sin n\alpha+2(1+2\cos^2\alpha)(1-\cos n\alpha)]\Big\}.$$

If the condition (2.1) holds, then

$$\frac{A''(\alpha)}{\pi} \geqslant \frac{1}{\sin^4 \alpha} \sum_{n=0}^{\infty} (a_{n+1}^2 + b_{n+1}^2)\{2n^2(n+2)^2(1-\cos\alpha)^2(2-\cos\alpha) + $$
$$+ n^3 \sin^3\alpha \sin(n+1)\alpha + 3n^2 \sin^2\alpha \cos n\alpha - 6n \sin\alpha \cos\alpha \sin n\alpha +$$
$$+ 2(1 + 2\cos^2\alpha)(1-\cos n\alpha)\}$$
$$= \frac{1}{\sin^4\alpha} \sum_{n=0}^{\infty} (a_{n+1}^2 + b_{n+1}^2)\, M_n(\alpha), \text{ where}$$

$$M_n(\alpha) = 2n^2(n+2)^2(1-\cos\alpha)^2(2-\cos\alpha) + n^3 \sin^3\alpha \sin(n+1)\alpha + \\ + 3n^2 \sin^2\alpha \cos n\alpha - 6n \sin\alpha\cos\alpha \sin n\alpha + 2(1+2\cos^2\alpha)(1-\cos n\alpha) \tag{2.3}$$

for abbreviation. One can see that $M_n(0) = 0$ for $n \in \mathbb{N}$ and $M_0(\alpha) = 0$. We will show that $M_n'(\alpha) \geqslant 0$. We have

$$M_n'(\alpha) = 2(n^4 + 4n^3 + 4n^2)\sin\alpha(1-\cos\alpha)(5-3\cos\alpha) +$$
$$+ n^4 \sin^3\alpha \cos(n+1)\alpha + 4n^3 \sin^3\alpha\cos(n+1)\alpha +$$
$$+ 8n \sin^2\alpha \sin n\alpha - 8\sin\alpha\cos\alpha(1-\cos n\alpha).$$

If $\alpha \in [0, \pi)$ then $\sin\alpha \geqslant 0$ and $|\sin n\alpha| \leqslant n \sin\alpha$ and $n^4 + 4n^3 \geqslant 5n^2$ for any positive integer n. Moreover, we have

$$M_n'(\alpha) \geqslant n^4 \sin\alpha(1-\cos\alpha)[2(5-3\cos\alpha) - (1+\cos\alpha)] +$$
$$+ 4n^3 \sin\alpha(1-\cos\alpha)[2(5-3\cos\alpha) - (1+\cos\alpha)] +$$
$$+ 8n^2 \sin\alpha(1-\cos\alpha)(5-3\cos\alpha) -$$
$$- 8n^2\sin^3\alpha - 8\sin\alpha\cos\alpha\cdot 2\sin^2 n\frac{\alpha}{2} \geqslant$$
$$\geqslant (n^4 + 4n^3)\sin\alpha(1-\cos\alpha)[9-7\cos\alpha] +$$
$$+ 8n^2 \sin\alpha(1-\cos\alpha)[5 - 3\cos\alpha - 1 - \cos\alpha] -$$
$$- 16n^2 \sin\alpha\cos\alpha \sin^2\frac{\alpha}{2}$$
$$\geqslant 5n^2\cdot 2\sin\alpha\sin^2\frac{\alpha}{2}(9-7\cos\alpha) - 16n^2\sin\alpha\cos\alpha\sin^2\frac{\alpha}{2} =$$
$$= 2n^2\sin\alpha\sin^2\frac{\alpha}{2}[45 - 35\cos\alpha - 8\cos\alpha] =$$

$$= 2n^2 \sin\alpha \sin^2\frac{\alpha}{2}(45 - 43\cos\alpha) \geqslant 0.$$

The function $M_n(\alpha)$ is not decreasing for each $\alpha \in [0, \pi)$ and $M_n(0) = 0$. This means that $M_n(\alpha) \geqslant 0$ for each $\alpha \in [0, \pi)$ and each positive integer n and $M_0(\alpha) = 0$. This completes the proof of the inequality

$$\frac{A''(\alpha)}{\pi} \geqslant \frac{1}{\sin^4\alpha} \sum_{n=0}^{\infty} (a_{n+1}^2 + b_{n+1}^2)\, M_n(\alpha) \geqslant 0$$

for each $\alpha \in [0, \pi)$. ■

EXAMPLE 1. Let $p(t) = a + b\cos 3t$, $a, b > 0$ then the curvature radius $R(t) = a - 8b\cos 3t$. If $a > 8b$ then the curve C is convex. From (2.1) the area function $A(\alpha)$ is convex if $a^2 \geqslant (8b)^2$.

EXAMPLE 2. Let $p(t) = a + b\cos 3t + c\cos 5t$ then from (2.1) $a^2 \geqslant (8b)^2 + (24c)^2$ and $R(t) = a - 8b\cos 3t - 24c\cos 5t > 0$ for $a > 8|b| + 24|c|$. We have

$$a^2 > (8|b| + 24|c|)^2 > (8b)^2 + (24c)^2.$$

For any convex curve C with support function $p(t) = a + b\cos 3t + c\cos 5t$ the area function $A(\alpha)$ is convex.

EXAMPLE 3. Consider a support function $p(t) = \frac{1}{1 - a\sin t}$, $t \in [0, 2\pi)$. Then the curvature radius of the corresponding oval is $R(t) = \frac{1 - 3a\sin t + 2a^2}{(1 - a\sin t)^3} \geqslant 0$ for $0 < |a| < \frac{1}{2}$. We will find the Fourier series of $p(t) = a_0 + \sum_{n=1}^{\infty} (a_n\cos nt + b_n\sin nt)$.

$$\frac{1}{1 - a\sin t} = \sum_{n=0}^{\infty} (a\sin t)^n,$$

where

$$\sin^m t = \begin{cases} \dfrac{(-1)^k}{2^{2k}}\left[2\displaystyle\sum_{j=0}^{k-1}\binom{2k}{j}(-1)^j\cos 2(k-j)\,t + (-1)^k\binom{2k}{k}\right] & \text{for } m = 2k \\ \dfrac{(-1)^k}{2^{2k}}\left[\displaystyle\sum_{j=0}^{k}\binom{2k+1}{j}(-1)^j\sin(2k-2j+1)\,t\right] & \text{for } m = 2k+1, \end{cases}$$

hence we have

$$\frac{1}{1-a\sin t}=\sum_{k=0}^{\infty}a^{2k}\frac{\binom{2k}{k}}{2^{2k}}+\sum_{k=0}^{\infty}(-1)^{k+1}\cos 2(k+1)\,t\sum_{j=k}^{\infty}\frac{a^{2j+2}}{2^{2j+1}}\binom{2j+2}{j-k}$$

$$+\sum_{k=0}^{\infty}(-1)^{k}\sin(2k+1)\,t\sum_{j=k}^{\infty}\frac{a^{2j+1}}{2^{2j}}\binom{2j+1}{j-k}.$$

The Fourier coefficients of $p(t)$ are equal to

$$a_n=\begin{cases}\displaystyle\sum_{k=0}^{\infty}a^{2k}\frac{\binom{2k}{k}}{2^{2k}} & \text{for } n=0\\ \displaystyle(-1)^{k+1}\sum_{j=k}^{\infty}\frac{a^{2j+2}}{2^{2j+1}}\binom{2j+2}{j-k} & \text{for } n=2k+2\\ 0 \text{ for } n=2k+1, \end{cases}$$

$$b_n=\begin{cases}0 \text{ for } n=2k\\ \displaystyle(-1)^{k}\sum_{j=k}^{\infty}\frac{a^{2j+1}}{2^{2j}}\binom{2j+1}{j-k} & \text{for } n=2k+1.\end{cases}$$

Using the inequalities

$$\sqrt{2\pi n}\left(\frac{n}{e}\right)^{n}e^{\frac{1}{12n+1}}<n!<\sqrt{2\pi n}\left(\frac{n}{e}\right)^{n}e^{\frac{1}{12n}},$$

$$\left(1+\frac{1}{n}\right)^{n}<e<\left(1+\frac{1}{n}\right)^{n+\frac{1}{2}},$$

we have

$$\frac{\binom{2j+2}{j+1}}{2^{2j+1}}<\frac{2}{\sqrt{\pi}\sqrt{j+1}}\leqslant\frac{2}{\sqrt{\pi}\sqrt{k+1}},$$

$$\frac{\binom{2j+1}{j+1}}{2^{2j}}<\frac{2}{\sqrt{\pi}\sqrt{j}}\leqslant\frac{2}{\sqrt{\pi}\sqrt{k}},\quad \text{for } j=k,\,k+1,\,\ldots$$

and

$$|a_{2k+2}| = \sum_{j=k}^{\infty} \frac{a^{2j+2}}{2^{2j+1}} \binom{2j+2}{j-k} \leqslant \sum_{j=k}^{\infty} \frac{a^{2j+2}}{2^{2j+1}} \binom{2j+2}{j+1} \leqslant \sum_{j=k}^{\infty} \frac{2a^{2j+2}}{\sqrt{\pi(k+1)}}$$

$$= \frac{2a^{2k+2}}{\sqrt{\pi(k+1)}} \frac{1}{1-a^2},$$

$$|b_{2k+1}| = \sum_{j=k}^{\infty} \frac{a^{2j+1}}{2^{2j}} \binom{2j+1}{j-k} \leqslant \sum_{j=k}^{\infty} \frac{a^{2j+1}}{2^{2j}} \binom{2j+1}{j+1} \leqslant$$

$$\leqslant \sum_{j=k}^{\infty} \frac{2a^{2j+1}}{\sqrt{\pi k}} = \frac{2a^{2k+1}}{\sqrt{\pi k}} \frac{1}{1-a^2}.$$

We can estimate the right hand side in the formula (2.1)

$$\sum_{n=0}^{\infty} (n(n+2))^2 (a_{n+1}^2 + b_{n+1}^2) =$$

$$= \sum_{k=0}^{\infty} (2k+1)^2 (2k+3)^2 a_{2k+2}^2 + \sum_{k=0}^{\infty} (2k)^2 (2k+2)^2 b_{2k+1}^2 <$$

$$< \sum_{k=0}^{\infty} (2k+1)^2 (2k+3)^2 \left(\frac{2a^{2k+2}}{\sqrt{\pi(k+1)}} \frac{1}{1-a^2} \right)^2 +$$

$$+ \sum_{k=0}^{\infty} (2k)^2 (2k+2)^2 \left(\frac{2a^{2k+1}}{\sqrt{\pi k}} \frac{1}{1-a^2} \right)^2 \leqslant$$

$$\leqslant \frac{1}{\pi(1-a^2)^2} \sum_{k=0}^{\infty} \left[64k(k+1)^2 a^{4k+2} + 8(2k+1)(2k+3)^2 a^{4k+4} \right] =$$

$$= \frac{1}{\pi(1-a^2)^2} \sum_{k=0}^{\infty} [(a^{4k+5})''' - 4(a^{4k+4})'' - 3(a^{4k+3})' - 3a^{4k+2} +$$

$$+ (a^{4k+7})''' - 4(a^{4k+6})'' - 3(a^{4k+5})' - 3a^{4k+4}] = \frac{8a^4(9-4a^2+a^4)}{\pi(1-a^2)^6}.$$

On the other hand

$$a_0^2 = \left(\sum_{k=0}^{\infty} \alpha^{2k} \frac{\binom{2k}{k}}{2^{2k}} \right)^2 = \left(1 + \sum_{k=1}^{\infty} \alpha^{2k} \frac{\binom{2k}{k}}{2^{2k}} \right)^2 > 1 .$$

Comparing both sides we obtain that if $0 < |\alpha| < \sqrt{\frac{2\sqrt{\pi^2 + 18\pi} - 3\pi}{-5\pi + 72}}$ then the condition (2.1) holds.

PROPOSITION 2.1. *Let $p(t)$ be the support function of the curve C. If there exists $\varepsilon > 0$ such that for $\alpha \in [0, \varepsilon]$ Fourier coefficients of $p(t)$ satisfy inequalities*

$$\sum_{n=0}^{\infty} (a_{n+1}^2 + b_{n+1}^2)[n^3 \sin^3\alpha \sin(n+1)\alpha + 3n^2 \sin^2\alpha \cos n\alpha - \\ - 6n \sin\alpha \cos\alpha \sin n\alpha + 2(1 + 2\cos^2\alpha)(1 - \cos n\alpha)] \geqslant 0 \tag{2.4}$$

and

$$a_0^2 \geqslant C(\varepsilon) \sum_{n=0}^{\infty} (n^3 + 3n^2 + 6n)(a_{n+1}^2 + b_{n+1}^2), \tag{2.5}$$

where $C(\varepsilon) = \dfrac{1}{2(1 - \cos\varepsilon)^2(2 - \cos\varepsilon)}$ *then $A(\alpha)$ is a convex function.*

PROOF. If $\alpha \in [0, \varepsilon]$ then from the assumption (2.4) we have $A''(\alpha) \geqslant 0$. For $\alpha \in (\varepsilon, \pi)$ we have

$$n^3 \sin^3\alpha \sin(n+1)\alpha + 3n^2 \sin^2\alpha \cos n\alpha - \\ - 6n \sin\alpha \cos\alpha \sin n\alpha + 2(1 + 2\cos^2\alpha)(1 - \cos n\alpha) \geqslant -n^3 - 3n^2 - 6n$$

and if a_0 satisfy (2.5) then we obtain

$$\frac{A''(\alpha)}{\pi} = \frac{2a_0^2(1 - \cos\alpha)^2(2 - \cos\alpha)}{\sin^4\alpha} + \\ + \frac{1}{\sin^4\alpha} \sum_{n=0}^{\infty} (a_{n+1}^2 + b_{n+1}^2)[n^3 \sin^3\alpha \sin(n+1)\alpha + \\ + 3n^2 \sin^2\alpha \cos n\alpha - 6n \sin\alpha \cos\alpha \sin n\alpha + 2(1 + 2\cos^2\alpha)(1 - \cos n\alpha)] \\ \geqslant \frac{1}{\sin^4\alpha} \left[C(\varepsilon) \sum_{n=0}^{\infty} (n^3 + 3n^2 + 6n)(a_{n+1}^2 + b_{n+1}^2) \frac{1}{C(\varepsilon)} - \right. \\ \left. - \sum_{n=0}^{\infty} (a_{n+1}^2 + b_{n+1}^2)(n^3 + 3n^2 + 6n) \right] \geqslant 0 . \quad \blacksquare$$

In particular, if the Fourier coefficients of $p(t)$ are such that

$$\exists_{n_0 \in \mathbb{N}} \ \forall_{n > n_0} \qquad a_n = b_n = 0$$

then $p(t)$ satisfy (2.4) for some $\varepsilon_0 > 0$. In fact if we put

$$N_{n+1}(\alpha) = \frac{1}{\sin^4 \alpha}[n^3 \sin^3 \alpha \sin(n+1)\alpha +$$

$$+3n^2 \sin^2 \alpha \cos n\alpha - 6n \sin \alpha \cos \alpha \sin n\alpha + 2(1 + 2\cos^2 \alpha)(1 - \cos n\alpha)]$$

in (2.2) we have $\lim_{\alpha \to 0^+} N_{n+1}(\alpha) = \frac{n^2(n+2)^2}{4} > 0$ for each $0 < n \leqslant n_0$ and $N_1 \equiv 0$. $N_{n+1}(\alpha)$ is a continuous function and therefore exists $\varepsilon_n > 0$ such that $N_{n+1}(\alpha) > 0$ for $\alpha \in [0, \varepsilon_n]$ we define $\varepsilon_0 = \min\{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{n_0}\}$. If for ε_0 conditon (2.5) holds then $A(\alpha)$ is a convex function.

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