Multiple Solutions of a Nonlinear Elliptic Equation Involving Neumann Conditions and a Critical Sobolev Exponent.

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Abstract - In this paper we prove the existence of two solutions of the nonhomogeneous Neumann problem (1.1) involving a critical Sobolev exponent. It is assumed that the coefficient $Q$ is positive and smooth on $\Omega$ and $\lambda > 0$ is a parameter which does not belong to the spectrum of $-\Delta$. We examine the common effect of the mean curvature of the boundary $\partial \Omega$ and the shape of the graph of the coefficient $Q$ on the existence of a second solution.

1. Introduction.

In this paper, we study the existence of multiple solutions of the superlinear problem

\[
\begin{aligned}
-\Delta u &= \lambda u + Q(x) u^{2^*-1} + f(x) \quad \text{in } \Omega \\
\frac{\partial u}{\partial \nu}(x) &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]

(1.1)

where $2^* = \frac{2N}{N-2}$, $N \geq 3$ is the critical Sobolev exponent, $\lambda \geq 0$ is a parameter and $\Omega \subset \mathbb{R}^N$ is a bounded domain with a smooth boundary.

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We assume that the coefficient $Q$ is smooth and positive on $\partial \Omega$ and $f \in L^r(\Omega)$ with $r > N$. We use the notation $u_+ = \max(u, 0)$.

This problem belongs to a class of problems referred to as the Ambrosetti-Prodi type. More precisely, in the case of the Dirichlet problem

\[
\begin{cases}
-\Delta u = g(u) + f(x) & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega
\end{cases}
\]

the limits

\[ g_- = \lim_{s \to -\infty} \frac{g(s)}{s} \quad \text{and} \quad g_+ = \lim_{s \to +\infty} \frac{g(s)}{s} \]

play an important role. We can basically distinguish three types of problems using the location of $g_-$ and $g_+$ with respect to the spectrum of the operator $-\Delta$ with the Dirichlet boundary conditions. Denoting by $\{\lambda_k\}$ the sequence of the eigenvalues of $-\Delta$ with the Dirichlet boundary conditions, the following types of problems have been considered:

(I) $-\infty \leq g_- < \lambda_1 < g_+ \leq +\infty$,

(II) $g_-$ and $g_+$ are both finite and the interval $(g_-, g_+)$ contains an eigenvalue. In this case the problem is asymptotically linear,

(III) $g_-$ lies between two consecutive eigenvalues and $g_+ = +\infty$.

We refer to the paper [12] where the extensive bibliography concerning these problems can be found. We point out here that conditions (I) and (III) cover the cases of subcritical, critical and supercritical growth for $g$. In the case of the Neumann problem the literature is rather scarce. In this paper we consider the nonlinear Neumann problem of type (III) with the nonlinearity of one-sided critical growth. We follow some ideas from [12], which considered a similar problem with the Dirichlet boundary conditions. First we consider the case $\lambda > 0$. The case $\lambda = 0$ will be treated separately.

Problem (1.1) may have constant solutions in contrast to the Dirichlet problem. We now discuss a number of conditions guaranteeing that a positive solution of (1.1) is not constant. If for some $\lambda > 0$ and a constant $c > 0$, the functions $Q$ and $f$ satisfy the equation

\[ (*) \quad \lambda c + Q(x) e^{c^{x-1}} + f(x) = 0 \]

for every $x \in \Omega$, then $u = c$ is a solution of (1.1). If $f$ and $Q$ are differentiable on some open subset of $\Omega$ then the following condition

(a) $\nabla f(\bar{x})$ is not parallel to $\nabla Q(\bar{x})$ for some $\bar{x} \in \Omega$
ensures that a positive solution of (1.1) is not constant. If \( f \) and \( Q \) are not differentiable we can proceed as follows. Integrating the equation \((*)\) we get

\[ (*) \quad \lambda c |f| + c^{2^* - 1} \int_{\Omega} Q(x) \, dx + \int_{\Omega} f(x) \, dx = 0, \]

where \(|\Omega|\) denotes the Lebesgue measure of \( \Omega \). From \((*)\) and \((***)\) we derive the equation

\[ c^{2^* - 1} \left( Q(x) |\Omega| - \int_{\Omega} Q(x) \, dx \right) + \left( f(x) |\Omega| - \int_{\Omega} f(x) \, dx \right) = 0. \]

We immediately obtain a contradiction if

(b) either \( Q(x) = \text{const} \) and \( f(x) \neq \text{const} \), or \( Q(x) \neq \text{const} \) and \( f(x) = \text{const} \).

If both functions \( Q(x) \) and \( f(x) \) are not constant we define a set

\[ \Omega_0 = \left\{ x; \frac{1}{|\Omega|} \int_{\Omega} Q(x) \, dx = Q(x) \right\}, \]

which is nonempty. Then a positive solution cannot be constant if

(c) either \( f(x) = \frac{1}{|\Omega|} \int_{\Omega} f(x) \, dx \) for all \( x \in \Omega - \Omega_0 \), or \( f(x) \neq \frac{1}{|\Omega|} \int_{\Omega} f(x) \, dx \) for some \( x \in \Omega_0 \).

Finally, if \((c)\) does not hold we require

(d) the ratio

\[ \frac{f(x) |\Omega| - \int_{\Omega} f(x) \, dx}{\int_{\Omega} Q(x) \, dx - Q(x) |\Omega|} \]

is either not constant on \( \Omega - \Omega_0 \), or it is constant and nonpositive on \( \Omega - \Omega_0 \).

Therefore one of these conditions will be assumed throughout this work.

We assume that \( f(x) = t + h(x) \), where \( t \) is a constant and \( h \in L^r(\Omega) \) with \( r > N \). We start by finding a negative solution of (1.1). We denote by \( \lambda_1 < \lambda_2 < \ldots \) the sequence of eigenvalues for \(-A\) with the Neumann boundary conditions. The first eigenvalue is simple and has constant eigenfunctions.
Let \( \lambda \neq \lambda_k \) for every \( k \). Then there exists a unique solution \( u_0 \in H^1(\Omega) \cap L^\infty(\Omega) \) of the problem

\[
\begin{align*}
-\Delta u &= \lambda u + h(x) \quad \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} &= 0 \quad \text{on } \partial \Omega.
\end{align*}
\]

The function \( u_t = -\frac{t}{\lambda} + u_0 \), with \( t > \lambda \sup_{\Omega} |u_0(x)| \) is negative and satisfies (1.1). We look for a second solution of the form \( u = v + u_t \), where \( v \) satisfies

\[
\begin{align*}
-\Delta v &= \lambda v + Q(x)(v + u_t)^{2^*-1} \quad \text{in } \Omega, \\
\frac{\partial v}{\partial \nu} &= 0 \quad \text{on } \partial \Omega.
\end{align*}
\]

(1.2)

Problem (1.2) will be solved through the min-max based on a topological linking. To this end, we define a variational functional

\[
J(v) = \frac{1}{2} \int_{\Omega} (|\nabla v|^2 - \lambda v^2) \, dx - \frac{1}{2^{*}} \int_{\Omega} Q(x)(v + u_t)^{2^{*}} \, dx
\]

for \( v \in H^1(\Omega) \). In the next section we examine Palais-Smale sequences for \( J \). In particular, we find the energy level of the functional \( J \) below which the Palais-Smale condition holds. In Section 3 we verify that the functional \( J \) has the geometry of a topological linking. Conditions guaranteeing the existence of critical points of \( J \) will be given in Sections 4 and 5. The existence results of this section depend on a relation between \( Q_m = \max_{x \in \Omega \setminus \partial \Omega} Q(x) \) and \( Q_M = \max_{x \in \Omega \setminus \partial \Omega} Q(x) \). Section 6 is devoted to the case \( \lambda = 0 \).

The existence of a critical point in this case is obtained through the implicit function theorem. The distinction of two cases involving the quantities \( Q_M \) and \( Q_m \) envisaged in Section 4 disappears in the case \( \lambda = 0 \).

2. The Palais-Smale condition.

We need two quantities:

\[
Q_m = \max_{x \in \partial \Omega} Q(x) \quad \text{and} \quad Q_M = \max_{x \in \Omega} Q(x).
\]
We set
\[ S = \min \left( \frac{S^{N/2}}{NQ_0^{(N-2)/2}}, \frac{S^{N/2}}{2NQ_0^{(N-2)/2}} \right), \]
where \( S \) denotes the best Sobolev constant, that is,
\[ S = \inf_{u \in D^{1,2}(\mathbb{R}^N)} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 \, dx}{\left( \int_{\mathbb{R}^N} |u(x)|^{2^*} \, dx \right)^{2/2^*}}. \]

Here \( D^{1,2}(\mathbb{R}^N) \) denotes a Sobolev space obtained as the completion of \( C_0^\infty(\mathbb{R}^N) \) with respect to the norm
\[ \|u\|^2_{D^{1,2}(\mathbb{R}^N)} = \int_{\mathbb{R}^N} |\nabla u|^2 \, dx. \]

In what follows, \( \| \cdot \| \) denotes the norm in \( H^1(\Omega) \), which is given by
\[ \|u\|^2 = \int_{\Omega} (|\nabla u|^2 + u^2) \, dx. \]

In this paper we frequently use the Sobolev inequality:
\[ \left( \int_{\Omega} |u|^{2^*} \, dx \right)^{2/2^*} \leq C_s \int_{\Omega} (|\nabla u|^2 + u^2) \, dx \]
for all \( u \in H^1(\Omega) \), where \( C_s > 0 \) is a constant.

**Proposition 2.1.** Let \( \lambda_k < \lambda < \lambda_{k+1} \). If
\[ J(u_n) \to c < S_m \quad \text{and} \quad J'(u_n) \to 0 \quad \text{in} \quad H^{-1}(\Omega) \]
then \( \{u_n\} \) is relatively compact in \( H^1(\Omega) \).

**Proof.** We commence by showing that \( \{u_m\} \) is bounded in \( H^1(\Omega) \). We write
\[ u_m = u_m^- + u_m^+, \quad u_m^- \in E^- \quad \text{and} \quad u_m^+ \in E^+, \]
where
\[ E^- = \text{span of all eigenfunctions corresponding to } \lambda_1, \ldots, \lambda_k, \]
and $E^+ = (E^-)^\perp$. If $\phi \in H^1(\Omega)$, then

$$
\int_\Omega \nabla u_u \nabla \phi \, dx - \lambda \int_\Omega u_u \phi \, dx = \int_\Omega Q(x)(u_u + u_t)^{2^* - 1} \phi \, dx + \varepsilon u \|\phi\|
$$

with $\varepsilon u \to 0$. Taking $\phi = u_u^+$, we get

$$
\int_\Omega |\nabla u_u^+|^2 - \lambda (u_u^+)^2 = \int_\Omega Q(x)(u_u + u_t)^{2^* - 1} u_u^+ \, dx + \varepsilon u \|u_u^+\|.
$$

Let $\delta > 0$ be such that $\lambda + \delta < \lambda_{k+1}$. Then

$$
\left(1 - \frac{\lambda + \delta}{\lambda_{k+1}}\right) \int_\Omega |\nabla u_u^-|^2 \, dx + \delta \int_\Omega (u_u^-)^2 \, dx \leq \int_\Omega Q(x)(u_u + u_t)^{2^* - 1} u_u^+ \, dx + \varepsilon u \|u_u^+\|.
$$

We now use (2.1) with $\phi = u_u^-$ and let $\delta_1 > 0$ be such that $\lambda - \delta_1 > \lambda_{k+1}$. Then

$$
\left(\frac{\lambda - \delta_1}{\lambda_{k+1}} - 1\right) \int_\Omega |\nabla u_u^-|^2 \, dx + \delta_1 \int_\Omega (u_u^-)^2 \, dx \leq - \int_\Omega Q(x)(u_u + u_t)^{2^* - 1} u_u^- \, dx + \varepsilon u \|u_u^-\|.
$$

On the other hand for $n \geq n_0$, we can write

$$
c + \varepsilon u \|u_u\| + 1 \geq J(u_u) - \frac{1}{2} \langle J'(u_u), u_u \rangle
$$

$$
= \frac{1}{2} \int_\Omega Q(x)(u_u + u_t)^{2^* - 1} u_u \, dx - \frac{1}{2^*} \int_\Omega Q(x)(u_u + u_t)^{2^*} \, dx
$$

$$
= \frac{1}{N} \int_\Omega Q(x)(u_u + u_t)^{2^*} \, dx - \frac{1}{2} \int_\Omega Q(x)(u_u + u_t)^{2^* - 1} u_t \, dx
$$

$$
\geq \frac{1}{N} \int_\Omega Q(x)(u_u + u_t)^{2^*} \, dx.
$$

Applying the Young inequality, we deduce from (2.2) and the above
estimate that for $\eta > 0$ we have

\begin{align*}
(2.4) \quad & \left(1 - \frac{\lambda + \delta}{\lambda_{k+1}}\right) \int_{\Omega} |\nabla u_n^+|^2 \, dx + \delta \int_{\Omega} (u_n^+)^2 \, dx \leq \\
& \leq \int_{\Omega} Q(x)(u_n + u_t)^{2^* - 1} u_n^+ \, dx + \eta \|u_n^+\| \\
& \leq \eta \left(\int_{\Omega} |u_n^+|^2 \, dx\right)^{2/2^*} + C_2 \left(\int_{\Omega} Q(x)(u_n + u_t)^{2^*} \, dx\right)^{2^* - 1/2^*} \, dx + \eta \|u_n^+\| \\
& \leq C_2 \left(\int_{\Omega} q_{2^*} \eta \|u_n^+\|^2 + \frac{C_2}{C_1}\left(\int (u_n + u_t)^{2^*} \, dx\right)^{(N+2)/N} + \eta \|u_n^+\| \right) \\
& \leq C_2 \left(\int_{\Omega} q_{2^*} \eta \|u_n^+\|^2 + \frac{C_2}{C_1}\left(\int (u_n + u_t)^{2^*} \, dx\right)^{(N+2)/N} + \eta \|u_n^+\| \right) + C_3
\end{align*}

for some constants $C_1 > 0$, $C_2 > 0$ and $C_3 > 0$. In a similar way, we obtain

\begin{align*}
(2.5) \quad & \left(\frac{\lambda - \delta}{\lambda_k} - 1\right) \int_{\Omega} |\nabla u_n^-|^2 \, dx + \delta \int_{\Omega} (u_n^-)^2 \, dx \leq \\
& \leq C_1 \left(\int_{\Omega} q_{2^*} \eta \|u_n^-\|^2 + \frac{C_2}{C_1}\left(\int (u_n + u_t)^{2^*} \, dx\right)^{(N+2)/N} + \|u_n^-\|^2 \right) + 1
\end{align*}

for some constant $C_1 > 0$. Estimates (2.4) and (2.5) imply that $\{u_n\}$ is bounded in $H^1(\Omega)$. We may therefore assume that $u_n \rightharpoonup u$ in $H^1(\Omega)$. By the concentration-compactness principle there exist sequences of points $\{x_j\} \subset \mathbb{R}^N$, sequences of numbers $\{\nu_j\}$ and $\{\mu_j\}$ such that

$$
|u_n|_{2^*} \rightharpoonup |u|_{2^*} + \sum_j \nu_j \delta_{x_j}
$$

and

$$
|\nabla u_n|_{2^*} \rightharpoonup |\nabla u|_{2^*} + \sum_j \mu_j \delta_{x_j}
$$

in the sense of measures, where

$$
S\nu_j^{2^*/2} \leq \mu_j \quad \text{if} \; x_j \in \Omega
$$
Fix \( x_j \). Let \( \{ \phi_\delta \} \) be a family of smooth and positive functions concentrating at \( x_j \) as \( \delta \to 0 \). Then using the Brézis-Lieb Lemma, we obtain

\[
\int_\Omega |\nabla u_n|^2 \phi_\delta \, dx + \int_\Omega \nabla u_n \nabla \phi_\delta \, dx + \lambda \int_\Omega u_n^2 \phi_\delta \, dx
= \int_\Omega Q(x)(u_n + u_t)^{2s-1} u_n \phi_\delta \, dx + o(1)
\]

\[
= \int_\Omega Q(x)|u_n + u_t|^{2s} \phi_\delta \, dx - \int_\Omega Q(x)(u_n + u_t)^{2s-1} u_t \phi_\delta \, dx + o(1)
\]

\[
\leq \int_\Omega Q(x)|u_n + u_t|^{2s} \phi_\delta \, dx - \int_\Omega Q(x)(u_n + u_t)^{2s-1} u_t \phi_\delta \, dx + o(1)
\]

\[
= \int_\Omega Q(x)|u_n|^{2s} \phi_\delta \, dx - \int_\Omega Q(x)|u_t|^{2s} \phi_\delta \, dx + \int_\Omega Q(x)|u + u_t|^{2s} \phi_\delta \, dx
\]

\[
- \int_\Omega Q(x)(u_n + u_t)^{2s-1} u_t \phi_\delta \, dx + o(1).
\]

Letting \( n \to \infty \) and then \( \delta \to 0 \) we deduce that in both cases \( x_j \in \partial \Omega \) and \( x_j \in \Omega \),

\[
\mu_j \leq Q(x_j) \nu_j.
\]

If \( \mu_j > 0 \) for some \( x_j \), then

\[
\mu_j \geq \frac{S^{N/2}}{Q(x_j)^{\gamma N - 2\gamma/2}} \text{ if } x_j \in \Omega \text{ and } \mu_j \geq \frac{S^{N/2}}{2Q(x_j)^{\gamma N - 2\gamma/2}} \text{ if } x_j \in \partial \Omega.
\]
We now write
\[ J(u_n) - \frac{1}{2^*} \langle J'(u_n), u_n \rangle \]
\[ = \frac{1}{N} \int_\Omega (|\nabla u_n|^2 - \lambda u_n^2)\, dx - \frac{1}{2^*} \int_\Omega Q(x)(u_n + u_t)^{2^*}_+\, dx \]
\[ + \frac{1}{2^*} \int_\Omega Q(x)(u_n + u_t)^{2^*}_- u_n + o(1) \]
\[ = \frac{1}{N} \int_\Omega (|\nabla u_n|^2 - \lambda u_n^2)\, dx - \frac{1}{2^*} \int_\Omega Q(x)(u_n + u_t)^{2^*}_+ u_t + o(1) \]
\[ \geq \frac{1}{N} \int_\Omega (|\nabla u_n|^2 - \lambda u_n^2)\, dx + o(1). \]

Since \( u \) is a solution of (1.1) we also have
\[ \int_\Omega (|\nabla u|^2 - \lambda u^2)\, dx = \int_\Omega Q(x)(u + u_t)^{2^*}_- u\, dx = 0. \]

We aim to show that \( \mu_j = 0 \) for every \( j \). If not, the concentration-compactness principle implies that
\[ c \geq \frac{1}{N} \int_\Omega (|\nabla u|^2 - \lambda u^2)\, dx + \frac{1}{N} \sum_j \mu_j \geq \frac{1}{N} \sum_j \mu_j. \]

If \( \mu_j > 0 \) for some \( j \) with \( x_j \in \partial \Omega \), then
\[ c \geq 2N \frac{S^{N^2}}{Q(x_j)^{(N-2)/2}} \geq \frac{1}{2N} \frac{S^{N^2}}{Q_{\text{min}}^{(N-2)/2}}. \]

This is obviously impossible. Similarly if \( \mu_j > 0 \) for some \( j \) with \( x_j \in \Omega \). Thus
\[ \int_\Omega Q(x)(u_n + u_t)^{2^*}_+\, dx \to \int_\Omega Q(x)(u + u_t)^{2^*}_+\, dx \]
and also
\[ \int_{\Omega} |\nabla u_n|^2 \, dx \to \int_{\Omega} |\nabla u|^2 \, dx \]
and the result follows. \( \blacksquare \)

3. Topological linking.

We assume that \( \lambda \in (\lambda_k, \lambda_{k+1}) \). Let
\[ E^- = \text{span} \{ e_1, \ldots, e_i \}, \]
where \( e_1, \ldots, e_i \) are eigenfunctions corresponding to \( \lambda_1, \ldots, \lambda_k \). We set \( E^+ = (E^-)^\perp \). Let
\[ S_\varrho = \partial B_\varrho \cap E^+ \text{ and } D = [0, Re] \oplus (B_e \cap E^-), \quad e \in E^+, \]
where \( B_e \) denotes the ball of radius \( r \) with centre at 0. To apply a topological linking we need to verify that
\[ J|_{S_\varrho} \geq \alpha > 0, \quad \varrho < R, \]
\[ J|_{\partial D} < \alpha \quad \text{and} \quad \max_{u \in D} J(u) < S_\varrho. \]

**Lemma 3.1.** There exist \( \varrho_0 > 0 \) and \( \alpha : (0, \varrho_0] \to (0, \infty) \) such that
\[ J(u) \geq \alpha(\varrho) \quad \text{for every } \quad v \in S_\varrho. \]

**Proof.** We choose \( \eta > 0 \) so that \( \lambda_1 < \lambda + \eta < \lambda_{k+1} \). Then
\[ \int_{\Omega} |\nabla u|^2 \, dx \geq \lambda_{k+1} \int_{\Omega} u^2 \, dx \]
for every \( u \in E^+ \). Since \( u_e < 0 \) on \( \Omega \), we have
\[ J(u) \geq \left( 1 - \frac{\lambda + \eta}{\lambda_{k+1}} \right) \|\nabla u\|^2 \, dx + \eta \int_{\Omega} u^2 \, dx - C_s^{-2\gamma/2} Q_M \left( \frac{\|\nabla u\|^2 + u^2}{\int_{\Omega} \|\nabla u\|^2 + u^2 \, dx} \right)^{2\gamma/2} \]
\[ \geq \beta \left( \|\nabla u\|^2 + u^2 \right) \, dx - C_s^{-2\gamma/2} Q_M \left( \frac{\|\nabla u\|^2 + u^2}{\int_{\Omega} \|\nabla u\|^2 + u^2 \, dx} \right)^{2\gamma/2}, \]
where

$$\beta = \min \left(1 - \frac{\lambda + \eta}{\lambda_{k+1}}, \eta \right).$$

Letting $q = \|u\|$ we obtain the following estimate

$$J(u) \geq \beta q^{2} - C_s^{-2s/2} Q_M q^{2*}.$$ 

To complete the proof we set

$$a(q) = \beta q^{2} - C_s^{-2s/2} Q_M q^{2*},$$ 

with $q_0 > 0$ such that $q_0^2 - C_s^{-2s/2} Q_M q_0^{2*} > 0.$

From now on, we use the instanton

$$U_\varepsilon(x) = \frac{c_N \varepsilon^{(N-2)/2}}{(\varepsilon^2 + |x|^2)^{N-2/2}},$$

in the definition of the set $D$, where $c_N > 0$ is a constant and we set $e_\varepsilon = P^+ U_\varepsilon$. It is well-known that $U = U_1$ satisfies the equation

$$-\Delta U = U^{2s-1} \quad \text{in} \quad \mathbb{R}^N$$

and moreover

$$\int_{\mathbb{R}^N} |\nabla U|^2 \, dx = \int_{\mathbb{R}^N} U^{2s} \, dx = S^{N/2}.$$ 

With the choice of $\varepsilon = e_\varepsilon$ we verify the remaining conditions of the topological linking. Without loss of generality we may assume in Lemma 3.2 below that $0 \in \Omega$.

**Lemma 3.2.** There exist $r_0 > 0$, $R_0 > 0$ and $\varepsilon_0 > 0$ such that for $r \geq r_0$, $R \geq R_0$ and $0 < \varepsilon \leq \varepsilon_0$ we have

$$J(u) < a \quad \text{for every} \quad u \in \partial D.$$ 

**Proof.** We set

$$\partial D = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3,$$
where

\[ \Gamma_1 = \overline{B}_r \cap E^- , \]

\[ \Gamma_2 = \{ v \in H^1(\Omega); \; v = w + se, \; w \in E^-, \|w\| = r, \; 0 \leq s \leq R \} , \]

\[ \Gamma_3 = \{ v \in H^1(\Omega); \; v = w + Re, \; w \in E^- \cap B_r \} . \]

For \( v \in E^- \) we have

\[ \int_{\Omega} |\nabla v|^2 \, dx \lesssim \lambda_k \int_{\Omega} v^2 \, dx \]

and

\[ J(v) \leq \frac{1}{2} \left( 1 - \frac{\lambda}{\lambda_k} \right) \int_{\Omega} |\nabla v|^2 \, dx - \frac{1}{2} \int_{\Omega} Q(x)(v + u_i)^2 \, dx \leq 0 . \]

We now consider \( \Gamma_2 \). Let \( v \in \Gamma_2 \) and define

\[ \delta^2 = \sup_{0 < r \leq 1} \int_{\Omega} |\nabla e_x|^2 \, dx . \]

Let \( r^2 = \|\nabla w\|^2 \) and choose \( \eta_1 > 0 \) so that \( \lambda_k < \lambda - \eta_1 \). Then for \( v = w + se \), we have

\[ J(v) \leq \frac{1}{2} \int_{\Omega} |\nabla v|^2 \, dx - \frac{\lambda}{2} \int_{\Omega} v^2 \, dx \]

\[ \leq \frac{1}{2} \int_{\Omega} |\nabla w|^2 \, dx - \frac{\lambda}{2} \int_{\Omega} w^2 \, dx + \frac{s^2}{2} \int_{\Omega} |\nabla e_x|^2 \, dx \]

\[ \leq \frac{1}{2} \left( 1 - \frac{\lambda - \eta_1}{\lambda_k} \right) \left( \int_{\Omega} |\nabla w|^2 \, dx \right) - \frac{\eta_1}{2} \int_{\Omega} w^2 \, dx + \frac{s^2}{2} \int_{\Omega} |\nabla e_x|^2 \, dx . \]

Let \( \eta_2 = \max \left( 1 - \frac{\lambda - \eta_1}{\lambda_k}, -\frac{\eta_1}{2} \right) < 0 \). We then have

\[ J(v) \leq \eta_2 r^2 + \frac{s^2}{2} \int_{\Omega} |\nabla e_x|^2 \, dx . \]

We set \( s_0 = \frac{\sqrt{2a}}{\delta} \). Then \( J(v) \leq \alpha \) for \( 0 \leq s \leq s_0 \). We now consider the
case $s > s_0$. Put

$$K = \sup \left\{ \left\| \frac{w + u_1}{s} \right\|_s ; s_0 \leq s \leq R, \left\| w \right\| = r, w \in E^− \right\}.$$ 

We now estimate $P^− U_i$. Let

$$P^− U_i = \sum_{j=1}^l \alpha_j e_j, \quad \alpha_j = \int_\Omega U_i e_j(x) \, dx.$$ 

Since the first eigenfunction corresponding to $\lambda = 0$ is constant, we see that $P^− U_i \neq 0$. Hence

$$\|P^− U_i\|_\mathcal{H}_s^2 = \sum_{j=1}^l \alpha_j^2 = \sum_{j=1}^l \left( \int_\Omega U_i e_j \right)^2 \leqslant \sum_{j=1}^l \|e_j\|_\mathcal{H}_s \|U_i\|_\mathcal{H}_s \leqslant C \varepsilon^{N−2}.$$ 

Therefore

$$P^+ U_i(0) = U_i(0) - P^− U_i(0) \geq C \varepsilon^{−(N−2)/2} - \|P^− U_i\|_\mathcal{H}_s \geq C \varepsilon^{−(N−2)/2}.$$ 

By the continuity of $P^+ U_i$ there exists a $\delta = \delta(K)$ such that

$$B_{\delta}(0) \subset \{ x \in \Omega ; P^+ U_i(x) > K \}.$$ 

We also need the following inequality: if $\omega \subset \Omega$ and $u + v > 0$ on $\omega$, then

$$\left| \int_\omega (u + v)^p dx - \int_\omega |u|^p dx - \int_\omega |v|^p dx \right| \leq C \int_\omega (|u|^{p−1} |v| + |u|^{p−1} |v|) dx,$$ 

where $C = C(p)$. We apply this estimate with $\omega = \Omega_s$, where

$$\Omega_s = \{ x \in \Omega ; P^+ U_i(x) > K \}.$$
Letting $Q_s = \min_{x \in \Omega} Q(x)$ we get
\[
\int_{\Omega} Q(x) \left( \frac{w + u_t}{s} + e_j \right)^2 \, dx \geq Q_s \int_{\Omega} \left( e_j + \frac{w + u_t}{s} \right)^2 \, dx
\]
\[
\geq Q_s \left( \int_{\Omega} |e_j|^2 \, dx + \int_{\Omega} \left| \frac{w + u_t}{s} \right|^2 \, dx \right) - C \int_{\Omega} \left( |e_j|^{2^* - 1} \left| \frac{w + u_t}{s} \right| + |e_j| \left| \frac{w + u_t}{s} \right|^{2^* - 1} \right) \, dx
\]
\[
\geq Q_s \left( \int_{\Omega} |e_j|^2 \, dx + \int_{\Omega} \left| \frac{w + u_t}{s} \right|^2 \, dx \right) - C \|e_j\|_{L^{2^* - 1}(\Omega)} + \|e_j\|_{L^1(\Omega)}.
\]
Since
\[
\|P^+ U_t\|_{L^{2^* - 1}} \leq C \epsilon^{-\frac{N-2}{2}} \quad \text{and} \quad \|P^+ U_t\|_{L^1} \leq C \epsilon^{-\frac{N-2}{2}}
\]
we deduce from the previous estimate that
\[
J(v) \leq \eta_2 s^2 + \frac{s^2}{2} S^{N/2} - \frac{s^{2^*}}{2} Q_s S^{N/2} + Cs^{2^*} \epsilon^{(N-2)/2}
\]
\[
= \eta_2 s^2 + \Phi_e(s).
\]
It is easy to check that
\[
\Phi_e(s) \leq \frac{1}{2} \left( \frac{S^{N/2}}{Q_s S^{N/2}} \right)^{1/2} + O(\epsilon^{(N-2)/2}).
\]
Increasing $r$, if necessary, we get
\[
J(v) < 0 \quad \text{for} \quad v \in \Gamma_2.
\]
If $v \in \Gamma_2$, then
\[
J(v) = \frac{1}{2} \left( 1 - \frac{\lambda}{k} \right) \int_{\Omega} |\nabla w|^2 \, dx + \frac{R^2}{2} \int_{\Omega} |\nabla e_j|^2 \, dx - \frac{R^{2^*}}{2^*} \int_{\Omega} \left( e_j + \frac{w + u_t}{R} \right)^{2^*} \, dx.
\]
Let $K > 0$ be such that $\|w + u_t\|_{L^\infty} \leq K$. Then there exists an $\epsilon_0 > 0$ (small enough) such that $P^+ e_j(0) > 2K$ for every $0 < \epsilon \leq \epsilon_0$. Then for
every \(0 < \varepsilon \leq \varepsilon_0\) we can find \(R_0 = R_0(\varepsilon)\) and \(\eta = \eta(\varepsilon) > 0\) such that
\[
\left| \left\{ x \in \Omega; \ e_x + \frac{w + e_{\varepsilon}}{R} > 1 \right\} \right| \geq \eta
\]
for \(R \geq R_0\). Hence \(J(v) \leq 0\) for \(v \in I_\varepsilon\) for \(\varepsilon \leq \varepsilon_0\) and \(R \geq R_0\).

4. Case \(Q_M < 2^{2/(N-2)} Q_m\).

Let \(H(y)\) denote the mean curvature of the boundary \(\partial \Omega\) at \(y \in \partial \Omega\).
Throughout this section we assume that:

\((\mathbf{A})\) the coefficient \(Q\) satisfies the following conditions:
\[
Q_M < 2^{2/(N-2)} Q_m,
\]
and \(|Q(x) - Q(y)| = o(|x - y|)\) for some \(y \in \partial \Omega\) with \(Q(y) = Q_m, H(y) > 0\) and \(x\) close to \(y\).

Obviously in this case we have \(S_\varepsilon = \frac{S^{N/2}}{2NQ_{2/(N-2)}^2}\).

**Proposition 4.1.** Let \(N \geq 5\) and suppose that \((\mathbf{A})\) holds. Then
\[
(4.1) \quad \max_{v \in D} J(v) < \frac{S^{N/2}}{2NQ_m^{2/(N-2)}}. 
\]

**Proof.** Without loss of generality we may assume that \(y = 0\). Let \(v \in D\). Then \(v = w + se_x\) and
\[
J(w + se_x) = \frac{1}{2} \int_\Omega \left( |\nabla w|^2 - \lambda w^2 \right) dx + \frac{s^2}{2} \int_\Omega \left( |\nabla e_x|^2 - \lambda e_x^2 \right) dx
\]
\[
- \frac{1}{2s} \int_\Omega Q(x)(w + se_x + u_x)_x^2 dx.
\]

For \(0 < s \leq s_0\), with \(s_0\) sufficiently small, we have
\[
J(w + se_x) \leq \frac{s^2}{2} \int_\Omega |\nabla e_x|^2 dx \leq \frac{S^{N/2}}{2NQ_m^{2/(N-2)}}.
\]

If \(s \geq s_0\), then repeating the estimates from Lemma 3.2 we get
\[
J(w + se_x) \leq \frac{s^2}{2} \int_\Omega \left( |\nabla e_x|^2 - \lambda e_x^2 \right) dx - \frac{s^2}{2s} \int_\Omega Q(x) e_x^2 dx + Cs^2 \varepsilon^{(N-2)/2}
\]
for some constant \( C > 0 \). Hence

\[
J(w + se) \leq \frac{1}{N} \left( \frac{\int (|\nabla e|^2 - \lambda e^2) \, dx}{\int Q(x) \, |e|^2 \, dx} \right)^{N/2} + O(e^{(N-2)/2}).
\]

Since

\[
\int_{\Omega} |P^+ U_\varepsilon|^2 \, dx = \int_{\Omega} U_\varepsilon^{2\varepsilon} \, dx + O(e^{N-2})
\]

and

\[
\int_{\Omega} |\nabla P^+ U_\varepsilon|^2 \, dx = \int_{\Omega} |\nabla U_\varepsilon|^2 \, dx + O(e^{N-2}),
\]

we obtain

\[
J(w + se) \leq \frac{1}{N} \left( \frac{\int (|\nabla U_\varepsilon|^2 - \lambda U_\varepsilon^2) \, dx}{\int Q(x) \, U_\varepsilon^{2\varepsilon} \, dx} \right)^{N/2} + O(e^{(N-2)/2}).
\]

We need the following asymptotic formulas (see [17])

\[
\int_{\Omega} |\nabla U_\varepsilon|^2 \, dx = \frac{K_1}{2} - I(\varepsilon) + o(\varepsilon),
\]

\[
\int_{\Omega} U_\varepsilon^{2\varepsilon} \, dx = \frac{K_2}{2} - II(\varepsilon) + o(\varepsilon),
\]

where \( K_1 = \int_{\mathbb{R}^N} |\nabla U|^2 \, dx \), \( K_2 = \int_{\mathbb{R}^N} U^{2\varepsilon} \, dx \), \( S = K_1 / K_2 (N-2) \), \( I(\varepsilon) = O(\varepsilon) \) and \( II(\varepsilon) = O(\varepsilon) \). Moreover, we have (see (3.17) in [17])

\[
\lim_{\varepsilon \to 0} \frac{I(\varepsilon)}{II(\varepsilon)} > \frac{N - 2}{N} \frac{K_2}{K_1}.
\]

By assumption \((A)\) we have

\[
\int_{\Omega} Q(x) \, U_\varepsilon^{2\varepsilon} \, dx = \frac{Qm K_2}{2} + o(\varepsilon).
\]
Thus

\[ J(w + se_\varepsilon) \leq \frac{1}{N} \left( \frac{K_1}{2} - I(\varepsilon) + o(\varepsilon) \right)^{N/2} + \frac{N - 2}{N} K_1 I(\varepsilon) + o(\varepsilon), \]

According to (4.2) we can find an \( \varepsilon_0 > 0 \) and a \( \varphi > 0 \) such that

\[ I(\varepsilon) > \frac{N - 2}{N} K_1 I(\varepsilon) + \varphi \]

for \( 0 < \varepsilon < \varepsilon_0 \). It then follows from (4.3) and (4.4) that

\[ J(w + se_\varepsilon) \leq \left( \left( \frac{K_1}{2} \right)^{N/2} - \frac{N - 2}{N} \left( \frac{K_1}{2} \right)^{(N - 2)/2} I(\varepsilon) + o(\varepsilon) \right)^{N/2} \]

\[ \times \left( \left( \frac{1}{2} K_2 Q_m \right)^{-(N - 2)/2} + \frac{N - 2}{2} Q_m I(\varepsilon) \left( \frac{1}{2} K_2 Q_m \right)^{-N/2} + o(\varepsilon) \right) \]

\[ < \frac{S^{N/2}}{2NQ_m^{(N - 2)/2}} - C_Q \]

for some constant \( C > 0 \) and the result follows.

We are now in a position to formulate the following result

**Theorem 4.2.** Suppose that the assumptions of Proposition 4.1 hold. Then problem (4.1) has at least two solutions.

5. Case \( Q_M \geq 2^{2(N - 2)} Q_m \).

If \( Q_M \geq 2^{2(N - 2)} Q_m \), then \( S_n = \frac{S^{N/2}}{NQ_m^{(N - 2)/2}} \). We assume that

\[ |Q(x) - Q(y)| = o(|x - y|^2) \]

for some \( y \in \Omega \) with \( Q(y) = Q_M \) and \( x \) close to \( y \).

Assuming that \( y = 0 \), we have

\[ \int_\Omega |\nabla U_\varepsilon|^2 dx = K_1 + O(\varepsilon^{N - 2}), \]

\[ \int_\Omega Q(x) U_\varepsilon^{2n} dx = K_2 Q_M + o(\varepsilon^2) \]
and
\[ \int_{\Omega} U_\epsilon^2 \, dx \geq c_1 \epsilon^2 \]
for some constant \( c_1 > 0 \) independent of \( \epsilon \). As in the proof of Proposition 4.1 we have
\[
J(\psi + se_\epsilon) \leq \frac{\left( \int (|\nabla U_\epsilon|^2 - \lambda U_\epsilon^2) \, dx \right)^{N/2}}{N \left( \int Q(x) U_\epsilon^{2 + e} \, dx \right)^{N - 2}/2} + O(\epsilon^{(N-1)/2})
\]
\[
\leq \frac{(K_1 + O(\epsilon^{N-2})) - c_1 \epsilon^2)^{N/2}}{N(K_2 Q_M + o(\epsilon^2))^{N - 2}/2} + O(\epsilon^{(N-1)/2}).
\]
If \( N \geq 7 \), taking \( \epsilon > 0 \) sufficiently small, we can check that
\[
\max_{\psi \in \mathcal{D}} J(\psi) \leq \frac{S^{N/2}}{Q_M^{N - 2}/2}.
\]

**Theorem 5.1.** Let \( N \geq 7 \). Suppose that \( Q_M \geq 2^{2(N-2)} Q_m \) and that (5.1) holds. Then problem (1.1) has two solutions.

6. **Existence of solutions in the case \( \lambda = 0 \).**

In this case problem (1.1) takes the form
\[
\begin{align*}
-\Delta u &= Q(x) u^{2^* - 1} + f(x) \quad \text{in } \Omega \\
\frac{\partial}{\partial \nu} u(x) &= 0 \quad \text{on } \partial \Omega,
\end{align*}
\]
(6.1)

Obviously a necessary condition for the existence of a solution of problem (6.1) is the condition
\[
\int_{\Omega} f(x) \, dx < 0.
\]
(6.2)

Since the eigenfunctions corresponding to \( \lambda = 0 \) are constant, we decompose \( H^1(\Omega) \) as \( H^1(\Omega) = \text{span} \{ 1 \} \oplus E^+ \), where
\[
E^+ = \left\{ v \in H^1(\Omega) ; \int_{\Omega} v \, dx = 0 \right\}.
\]
Thus for every function \( u \in H^1(\Omega) \) we have \( u = t + v \), where \( t \in \mathbb{R} \) and
\[ \int_{\Omega} v \, dx = 0. \] The variational functional \( J \) for (6.1) is given by

\[
J(u) = \frac{1}{2} \int_{\Omega} |\nabla v|^2 \, dx - \frac{1}{2^n} \int_{\Omega} Q(x)(t + v)^{2^*_s} \, dx - \int_{\Omega} f(x)(t + v) \, dx.
\]

It is easy to show that the function \( t \to J(t + v) \) is bounded above. Let \( v \in E^+ \) and set

\[ g(t) = J(t + v). \]

It is clear that for every \( v \in E^+ \) there exists \( t(v) > 0 \) such that

\[ g(t(v)) = \max_{t \in \mathbb{R}} g(t), \]

that is, \( J(t + v) \leq J(t(v) + v) \) for every \( t \in \mathbb{R} \). Thus by the implicit function theorem we can define a continuously differentiable mapping

\[ v \in E^+ \mapsto t(v) \in \mathbb{R}, \]

such that \( J(t + v) \leq J(t(v) + v) \) for every \( t \neq t(v) \). Since

\[ 0 = g'(t(v)) = -\int_{\Omega} Q(x)(t(v) + v)^{2^*_s - 1} \, dx - \int_{\Omega} f(x) \, dx, \]

we see that

\[ (6.3) \quad -\int_{\Omega} Q(x)(t(v) + v)^{2^*_s - 1} \, dx + \int_{\Omega} f(x) \, dx = 0 \]

for every \( v \in E^+ \). In particular, if \( v = 0 \), then

\[ -\int_{\Omega} Q(x)(t(0))^{2^*_s - 1} \, dx = -\int_{\Omega} f(x) \, dx. \]

This combined with (6.2) yields \( t(0) > 0 \) and we have

\[ (6.4) \quad t(0)^{2^*_s - 1} \int_{\Omega} Q(x) \, dx = -\int_{\Omega} f(x) \, dx. \]

We now claim that the functional

\[ F(v) = J(v + t(v)) \]

attains its minimum on some ball \( B_{\rho}(0) \). We set

\[ A = -\int_{\Omega} f(x) \, dx \quad \text{and} \quad B = \int_{\Omega} Q(x) \, dx. \]
By easy computations using (6.4), we verify that
\[ F(0) = \frac{N + 2}{2N} A^{2N(N + 2)} B^{(N - 2)(N + 2)} . \]

We now estimate \( F(v) \) from below:
\[
F(v) \geq J(v) = \frac{1}{2} \int_\Omega |\nabla v|^2 dx - \frac{1}{2^*} \int_\Omega Q v^{2^*}_+ dx - \int_\Omega f v dx
\]
\[
\geq \frac{1}{2} \int_\Omega |\nabla v|^2 dx - \frac{1}{2^*} \int_\Omega Q v^{2^*}_+ dx - \|f\| \|v\|_2.
\]

We now observe that
\[
\int_\Omega |\nabla v|^2 dx \geq \lambda_2 \int_\Omega v^2 dx
\]
for every \( v \in E^+ \). Since \( \int_\Omega v dx = 0 \), we can use the Sobolev inequality to obtain
\[
F(v) \geq \frac{1}{2} \int_\Omega |\nabla v|^2 dx - \frac{Q_M}{2^*} S^{-N/2} \left( \int_\Omega |\nabla v|^2 dx \right)^{2^*/2}
\]
\[
- \|f\|_2 \lambda_2^{1/2} \left( \int_\Omega |\nabla v|^2 dx \right)^{1/2}.
\]

Letting \( q = \|v\|_2 \) we derive from the above estimate
\[
F(v) \geq \frac{q^2}{2} - \frac{Q_M}{2^*} S^{-N/2} q^{2^*} - \|f\|_2 \lambda_2^{1/2} q
\]
\[
= q^z \left( \frac{q}{2} - \frac{Q_M}{2^*} S^{-N/2} q^{2^* - 1} - \|f\|_2 \lambda_2^{1/2} \right) = q j(q).
\]

Since \( j(q) \) achieves its maximum at
\[
q_0 = \left( \frac{N}{(N + 2) Q_M} \right)^{(N - 2)/4} S^{N/4}
\]
we see that

\[ F(v) \geq q_0 \left( \frac{1}{2} - \frac{N - 2}{2(N + 2)} \right) - \left\| f \right\|_2 \lambda_{\alpha}^{1/2} \]

\[ = q_0 \left( \frac{2q_0}{N + 2} - \left\| f \right\|_2 \lambda_{\alpha}^{1/2} \right). \]

We now assume that

\[ \left\| f \right\|_2 \leq \lambda_{\alpha}^{1/2} \left( \frac{N}{N + 2} \right)^{(N - 2)/4} S^{N/4} \]

and

\[ - \int_{\Omega} f(x) \, dx \leq \left( \frac{2N}{N + 2} \right)^{(N + 2)/2N} N^{-(N + 2)/2N} Q_m^{-1/4} S^{(N + 2)/4} \left( \int_{\Omega} Q(x) \, dx \right)^{1/2}. \]

As an immediate consequence of (6.5), (6.6), (6.7) and (6.8) we can state the following lemma:

**Lemma 6.1.** Suppose that (6.7) and (6.8) hold. Then \( F(v) > F(0) \) for all \( v \in E^+ \) such that \( \| v \| = q_0 \), and moreover

\[ F(0) < \frac{S^{N/2}}{N Q_m^{N - 2)/2}}. \]

We can now formulate the existence result in the case \( \lambda = 0 \).

**Theorem 6.2.** Suppose that (6.7) and (6.8) hold. Then problem (6.1) has a solution.

**Proof.** It follows from Lemma 6.1 that

\[ m = \inf_{v \in H_0^1(\Omega)} F(v) < \frac{S^{N/2}}{N Q_m^{(N - 2)/2}}. \]

Let \( \{ v_n \} \) be a minimizing sequence for (6.9). Since \( \{ v_n \} \) is bounded in \( H^1(\Omega) \), we may assume that \( v_n \to v_0 \) in \( H^1(\Omega) \) and \( v_n \to v_0 \) in \( L^q(\Omega) \) for every \( 2 \leq q < 2^* \). By the low semicontinuity of norm with respect to
weak convergence we have
\[ \|v_0\| \leq \liminf_{n \to \infty} \|v_n\| \leq \varrho_0. \]

Estimate (6.6) shows that \( F \) is bounded away from 0 near the boundary of \( B_{v_0}(0) \) for \( f \) small enough. On the other hand \( m \leq F(0) \) and \( F(0) \) is close to 0 for small \( f \). Therefore we can always assume that the minimizing sequence \( \{v_n\} \) is contained in the interior of the ball \( B_{v_0}(0) \), say \( v_n \subset cB_{v_0}(0) \). It then follows from the Ekeland variational principle that
\[ F(v_n) \to m \quad \text{and} \quad F'(v_n) \to 0. \]

Since \( F'(v_n) \to 0 \) means that \( f' + t(v_n) \to 0 \) we obtain
\[ \frac{1}{2} \int_{\Omega} |\nabla v_n|^2 \, dx - \frac{1}{2^*} \int_{\Omega} Q(x)(v_n + t(v_n))^{2^*-1} \, dx - \int_{\Omega} f(v_n + t(v_n)) \, dx = m + o(1) \]
and also by (6.3) we have
\[ \int_{\Omega} |\nabla v_n|^2 \, dx - \int_{\Omega} Q(x)(v_n + t(v_n))^{2^*-1} v_n - \int_{\Omega} f v_n \, dx = o(1). \]

Since \( v_0 \) is a weak solution of (6.1) we have
\[ \int_{\Omega} (|\nabla v_0|^2 - Q(x)(v_0 + t(v_0))^{2^*-1} v_0 - f v_0) \, dx = 0 \]
and
\[ \int_{\Omega} (Q(x)(v_0 + t(v_0))^{2^*-1} + f) \, dx = 0. \]

We need to show that \( v_n \to v_0 \) in \( H^1(\Omega) \). As in [12] we show that \( \lim_{n \to \infty} t(v_n) = t(v_0) \). We set \( w_n = v_n - v_0 \). By the Brézis-Lieb Lemma, we have
\[ F(v_0) + \frac{1}{2} \int_{\Omega} |\nabla w_n|^2 \, dx - \frac{1}{2^*} \int_{\Omega} Q(x)(w_n)^{2^*} \, dx = m + o(1) \]
and
\[ \int_{\Omega} |\nabla w_n|^2 \, dx - \int_{\Omega} Q(x)(w_n)^{2^*} \, dx - \int_{\Omega} Q(x)(v_0 + t(v_0))^{2^*} \, dx + \int_{\Omega} f v_0 + t(v_0) \, dx = o(1). \]
It then follows from (6.10) and (6.11) that
\[ \int_{\Omega} |\nabla w_n|^2 \, dx - \int_{\Omega} Q(x)(w_n)^2 \, dx = o(1). \]
Hence by (6.12) we get
\[ F(v_0) + \frac{1}{N} \int_{\Omega} |\nabla w_n|^2 \, dx = m + o(1). \]
Since \( F(v_0) \geq m \), this implies that \( \int_{\Omega} |\nabla w_n|^2 \, dx = o(1) \) and consequently \( v_n \to v_0 \) in \( H^1(\Omega) \). ■

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