Connections on Distributional Bundles.

DANIEL CANARUTTO(*)

ABSTRACT - A general approach to the geometry of distributional bundles is presented. In particular, the notion of connection on these bundles is studied. A few examples, relevant to quantum field theory, are discussed.

Introduction.

The notion of smoothness introduced by Frölicher [Fr] provides a general setting for calculus in functional spaces [FK, KM] and differential geometry in functional bundles [JM, KM, CK, MK]. An important aspect of that approach is that the essential results can be formulated in terms of finite-dimensional spaces and maps, without heavy involvement in infinite-dimensional topology and other intricated questions. In particular, the notion of a smooth connection on a functional bundle has been applied in the context of the «covariant quantization» approach to Quantum Mechanics [JM, CJM].

In a previous paper [C00a] I applied these ideas to the differential geometry of certain bundles whose fibres are distributional spaces, more specifically scalar-valued generalized half-densities. The main purpose of the present paper is to extend those results to the general case of the bundle of generalized «tube» sections of a 2-fibred «classical» (i.e. finite dimensional) bundle; basic notions of standard differential geometry – such as tangent space, jet space, connection and curvature – are intro-

E-mail: canarutto@dma.unifi.it; http://www.dma.unifi.it/~canarutto
duced for this case; adjoint connections and tensor product connections are shown to exist. Furthermore, a suitable connection on the underlying classical bundle is shown to yield a connection on the corresponding distributional bundle; some particularly important cases are the vertical bundle and its tensor algebra, which turn out to be closely related to the notion of adjoint connection. Finally, I consider a few examples which are relevant in view of applications to quantum field theory: the «Dirac connection» on the bundle of 1-electron states for a given observer, and the connections induced on the phase-distributional bundles describing electron and photon fields.

1. Generalized sections.

Let \( p : Y \to \mathcal{Y} \) be a real or complex classical vector bundle, namely a finite-dimensional vector bundle over the Hausdorff paracompact smooth real manifold \( \mathcal{Y} \). Moreover assume that \( \mathcal{Y} \) is oriented, let \( n := \dim \mathcal{Y} \), and denote the positive component of \( \wedge^n \mathcal{T} \mathcal{Y} \) by \( \mathcal{V} \mathcal{Y} := (\wedge^n \mathcal{T} \mathcal{Y})^+ \). Then \( \mathcal{V} \mathcal{Y} \to Y \) is a semi-vector bundle [C98, C00a, C00b, CJM], as well as its dual bundle \( \mathcal{V}^* \mathcal{Y} \equiv (\wedge^n \mathcal{T}^* \mathcal{Y})^+ \to Y \) which is called the bundle of positive densities on \( \mathcal{Y} \).

Let \( \mathcal{Y}_0 \equiv \mathcal{O}_0(\mathcal{Y}, \mathcal{V}^* \mathcal{Y} \otimes \mathcal{Y}^* \mathcal{Y}) \) be the vector space of all smooth sections \( \mathcal{Y} \to \mathcal{V}^* \mathcal{Y} \otimes \mathcal{Y}^* \mathcal{Y} \) which have compact support. A topology on this space can be introduced by a standard procedure [Sc]; its topological dual will be denoted as \( \mathcal{Y} \equiv \mathcal{O}(\mathcal{Y}, \mathcal{Y}) \) and called the space of generalized sections, or distribution-sections of the given classical bundle, while \( \mathcal{Y}_0 \) is called the space of test sections. In particular, a sufficiently regular ordinary section \( s : \mathcal{Y} \to \mathcal{Y} \) can be seen as a generalized section by the rule

\[
\langle s, u \rangle = \int (s(y)u(y)), \quad u \in \mathcal{Y}_0.
\]

On turn, \( \mathcal{Y} \) has a natural topology [Sc], and its subspace \( \mathcal{Y}_0^\circ \equiv \mathcal{O}_0(\mathcal{Y}, \mathcal{Y}) \) of all smooth sections with compact support is dense in it. Some particular cases of generalized sections are that of \( r \)-currents \( \mathcal{Y} \equiv \wedge^r \mathcal{T} \mathcal{Y}, \quad r \in \mathbb{N} \) and that of half-densities \(^1(\mathcal{Y} \equiv (\mathcal{V}^* \mathcal{Y})^{1/2})\).

\(^1\) The «square root» bundle \((\mathcal{V}^*)^{1/2}\) is characterized, up to isomorphism, by \((\mathcal{V}^*)^{1/2} \otimes (\mathcal{V}^*)^{1/2} \cong \mathcal{V}^*\).
The topological dual of $\mathcal{Y}^*$ is $\mathcal{Y}^* \equiv \mathcal{O}(\mathcal{Y}, \mathcal{V}^* \otimes \mathcal{Y}^*)$, that is the space of generalized $\mathcal{Y}^*$-valued densities on $\mathcal{Y}$, or the adjoint space of $\mathcal{Y}$.

**Remark.** If $\theta \in \mathcal{Y}$ and $\phi \in \mathcal{Y}^*$ then, possibly, the contraction $\langle \theta, \phi \rangle$ may be defined even if neither one is a test section.

Generalized sections can be naturally restricted to any open subset $\mathcal{Y} \subset \mathcal{Y}$ of the base manifold, namely there is a natural linear projection $\mathcal{Y} \to \mathcal{Y} \equiv \mathcal{O}(\mathcal{Y}, \mathcal{Y})$, where $\mathcal{Y} := p^{-1}(\mathcal{Y})$. Accordingly, if $(b_i)$ is a local frame of $\mathcal{Y}$, a generalized section $\xi \in \mathcal{Y}$ has the local expression $\xi = \xi^i b_i$ with $\xi^i \in \mathcal{O}(\mathcal{Y}, \mathcal{C})$.

There is no inclusion $\mathcal{Y} \hookrightarrow \mathcal{Y}$, since elements in $\mathcal{Y}$ cannot be naturally extended to generalized sections on $\mathcal{Y}$ (such extension may not exist at all). However, a gluing property holds: if $\{\mathcal{Y}_i\}$ is a covering of $\mathcal{Y}$ and $\{\theta_i \in \mathcal{Y}_i\}$ is a family of generalized sections such that $\theta_i$ and $\theta_j$ coincide on $\mathcal{Y}_i \cap \mathcal{Y}_j$, then there exists a unique $\theta \in \mathcal{Y}$ whose restriction to $\mathcal{Y}_i$ coincides with $\theta_i$.

Let $p'\colon \mathcal{Y}' \to \mathcal{Y}$ be another classical vector bundle and $q : \mathcal{Y} \to \mathcal{Y}'$ a smooth fibred isomorphism over the diffeomorphism $q : \mathcal{Y} \to \mathcal{Y}'$; namely, $p' \circ q = q \circ p$. Clearly, $q$ determines a natural isomorphism between the spaces of ordinary sections of the two bundles; one easily sees that this restricts to an isomorphism of the corresponding spaces of test sections, and extends to an isomorphism $q^* : \mathcal{Y} \to \mathcal{Y}'$. One also has the adjoint construction. It is not difficult to see that $q^*$ turns out to be a continuous isomorphism (the proof is essentially the same as given in [C00a] for the particular case of scalar-valued half-densities).

### 2. F-smoothness in distributional spaces.

Let $I \subset \mathbb{R}$ be an open interval. A curve $\alpha : I \to \mathcal{Y}$ is said to be F-smooth (Frölicher-smooth) if the map

$$\langle \alpha, u \rangle : I \to \mathbb{C} : t \mapsto \langle \alpha(t), u \rangle$$

is smooth for every $u \in \mathcal{Y}_0$. Accordingly, a function $\phi : \mathcal{Y} \to \mathbb{C}$ is called F-smooth if $\phi \circ \alpha : I \to \mathbb{C}$ is smooth for all F-smooth curve $\alpha$, and a map $\Phi : \mathcal{Y} \to \mathcal{W}$ between any two distributional spaces is called F-smooth if $\Phi \circ \Phi \circ \alpha$ is smooth for all F-smooth $\alpha : I \to \mathcal{Y}$ and $\phi : \mathcal{W} \to \mathbb{C}$. 
It can be proved [Bo] that a function \( f : M \to \mathbb{R} \), where \( M \) is a classical manifold, is smooth (in the standard sense) iff the composition \( f \circ c \) is a smooth function of one variable for any smooth curve \( c : I \to M \). Thus one has a unique notion of smoothness based on smooth curves, including both classical manifolds and distributional spaces. This is convenient for dealing with smoothness relatively to product spaces such as \( M \times \mathcal{Y} \); moreover, one has a natural notion of smoothness for maps \( M \to \mathcal{Y} \) and \( \mathcal{Y} \to M \). Hence, one may simply write smooth for \( F \)-smooth.

Let \( \mathcal{C}_\mathcal{Y} \) be the set of all \( F \)-smooth curves in \( \mathcal{Y} \); take any \( i \in \mathbb{N} \cup \{ 0 \} \) and consider the following binary relation in \( \mathbb{R} \times \mathcal{C}_\mathcal{Y} \):

\[
(t, a) \sim_i (s, b) \iff D^k(\alpha, u)(t) = D^k(\beta, u)(s) \quad \forall u \in \mathcal{Y}_0, \ k = 0, \ldots, i.
\]

Then clearly \( \sim_i \) is an equivalence relation; the quotient

\[
T_i \mathcal{Y} := \mathcal{C}_\mathcal{Y} / \sim_i
\]

will be called the tangent space of order \( i \) of \( \mathcal{Y} \). The equivalence class of \( (t, a) \in \mathcal{C}_\mathcal{Y} \) will be denoted by \( \tilde{a}_t \). Obviously, \( T_i \mathcal{Y} \) is a fibred set over \( \mathcal{Y} \); the fibre over some \( \lambda \in \mathcal{Y} \) will be denoted by \( T_i \lambda \). In particular

\[
T^0 \mathcal{Y} = \mathcal{Y}.
\]

The set \( T \mathcal{Y} := T^1 \mathcal{Y} \) is called simply the tangent space of \( \mathcal{Y} \), and \( \tilde{\alpha}(t) := \tilde{\alpha}_t(\alpha(t)) \) is called the tangent vector of \( \alpha \) at \( \alpha(t) \). Any element in \( T \mathcal{Y} \) can be represented as \( \tilde{\alpha}(0) \), for a suitable curve \( \alpha \) defined on a neighbourhood \( I \) of \( 0 \). It is not difficult to see that there is a natural isomorphism

\[
\mathcal{Y} \times \mathcal{Y} \to T \mathcal{Y} : (\lambda, \mu) \mapsto \tilde{\alpha}(\lambda + t\mu)_{t=0}.
\]

**Proposition 2.1.** Let \( \mathfrak{A} \) and \( \mathfrak{B} \) be smooth spaces (each one is either a classical manifold or a distributional space) and \( \Phi : \mathfrak{A} \to \mathfrak{B} \) a smooth map. Then there exists a unique smooth map \( T\Phi : T\mathfrak{A} \to T\mathfrak{B} \), called the tangent prolongation of \( \Phi \), such that for every smooth curve \( \alpha : I \to \mathfrak{A} \) one has

\[
\tilde{\Phi}_\alpha(t) = T\Phi(\tilde{\alpha}(t)), \quad t \in I.
\]

The proof of this non-trivial statement is omitted because it is essentially similar to that of the particular case considered in [C00a]. It is not difficult to see that tangent prolongations behave naturally in terms of any compositions.
3. Distributional bundles.

The basic classical geometric setting underlying distributional bundles is the following. One considers a classical 2-fibred bundle

\[ V \xrightarrow{q} E \xrightarrow{q} B, \]

where \( q : V \rightarrow E \) is a complex (or real) vector bundle, and the fibres of the bundle \( E \rightarrow B \) are smoothly oriented. Moreover, one assumes that \( q \circ q : V \rightarrow B \) is also a bundle (not a vector bundle in general), and that for any sufficiently small open subset \( X \subset B \) there are bundle trivializations

\[ (q, y) : E_X \rightarrow X \times Y, \quad (q \circ q, y) : V_X \rightarrow X \times Y \]

(here \( E_X := q^{-1}(X) \) and the like) with the following projectability property: there exists a surjective submersion \( p : Y \rightarrow Y \) such that the diagram

\[
\begin{array}{ccc}
V_X & \xrightarrow{(q \circ q, y)} & X \times Y \\
\downarrow q & & \downarrow_{1_X \times p} \\
E_X & \xrightarrow{(q, y)} & X \times Y \\
\end{array}
\]

commutes; this implies that \( Y \rightarrow Y \) is a vector bundle, not trivial in general.

The above conditions are easily checked to hold in many cases which are relevant for physical applications (as in the cases considered in § 11 and § 12). In particular, the above conditions hold if \( V = E \times W \) where \( W \rightarrow B \) is a vector bundle, if \( V = VE \) (the vertical bundle of \( E \rightarrow B \)) and if \( V \) is any component of the tensor algebra of \( VE \rightarrow E \).

Let \( n \) be the dimension of the fibres of \( E \rightarrow B \). The orientation requirement implies that \( ^n V E \rightarrow E \) is a trivializable bundle with smoothly oriented fibres, and one has the smooth bundle \( ^n V E := (\wedge^n V^* E)^\ast \rightarrow E \). Then for each \( x \in B \) one may consider the distributional space \( \mathfrak{v}_x := \mathcal{Q}(E_x, V_x) \), and obtains the fibred set

\[ \rho : \mathfrak{v} \equiv \mathcal{Q}_B(E, V) := \coprod_{x \in B} \mathfrak{v}_x \rightarrow B. \]

For any two classical local bundle trivializations \((q, y)\) and \((q \circ q, y)\) as
above, let
\[ Y : \mathcal{V}_X = \rho^{-1}(X) \to Y = \mathcal{O}(\mathcal{V}, Y), \]
\[ Y_x := (y_x)_x, \quad x \in X. \]
Then \((\rho, Y) : \mathcal{V}_X \to X \times Y\) is a local bundle trivialization of \(\mathcal{V} \to B\). Moreover, if \((q, q') : E_X \to X' \times Y'\) and \((q \circ q, q') : V_X \to X' \times Y'\) are two other classical bundle trivializations related by the same projectability property, then \((\rho, Y') \circ (\rho, Y)^{-1} : X \cap X' \times Y \to X \cap X' \times Y'\) is \(F\)-smooth and linear. Hence, suitable classical bundle atlases on \(V \to B\) and \(E \to B\) determine a linear \(F\)-smooth bundle atlas on \(\mathcal{V} \to B\), which is said to be an \(F\)-smooth distributional bundle\(^2\). Clearly, \(\mathcal{V}\) turns out to be an \(F\)-smooth space in a natural way: a curve \(\alpha : I \to \mathcal{V}\) is defined to be \(F\)-smooth if \((\rho, Y) \circ \alpha\) is such for any local \(F\)-smooth trivialization; in general, the \(F\)-smoothness of any map from or to \(\mathcal{V}\) is equivalent to the \(F\)-smoothness of its local trivialized expressions.

If \(\alpha\) is \(F\)-smooth then it is natural to set
\[ T((\rho, Y) \circ \alpha) = (T(\rho \circ \alpha), T(Y \circ \alpha)) : I \times \mathbb{R} \to TX \times TY. \]
One says that two \(F\)-smooth curves are first-order equivalent at some point if their trivialized expressions are such; in this way one obtains the definition of the tangent space \(T\mathcal{V}\). Obviously, this is a fibred set over \(\mathcal{V}\); a local bundle trivialization \((\rho, Y)\) of \(\mathcal{V}\) yields the local bundle trivialization
\[ T(\rho, Y) : T\mathcal{V}_X \to T(X \times Y) = TX \times TY, \]
and the transition maps between two induced trivializations are \(F\)-smooth and linear. Hence \(T_\mathcal{V} : T\mathcal{V} \to \mathcal{V}\), the tangent bundle of \(\mathcal{V}\), is an \(F\)-smooth vector bundle. One has another \(F\)-smooth bundle with the same total \(F\)-smooth space, namely
\[ T\rho : T\mathcal{V} \to TB : \partial \alpha \mapsto \partial(q \circ \alpha). \]
Moreover one has the \textit{vertical subbundle}
\[ V\mathcal{V} := \text{Ker } T\rho \subset T\mathcal{V}, \]
the natural identification \(V\mathcal{V} = \mathcal{V} \times \mathcal{V}_B\) and the exact sequence over \(\mathcal{V}\)
\[ 0 \to V\mathcal{V} \to T\mathcal{V} \to \mathcal{V} \times TB \to 0. \]

\(^2\) Not every trivialization of a distributional bundle derives from trivializations of the underlying classical 2-fibred bundle.
The subbundle of $T^* B \otimes T \mathfrak{V}$ which projects over the identity of $TB$ is called the first jet bundle, denoted by $J \mathfrak{V} \to \mathfrak{V}$. This is an affine bundle over $\mathfrak{V}$, with «derived» vector bundle $T^* B \otimes \mathfrak{V}$. The restriction of $T^* \rho \otimes T(\rho, \gamma)$ is a local bundle trivialization which is denoted by

$$J(\rho, \gamma) : J \mathfrak{V} \to J(X \times \mathfrak{Y}) \equiv \mathfrak{Y} \times (T^* X \otimes \mathfrak{Y}).$$

If $x \equiv (x^a) : X \to \mathbb{R}^m$ is a coordinate chart then one has the fibred charts

$$(x, \gamma) : \mathfrak{V} \to \mathbb{R}^m \times \mathfrak{Y},$$

$$(x^a, \gamma, \dot{x}^a, \dot{\gamma}) := T(x, \gamma) : T \mathfrak{V} \to \mathbb{R}^m \times \mathfrak{Y} \times \mathbb{R}^m \times \mathfrak{Y},$$

$$(x^a, \gamma, \gamma^a) := J(x, \gamma) : J \mathfrak{V} \to \mathbb{R}^m \times \mathfrak{Y} \times (\mathbb{R}^m \otimes \mathfrak{Y}).$$

Tangent prolongations of F-smooth maps involving $\mathfrak{V}$ can be expressed through local trivializations; in particular, if $\sigma : B \to \mathfrak{V}$ is an F-smooth section, then $T \sigma : TB \to T \mathfrak{V}$ projects over the identity of $TB$, so that it can be viewed as a section $j \sigma : B \to J \mathfrak{V}$. Setting $\sigma^\gamma := \gamma \circ \sigma : B \to \mathfrak{Y}$ one has

$$(x^a, \gamma, \dot{x}^a, \dot{\gamma}) \circ T \sigma = T \sigma^\gamma = (x^a, \sigma^\gamma, \dot{x}^a, \dot{\gamma}^a \partial_a \sigma^\gamma),$$

$$(x^a, \gamma, \gamma^a) \circ j \sigma = J \sigma^\gamma = (x^a, \gamma^a, \partial_a \sigma^\gamma).$$

For maps $f : \mathfrak{V} \to \mathbb{R}$ one introduces the notation

$$\partial \gamma f := V f \circ (1 \mathfrak{V} \times (\rho, \gamma)^{-1}) : \mathfrak{V} \times \mathfrak{Y} \to \mathbb{R},$$

and obtains the local coordinate expression

$$df := \text{pr}_1 \circ T f = \partial_a f dx^a + (\partial \gamma f) \circ d \gamma.$$

**Remark.** If $\tilde{Y} \subset \mathfrak{Y}$ is an open subset such that $\tilde{Y} := p^{-1}(\tilde{Y})$ is trivializable, and $(\gamma', \mathfrak{Y}^d) : \tilde{Y} \to \mathbb{R}^n \times \mathbb{R}^p$ is a linear bundle chart, then $\sigma^\gamma$ has a coordinate expression whose components are scalar-valued distributions $\sigma^a \in \mathcal{O}_Y(\tilde{Y}, \mathbb{R})$.

4. **F-smooth fibred morphisms.**

Let $V' \to E' \to B'$ another 2-fibred bundle with the same properties, and $\rho' : \mathfrak{V}' \to B'$ the induced distributional bundle. Let moreover
\( \Phi : \mathfrak{V} \rightarrow \mathfrak{V}' \) be a fibred \( F \)-smooth map over the smooth map \( \phi : \mathcal{B} \rightarrow \mathcal{B}' \). Then, similarly to the classical case, the tangent prolongation

\[ T\Phi : T\mathfrak{V} \rightarrow T\mathfrak{V}' \]

is a linear fibred morphism over \( \Phi \) and a fibred morphism over \( T\phi : TB \rightarrow TB' \). setting \( \Phi' := \gamma' \circ \Phi : \mathfrak{V} \rightarrow \mathfrak{Y}' \) one has \(^{(2)}\)

\[ (x', \gamma', \dot{x}', \dot{\gamma}) \circ T\Phi = \left( \phi^n, \Phi' , \dot{x} \partial_n \phi^X, \dot{\gamma} \partial_n \Phi' + \partial_n \Phi' \circ \dot{\gamma} \right). \]

If moreover \( \phi \) is a diffeomorphism, then the restriction of \( \phi \circ T\Phi \) determines a fibred morphism \( J\Phi : J\mathfrak{V} \rightarrow J\mathfrak{V}' \) over \( \Phi \).

If \( \Phi \) is linear over \( \phi \), then one writes

\[ \Phi' = \partial_n \Phi' , \quad \Phi' \circ (\gamma, \gamma')^{-1} : X \rightarrow \text{Lin}(\mathfrak{Y}, \mathfrak{Y}'), \]

which is analogous to the matrix expression of a linear morphism in finite-dimensional case.

Let now \( \varphi : \mathfrak{V} \rightarrow \mathfrak{V}' \) be a classical linear isomorphism over the fibred diffeomorphism \( \varphi : \mathfrak{E} \rightarrow \mathfrak{E}' \), which in turn is projectable over the diffeomorphism \( \phi : \mathcal{B} \rightarrow \mathcal{B}' \). Then one has the induced linear isomorphism \( \Phi := \varphi \circ \mathfrak{V} \rightarrow \mathfrak{V}' \) over \( \mathcal{B} \). In the domain of a local coordinate chart one has \(^{(4)}\)

\[ (\Phi \lambda)^A = (\Phi' \lambda')^A \circ \frac{\partial}{\partial \lambda'}, \quad \lambda \in \mathfrak{V}, \]

\[ \left( \partial_a \Phi' \lambda' ight)^A = \left( \partial_a \varphi' \lambda' \circ \frac{\partial}{\partial \lambda'} \right) \left( \partial_a \varphi' \lambda' \circ \frac{\partial}{\partial \lambda'} \right) \left( \partial_a \varphi' \lambda' \circ \frac{\partial}{\partial \lambda'} \right) \left( \partial_a \varphi' \lambda' \circ \frac{\partial}{\partial \lambda'} \right) \left( \partial_a \varphi' \lambda' \circ \frac{\partial}{\partial \lambda'} \right), \]

where back pointing arrows indicate the inverse maps. By using these formulas one can write down the coordinate expressions of \( T\Phi \) and \( J\Phi \).

As a special case, one also gets the transformation formulas in \( T\mathfrak{V} \) and \( J\mathfrak{V} \) between any two charts induced by classical charts; a detailed treatment of these aspects lies outside the scope of a short paper and will be exposed in a future survey paper.

When \( \mathcal{V} = \mathcal{G}E, \mathcal{V}' = \mathcal{E}'E \) and \( \varphi \) is a fibred diffeomorphism over \( \phi \), then one has the special case \( \varphi = \mathcal{G} \varphi, \) which extends to any component of the tensor algebra of \( \mathcal{E}E \rightarrow \mathcal{E} \). In particular, one is interested in the bun-

\(^{(2)}\) These partial derivatives are naturally defined as a consequence of proposition 2.1.

\(^{(4)}\) The proof of the second formula is not difficult but somewhat delicate, as one must take carefully into account the various involved compositions.
Connections on distributional bundles

dles of scalar $q$-densities, where $q$ is a rational number, namely in the distributional bundles $\mathcal{O}_{df}(E, C \otimes V^qE)$ where $V^qE \equiv (V^qE)^q$ and the like. One gets

$$\mathcal{C}_a \phi^Y(\lambda) = (\partial_1 \lambda \circ \bar{\phi}) \mathcal{C}_a \bar{\phi} \cdot (\partial_a \phi^a \circ \bar{\phi}) |V\bar{\phi}|^q + q(\lambda \circ \bar{\phi}) \cdot |V\bar{\phi}|^q (\partial_a \phi^a \circ \bar{\phi}) \partial_a \phi^a \circ \bar{\phi} \cdot \bar{\phi}$$

where $|V\bar{\phi}|$ denotes the vertical Jacobian determinant of $\bar{\phi}$.

5. Distributional connections.

Similarly to the standard finite-dimensional case, a connection on the distributional bundle $\mathbf{V}$ is defined to be an $F$-smooth section

$$\mathcal{C} : \mathbf{V} \to J \mathbf{V}.$$ 

In the domain $X \subset B$ of a local bundle chart $(x, y) : \mathbf{V}_x \to \mathbb{R}^m \times \mathbf{Y}$ one has the local expression

$$\mathcal{C}_a^Y := Y_a \circ \mathcal{C} : \mathbf{V} \to \mathbf{Y}.$$ 

The existence of global connections then follows from standard arguments, using the paracompactness of $B$.

Basically, one deals with linear connections, that is connections $\mathcal{C}$ which are linear morphisms over $B$. Then one writes

$$\mathcal{C}_a^Y = \mathcal{C}_a^Y \circ Y, \quad \mathcal{C}_a^Y : X \to \text{End}(\mathbf{Y}).$$

If $\mathcal{C}_a^Y$ is the expression of $\mathcal{C}$ in a different fibred chart $(x', y')$ over the same domain $X$, then

$$\mathcal{C}_a^{y'^Y} = \partial_a \mathcal{K} \cdot (\partial_a \mathcal{Y} \circ \mathcal{K}^Y \circ \mathcal{C}_a^Y) \circ \mathcal{Y}^{y'^Y},$$

where

$$\mathcal{K} := (k, \mathcal{Y}^{y'^Y}) : (x', y') \circ (x, y)^{-1} : \mathbb{R}^m \times \mathbf{Y} \to \mathbb{R}^m \times \mathbf{Y}$$

denotes the transition map.

As in the finite-dimensional case, a connection yields a number of structures (whose assignment is actually equivalent to that of the connection itself). First, $\mathcal{C}$ can be viewed as a linear map $\mathbf{V} \otimes TB \to T \mathbf{V}$. 

and $(\pi, T\rho) \circ \overline{\xi}$ is the identity of $\mathfrak{v}_B \times TB$. The image

$$H_\xi \mathfrak{v} := \overline{\xi}(\mathfrak{v}_B \times TB)$$

is a vector subbundle of $T\mathfrak{v} \to \mathfrak{v}$, with $m$-dimensional fibres; the restriction of $\overline{\xi} \circ (\pi, T\rho)$ is the identity of $H_\xi \mathfrak{v}$. If $v : B \to TB$ is a smooth vector field, then $\overline{\xi} : \mathfrak{v} \to T\mathfrak{v}$ is an $F$-smooth vector field, called its \textit{horizontal lift}, with coordinate expression

$$\dot{x}^a \circ \overline{\xi}_v = v^a, \quad \dot{y} \circ \overline{\xi}_v = v^a \xi_y^a.$$

One also has the complementary map

$$\Omega := 1 - \overline{\xi} : T\mathfrak{v} \to \nabla \mathfrak{v} \equiv \mathfrak{v}_B \times \mathfrak{v},$$

so that the map $(\overline{\xi} \circ (\pi, T\rho), \Omega)$ determines the decomposition

$$T\mathfrak{v} = H_\xi \mathfrak{v} \oplus \nabla \mathfrak{v}.$$

Let $\sigma : B \to \mathfrak{v}$ be an $F$-smooth section. The \textit{covariant derivative} of $\sigma$ is defined to be the linear morphsim over $B$

$$\nabla \sigma := \nabla \overline{\xi} \sigma := \text{pr}_2 \circ \Omega \circ T\sigma : TB \to \mathfrak{v}.$$  

If $v : B \to TB$ is a vector field, then one also writes $\nabla_v \sigma := \nabla \sigma \circ v$. The local coordinate expression of the covariant derivative is

$$(\nabla \sigma)^Y := Y \circ \nabla \sigma = \dot{x}^a (\partial_a \sigma^Y - \xi^Y_a \circ \sigma).$$

The \textit{curvature tensor} of a linear connection $\overline{\xi}$ can be defined, as in the finite-dimensional case, as the section $\mathcal{R} : B \to \wedge^2 T^a B \otimes \text{End}(\mathfrak{v})$ given by

$$\mathcal{R}(u, v) := \nabla_u \nabla_v \sigma - \nabla_v \nabla_u \sigma - \nabla_{[u, v]} \sigma, \quad u, v : B \to TB, \sigma : B \to \mathfrak{v},$$

which has the local chart expression

$$\mathcal{R}^{Y} = \mathcal{R}_{ab}^{Y} \text{d}x^a \wedge \text{d}x^b = 2(\partial_0 \xi^a \xi^Y_a + \xi^a \xi^Y_a \circ \xi^Y_a \circ \sigma) \text{d}x^a \wedge \text{d}x^b.$$

A more general definition of curvature, valid also in the non-linear case, can be given in terms of the Frölicher-Nijenhuis bracket $[\text{FN}, \text{MK}, \text{MM}, \text{KMS}]$. First, one must define the Lie bracket of any two $F$-smooth vector fields $W, Z : \mathfrak{v} \to T\mathfrak{v}$. Using the canonical involution $s : TT\mathfrak{v} \to$
→ TT ψ, and TZ ⊲ W − s(TW ⊲ Z): ψ → VT ψ ≡ T ψ × T ψ, one sets

\[[W, Z] := \text{pr}_2(TZ ⊲ W − s(TW ⊲ Z)): ψ → T ψ,\]

which has the local expression

\[[W, Z]^w = W^b \partial_b Z^w − Z^b \partial_b W^w + \partial_Y Z^w ⋅ W^Y − \partial_Y W^w ⋅ Z^Y,\]

\[[W, Z]^\nu = W^b \partial_b Z^\nu − Z^b \partial_b W^\nu + \partial_Y Z^\nu ⋅ W^Y − \partial_Y W^\nu ⋅ Z^Y.\]

The Frölicher-Nijenhuis bracket of F-smooth tangent-valued forms ψ → ∧ T* B ⊗ T ψ can now be introduced by a straightforward extension of the standard definition, namely by the requirement that for decomposable forms one has

\[[a ⊲ W, ⊲ Z] = a ∧ ⊲ [W, Z] + a ∧ (W, β) ⊲ Z − (Z, a) ∧ ⊲ W +

\[+ (−1)^r(Z|a) ∧ dβ ⊲ W + (−1)^rα(W|β) ⊲ Z,\]

where α : B → ∧ T* B, β : B → ∧ T* B, and W, Z : B → T B.

If C : ψ → J ψ is an F-smooth connection then its curvature is defined to be

\[\mathcal{R} := −[C, C]: ψ → ∧^2 T* B ⊗ T ψ.\]

6. Adjoint connections.

The distributional bundle ψ* := O_B(E, V* ⊗ E* → B) → B is called the adjoint bundle of ψ → B; its fibre type is Y*, the adjoint of Y (§1).

An endomorphism A ∈ End(O) of an arbitrary distributional space O determines a dual endomorphism A′ ∈ End(O*) of the test space, defined by A′ u := u ⋅ A, that is (A′ u, φ) = ⟨u, Aφ⟩. Moreover it may happen that A′ can be extended to an endomorphism A* of the distributional completion O* of O; this possible extension is called the adjoint of A. This requirement is fulfilled, in particular, by the polynomial derivation operators [C01].

**Proposition 6.1.** Let the F-smooth connection C : ψ → J ψ be such that, in every local F-smooth chart (x, Y): ψ → X × Y, the
local expression $\xi^\gamma \colon TB \to \text{End}(\mathcal{Y})$ admits an adjoint $(\xi^\gamma)^* \colon TB \to \text{End}(\mathcal{Y}^*)$.

Then, there exists a unique $F$-smooth connection $\xi^\gamma \colon \mathcal{V}^* \to J\mathcal{V}^*$ such that $Je \circ (\xi^\gamma, \xi^\gamma) = 0$, where $e : \mathcal{V}_B \times \mathcal{V}^* \to B \times C : (\sigma, \lambda) \mapsto (\psi(\sigma), (\lambda), \sigma)$. Its chart expression is

$$(\xi^\gamma_{\alpha \gamma})^* = -((\xi^\gamma_{\alpha \gamma})^\star),$$

that is

$$(\nabla^\gamma_\alpha \lambda)^\gamma = \psi^* (\partial_{\alpha} \lambda^\gamma - (\xi^\gamma_{\alpha \gamma} \circ \lambda^\gamma) = \psi^* (\partial_{\alpha} \lambda^\gamma + \lambda^\gamma \circ (\xi^\gamma_{\alpha \gamma})).$$

Equivalently, $\xi^\gamma$ is determined by the requirement that

$$\psi. (\lambda, \sigma) = (\nabla^\gamma_\alpha \lambda, \alpha) + (\lambda, \nabla_v \sigma)$$

hold for all smooth sections $\lambda : B \to \mathcal{V}^*$ and $\sigma : B \to \mathcal{V}$, and for all vector field $v : B \to T_B$, whenever all contractions are well-defined.

**Proof.** Let $\xi^\gamma : \mathcal{V}^* \to J\mathcal{V}^*$ be any linear connection; denote by $z \equiv \text{pr}_2$ the (trivial) fibre coordinate on $B \times C \to B$, and observe that

$$Je \circ (\xi^\gamma, \xi^\gamma) : \mathcal{V}_B \times \mathcal{V}^* \to C \times T^B$$

has the chart expression

$$z_{\alpha} \circ Je \circ (\xi^\gamma, \xi^\gamma)(\sigma, \lambda) = (\lambda^\gamma, \xi^\gamma_{\alpha \gamma}(\sigma^\gamma)) + (\xi^\gamma_{\alpha \gamma}(\lambda^\gamma), \sigma^\gamma),$$

which holds for any $(\sigma, \lambda) \in \mathcal{V}_B \times \mathcal{V}^*$ whenever all contractions are well-defined. This expression vanishes iff $\xi^\gamma_{\alpha \gamma} = -((\xi^\gamma_{\alpha \gamma})^\star)$. If $s : B \to \mathcal{V}^* \subset \mathcal{V}$ is a section of the subbundle of test maps in $\mathcal{V}$, one has $\nabla_v s : B \to \mathcal{V}$ in general. For every $u : B \to \mathcal{V}^*$, the map

$$\nabla^\gamma_\alpha u : \mathcal{V}^*_B \to C : s \mapsto \langle s, u \rangle - \langle \nabla_v s, u \rangle$$

is linear continuous, hence $\nabla^\gamma_\alpha u : B \to \mathcal{V}^*$. Its chart expression is

$$\langle s, \nabla^\gamma_\alpha u \rangle = \psi^* (\partial_{\alpha} s^\gamma, u_{\gamma}) - \langle v^* (\partial_{\alpha} s^\gamma, u_{\gamma}) + \langle \psi^* (\xi^\gamma_{\alpha \gamma} \circ s^\gamma, u_{\gamma}) =$$

$$= \langle s^\gamma, v^* (\partial_{\alpha} u_{\gamma} + (\xi^\gamma_{\alpha \gamma})^\star u_{\gamma}) \rangle.$$ By continuity, the operation $\nabla^\gamma_\alpha$ can be extended to all sections $\lambda : B \to \mathcal{V}^*$, and is seen to define a covariant derivative. □
REMARK. The adjoint connection $\xi^*$ is not reducible to the sub-bundle $\nabla_0 \rightarrow B$.

REMARK. Similarly to the finite-dimensional case, a distributional connection $\xi$ determines connections on any tensor bundle over $B$ constructed from $\mathcal{V} \rightarrow B$. Together with its possible adjoint $\xi^*$, it determines connections on the tensor algebra of $\mathcal{V} \rightarrow B$ and its subspaces.

7. Connection induced by a classical connection.

In this section, I’ll show that a suitable underlying classical structure determines a connection on a distributional bundle (though not all distributional connections arise in this way).

Consider again the classical 2-fibred bundle $V \rightarrow E \rightarrow B$ as before. By $VV$ and $JV$ one denotes the vertical and jet spaces of $V$ relatively to base $B$, while vertical and jet spaces relatively to base $E$ will be denoted by $V_E V$ and $J_E V$.

A connection $\Gamma : V \rightarrow JV$ is said to be projectable if there is a connection $\Gamma : E \rightarrow JE$ such that the diagram

$$
\begin{array}{ccc}
V & \xrightarrow{\Gamma} & JV \\
\downarrow q & & \downarrow Jq \\
E & \xrightarrow{\Gamma} & JE
\end{array}
$$

commutes; moreover, $\Gamma$ is said to be linear if it is a linear morphism over $\nabla$.

Let $(x^a, y^i, y^A) : V \rightarrow \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^p$ be a local 2-fibred coordinate chart, linear over $(x^a', y^i') : E \rightarrow \mathbb{R}^m \times \mathbb{R}^n$; the coordinate expression of a linear projectable connection is then

$$
\Gamma = dx^a \otimes (\partial x_a + \Gamma^i_a \partial y_i + \Gamma^A_{aB} y^B \partial y_A),
$$

$$
\Gamma = dx^a \otimes (\partial x_a + \Gamma^i_a \partial y_i),
$$

with $\Gamma^i_a, \Gamma^A_{aB} : E \rightarrow \mathbb{R}$.

A smooth section $s : E \rightarrow V$ can be viewed as a section of a functional bundle, whose fibre over each $x \in M$ is the space of all smooth sections $\mathbb{E}_x \rightarrow V_x$; in the case when one considers local sections $E \rightarrow V$, these must be defined on a «tubelike» open subset of $E$. Moreover, this functional bundle can be viewed as a subbundle of $\mathcal{V} := \mathcal{O}_B(E, V) \rightarrow B$. 
Observe now that the above $\sigma$ can be viewed as the vertical-valued 0-form

$$ (1_V, \sigma) : V \to V \times \mathbb{E} V \equiv V \mathbb{E} V \subset TV, $$

which has the same coordinate expression $\sigma = \sigma^A \partial A$. One may also view $\Gamma$ as a projectable tangent-valued 1-form

$$ \Gamma : V \to T^* M \otimes TV \subset T^* V \otimes TV, $$

and consider the Frölicher-Nijenhuis bracket

$$ \lbrack \Gamma, \sigma \rbrack : V \to T^* V \otimes TV. $$

Actually, $\lbrack \Gamma, \sigma \rbrack$ turns out to be a basic vertical-valued form $V \mathbb{E} V$, as one immediately sees from its coordinate expression

$$ \lbrack \Gamma, \sigma \rbrack = (\partial_a \sigma A + \Gamma^i_a \partial_i \sigma A - \Gamma^A_a B \sigma B) \circ \partial a \otimes \partial A. $$

From this, it is clear that $\lbrack \Gamma, \sigma \rbrack$ can be extended to the case when $\sigma$ is a section $B \to \mathbb{V}$; moreover, it can be seen as the covariant derivative of a linear connection $\mathbb{C} : \mathbb{V} \to J \mathbb{V}$, which in the considered chart has the expression $\mathbb{C}_a^Y \sigma^Y = \Gamma^A_a B \sigma B - \Gamma^A_a \partial_i \sigma A$, that is

$$ (\mathbb{C}_a^Y )^I_B = \Gamma^A_a B - \partial_A \Gamma^A_a B \partial_i. $$

It is not difficult (just a somewhat intricated calculation) to check that the above expression transforms in the right way under the distributional bundle chart transformation induced by a classical chart transformation.

There is a natural relation between the curvature $R$ of $\Gamma$ and the curvature $\mathfrak{R}$ of the induced distributional connection $\mathbb{C}$. Actually one has $R = \circ \partial a \wedge \circ \partial b (R_{ab}^i \partial i + R_{ab}^A B \partial B) \circ \mathbb{C} \circ \mathbb{C}$ with

$$ R_{ab}^i = - \partial_a \Gamma_b^i + \partial_b \Gamma_a^i - \Gamma^i_a \partial_a \Gamma_b^i + \Gamma^i_b \partial_b \Gamma_a^i, $$

$$ R_{ab}^A B = - \partial_a \Gamma_b^A B + \partial_b \Gamma_a^A B - \Gamma^A_a B \partial_a \Gamma_b^A B + \Gamma^A_b \partial_b \Gamma_a^A B - \Gamma^A_a C \Gamma_a^B C + \Gamma^A_a C \Gamma_b^A C. $$

A direct calculation then gives

$$ \mathfrak{R}_{ab}^Y \sigma^Y = R_{ab}^A B \sigma B - R_{ab}^i \partial i \sigma A, $$
that is, simply, the Frölicher-Nijenhuis bracket
\[ \mathcal{R}(\sigma) = -[R, \sigma]. \]

8. Induced connection and horizontal transport.

In this section it will be showed that the notion of distributional connection induced by a classical connection arises in a natural and somewhat more intuitive way in terms of the parallel (i.e. horizontal) transports related to the two connections.

Let \( I \subset \mathbb{R} \) be an open neighbourhood of 0, and \( c : I \to B \) a smooth curve. For any \( v_0 \in V_{c(0)} \), one has, locally, a unique \( \Gamma \)-horizontal curve \( C_{v_0} : I \to V \), with \( I_0 \subset I \), such that \( C_{v_0}(0) = v_0 \). Moreover \( C_{v_0} \) is linear projectable over \( C_{v_0} : I \to E \), the horizontal \( \Gamma \)-lift of \( c \) starting from \( v_0 \equiv q(v_0) \).

If \( t \in I_0 \), so that the horizontal transport of \( v_0 \in V_{c(0)} \) to \( V_{c(t)} \) is defined, then there is a neighbourhood \( U \subset V_{c(0)} \) of \( v_0 \) such that the horizontal transport of every \( u \in U \) to \( V_{c(t)} \) is defined too (this is a consequence of the continuity of \( \Gamma \)). From a general result in the theory of ordinary differential equations, on the other hand, it follows that horizontal transport relatively to a linear connection on a vector bundle determines an isomorphism of any two fibres along any smooth curve connecting their base points. This is not the case of the presently considered setting, since \( V \to B \) is not a vector bundle in general. But the whole fibre \( V_2 \) is linearly sent to the whole fibre \( V_2 \), where \( C_{v_0}(t) \equiv v_t \in E_{c(t)} \); namely horizontal transport determines an isomorphism between these two fibres.

Momentarily forgetting these locality issues, assume horizontal transport along \( c \) determines, for all \( t \in I \), a fibred isomorphism \( C_t : V_{c(0)} \to V_{c(t)} \) over a diffeomorphism \( C_t : E_{c(0)} \to E_{c(t)} \). In other terms one has a 1-parameter family of fibred isomorphisms over a 1-parameter family of diffeomorphisms, denoted by

\[ C : I \times V_{c(0)} \to V, \quad C : I \times E_{c(0)} \to E. \]

Let now \( \lambda \in \mathcal{V}_{c(0)} = \mathcal{O}(E_{c(0)}, V_{c(0)}) \). Then

\[ (C_t)_* \lambda \in \mathcal{V}_{c(t)} = \mathcal{O}(E_{c(t)}, V_{c(t)}). \]
Namely, the classical horizontal transport locally determines a lift

$$C_x : I \times \nabla_{(0)} \to \nabla$$

of the base curve $c$. It can be seen that this is exactly the horizontal lift of $c$ relatively to the distributional connection $\nabla$ induced by $\Gamma$, namely that

$$\nabla : TM \times \nabla \to T \nabla : (\nabla c(0), \lambda) \mapsto \nabla(C_x \lambda)(0).$$

This result follows from a coordinate calculation; from the definition of a horizontal curve one has

$$\frac{\partial}{\partial t} C(t, v_0) = \dot{c}(t) \Gamma_a(v_0), \quad \frac{\partial}{\partial t} C^A_B(0, v_0) = \dot{c}(0) \Gamma^A_B(v_0),$$

while the induced horizontal curve $C_x \lambda : I \to \nabla$ can be written, by some abuse of language, as

$$(C_x \lambda)^A(t, y) = C^A_B(t, \overline{C}(t, y)) \lambda^B(\overline{C}(t, y)).$$

Calculating the tangent vector $\nabla(C_x \lambda) : I \to T \nabla$ is now a straightforward (though not immediate) task; using the relation between $\Gamma$ and $\nabla$ one gets the claimed result.

As already observed, in general this horizontal lift of $c$ through $\nabla$ may not exist for every $\lambda \in \nabla_{(0)}$, but it can defined for the restriction of $\lambda$ to a suitable open subset. Furthermore, the horizontal lift construction can be done whenever $\lambda$ has compact support $K \subset E_{(0)}$, by the following argument. For every $e \in E_{(0)}$ choose an open neighbourhood of $e$, $U \subset E_{(0)}$, such that the restriction of $\lambda$ to $U$ is horizontally transported over $c$ up to $t = t_U > 0$; from this open covering of $K$ select a finite subcovering $U$, and define $t_K := \min \{t_U, U \in U\}$. Then by a partition of unity subject to $U$ one has horizontal transport of $\lambda$ over $c$ up to $t = t_K$.

9. Induced connections and tensor products.

Consider another 2-fibred bundle $V' \to E' \to B$. The fibred tensor product of $V$ and $V'$ is defined to be the 2-fibred bundle

$$W := V \otimes_B V' \to F := E \times E' \to B.$$
Let \((x^a, y^i, y^A)\) and \((x^a, y^i, y^A)\) be 2-fibred coordinate charts on \(V\) and \(V^8\); then one has induced coordinates \((x^a, y^i, y^A, y^i, y^A, \omega^{AA})\) on \(W\), where
\[
\omega^{AA} \equiv y^A \otimes y^A\quad \text{i.e.}\quad \omega^{AA} \circ \otimes = y^A y^A,
\]
\[
\otimes : V \times V^8 \to W: (v, \nu^8) \mapsto v \otimes \nu^8.
\]

The jet prolongation \(\text{j} \otimes : \text{j}V \times \text{j}V^8 \to \text{j}W\) is characterized by the requirement that the diagram
\[
\begin{array}{ccc}
\text{j}V \times \text{j}V^8 & \xrightarrow{\text{j} \otimes} & \text{j}W \\
\downarrow_{(j\sigma, j\nu^8)} & & \uparrow_{j(\sigma \otimes \nu^8)} \\
B & & B
\end{array}
\]
commutes for any two sections \(\sigma : B \to V\), \(\sigma^8 : B \to V^8\). Thus one finds the coordinate expression
\[
\omega^{AA} \circ \text{j} = y^A y^A + y^A y^A\).
\]

Let now \(\Gamma : V \to \text{j}V\) and \(\Gamma^8 : V^8 \to \text{j}V^8\) be linear projectable connections over \(\text{j}E \to \text{j}E\) and \(\text{j}E^8 \to \text{j}E^8\), respectively; then there exists a unique connection \(\Gamma \otimes \Gamma^8 : W \to \text{j}W\) such that the diagram
\[
\begin{array}{ccc}
\text{j}V \times \text{j}V^8 & \xrightarrow{\text{j} \otimes} & \text{j}W \\
\downarrow_{(j\Gamma, j\Gamma^8)} & & \uparrow_{(\Gamma \otimes \Gamma^8)} \\
V \times V^8 & \xrightarrow{\otimes} & W
\end{array}
\]
commutes; moreover, \(\Gamma \otimes \Gamma^8\) is linear projectable over
\[
(\Gamma, \Gamma^8) : E \times E^8 \to \text{j}E \times \text{j}E^8,
\]
and its coordinate expression is
\[
(y_a^i, y_a^A, y_a^i, y_a^A, \omega_a^{AA}) \circ (\Gamma \otimes \Gamma^8) =
\]
\[
= (\Gamma_a^i, \Gamma_a^A y^B, \Gamma_a^i, \Gamma_a^A y^B, \Gamma_a^A y^B + y^A \Gamma_a^A y^B),
\]
where the components of \(\Gamma^8\) are recognized by primed indices.

The distributional bundle \(\mathcal{W} := \mathcal{O}_B(F, W) \to B\) is easily seen to co-
incide with the fibred tensor product of $\mathcal{V}$ and $\mathcal{V}'$, namely

$$\mathcal{W} := \mathcal{O}_M(F, W) = \mathcal{O}_M(E \times E', V \otimes E) = \mathcal{O}_M(E, V) \otimes \mathcal{O}_M(E', V') \equiv \mathcal{V} \otimes \mathcal{V'}.$$ 

Let $\mathcal{C} : \mathcal{V} \rightarrow J \mathcal{V}$ and $\mathcal{C}' : \mathcal{V}' \rightarrow J \mathcal{V}'$ be the distributional connections induced by $\Gamma$ and $\Gamma'$. These yield, exactly by the same argument which is valid in the finite-dimensional case, a linear connection $\mathcal{C} \otimes \mathcal{C}' : \mathcal{W} \rightarrow J \mathcal{W}$; it is not difficult to prove:

**Proposition 9.1.** The tensor product connection $\mathcal{C} \otimes \mathcal{C}'$ is exactly the distributional connection associated with the classical connection $\Gamma \otimes \Gamma'$. For $\omega \in \mathcal{W}$ one has

$$(\mathcal{C} \otimes \mathcal{C}')_* \mathcal{V} \omega^{VV'} = \Gamma_a \indices{^A_B} \omega^{BA'} - \Gamma_a \indices{^A_i} \partial_i \omega^{AA'} + \Gamma_a \indices{^A'B} \omega^{AB'} - \Gamma_a \indices{^A}_{i'} \partial_{i'} \omega^{AA'}. $$

If $E = E'$ then one also has the 2-fibred bundle $V \otimes V \rightarrow E \rightarrow B$. The distributional bundle $\mathcal{O}_M(E, V \otimes V)$ is different from $\mathcal{V} \otimes \mathcal{V}'$. If $\Gamma : V \rightarrow J V$ and $\Gamma' : V' \rightarrow J V'$ are now linear projectable connections over the same connection $\Gamma : E \rightarrow J E$, then, besides $\Gamma \otimes \Gamma'$, they also determine a different kind of tensor connection, that is

$$\Gamma \otimes \Gamma' : V \otimes V \rightarrow J(V \otimes EV'),$$

which is characterized by the commuting diagram

$$J(V \times V') \equiv JV \times JV' \xrightarrow{J \otimes} J(V \otimes V') \xrightarrow{\phi_{\Gamma \otimes \Gamma'}}  J(V \otimes V')$$

and has the coordinate expression

$$(y_a^i, y_a^i, w_a^{iA'}) \circ (\Gamma \otimes \Gamma') =$$

$$(\Gamma^i_a, \Gamma^i \indices{^A_B} y^B_a, \Gamma^i \indices{^A'B} y^B_a, \Gamma^i \indices{^A_B} y^B_a y^{A'} + y^A \Gamma^i \indices{^A'B} y^B_a).$$

The induced distributional connection

$$\mathcal{C} \otimes \mathcal{C}' : \mathcal{O}_M(E, V \otimes V') \rightarrow J \mathcal{O}_M(E, V \otimes V')$$
has the coordinate chart expression
\[
(\tilde{\xi} \otimes \tilde{\xi'})_a^{yy} \omega^{yy} = \Gamma^A_a \omega^{BA} + \Gamma^{A'}_a \omega^{A'B'} - \Gamma^i_a \partial_i \omega^{AA'}.
\]

10. Induced connection: vertical bundle and adjoint case.

A linear projectable connection \( \Gamma : V \rightarrow JV \), as considered in the previous sections, determines a unique «dual» connection \( \Gamma^* : V^* \rightarrow JV^* \); this is again linear projectable over the same \( L \) and is characterized by
\[
Jc \circ (\Gamma, \Gamma^*) = 0,
\]
where \( c : V \times V^* \rightarrow \mathbf{E} \times \mathbf{C} \) denotes the duality contraction; it has the coordinate expression
\[
\Gamma^*_a = - \Gamma^A_a.
\]

On turn, \( \Gamma^* \) determines a connection on the distributional bundle \( \mathcal{O}_B(E, V^*) \). In general, this is not the adjoint connection \( \xi^* \) of \( \xi \), which is actually a connection on a different distributional bundle. In order to study the relation between \( \xi^* \) and the classical connection \( \Gamma \) one has to perform some further constructions.

The first step consists in the vertical extension of \( \Gamma : E \rightarrow JE \). Recalling the natural isomorphism \( JUE \equiv VJE \), one gets the morphism
\[
\bar{\Gamma} := V\overline{\Gamma} : VE \rightarrow JVE,
\]
which turns out to be a linear projectable connection over \( L \). Its coordinate expression is
\[
\bar{\Gamma}^a = \tilde{\partial} \Gamma^a.
\]
Its dual connection \( \bar{\Gamma}^* : V^* E \rightarrow JV^* E \) has the coordinate expression
\[
(\bar{\Gamma}^*)_a = - \bar{\Gamma}^a = - \tilde{\partial} \Gamma^a.
\]

Now one finds induced linear projectable connections over \( \Gamma \) in all tensor product bundles over \( E \rightarrow B \) constructed from \( VE \) and \( V^* E \). Most noticeably, one has projectable linear connections over \( L \) on the 2-fibre bundles
\[
\wedge^r V^* E \rightarrow E \rightarrow B, \quad r \in \mathbb{N},
\]
\[
\vee^r E \rightarrow E \rightarrow B,
\]
\[
\vee^r E \rightarrow E \rightarrow B,
\]
and, using $\Gamma$, in their tensor products with $V$ and $V^*$ over $E$. In particular, the connection $\bar{\Gamma}: {}^V/E \to J {}^V/E$ has the coordinate expression

$$\bar{\Gamma}_a = (\bar{\Gamma}^a)_a = -\partial_i \Gamma^a_i.$$ 

All these classical connections determine linear connections on the corresponding distributional bundles, and, in particular, in the distributional bundle

$$\mathfrak{g}^* := \mathcal{O}(E, {}^V/E \otimes V^*) .$$

The classical connection

$$\Gamma' = (\bar{\Gamma} \otimes \mathfrak{g}^*): {}^V/E \otimes V^* \to J(\mathfrak{g}^*),$$

which is again linear projectable over $\Gamma$, has the coordinate expression

$$z_B a = (\bar{\Gamma}^a a) y_A = \left(\delta_B^A \delta_i^a \partial_i + \Gamma^a_i \right) y_A,$$

where $(z_B)$ and $(y_A)$ are the induced coordinates in the fibres of $\mathfrak{g}^* \otimes V^* \to E$ and $V^* \to E$, respectively.

Now, $\Gamma'$ induces a linear distributional connection $\mathcal{C}' : \mathfrak{g}^* \to J \mathfrak{g}^*$; if $\tau : B \to \mathfrak{g}^*$ is an $F$-smooth section, with coordinate expression $\tau = \tau_A d^i y^A$, then one finds

$$\mathcal{C}_{\tau_B}^a : \tau_B = \bar{\Gamma}^a_i \partial_i \tau_B = -\Gamma^a_i \tau_B - \Gamma^a_i \partial_i \tau_B .$$

Now it is a straightforward matter to proof:

**Proposition 10.1.** The distributional connection $\mathcal{C}' : \mathfrak{g}^* \to J \mathfrak{g}^*$ coincides with the adjoint connection $\mathcal{C} : \mathfrak{g} \to J \mathfrak{g}$ (proposition 6.1).

11. Quantum Dirac connection.

Let $(M, g)$ be an Einstein spacetime. A time map is a bundle $t : M \to T$, where $T$ is an oriented 1-dimensional real manifold whose fibres $T_{t(t)} \equiv \partial^1(t), \ t \in T$, are spacelike (this is one possible extension of the notion of observer to the curved spacetime case). The assignment of $t$ determines a splitting of the spacetime’s tangent bundle as $TM = T^1_M \oplus T^\perp_M$, where, for each $x \in M$, $T^1_M$ is defined to be the timelike subspace of $T_x M$.
which is orthogonal to the spacelike fibre through \( x \), and \( T^+_x M \) is the subspace orthogonal to \( T^x M \); namely \( T^\perp M \equiv VM \) is constituted by all vectors tangent to the spacelike fibres.

The bundle \( M \to T \) has a natural trivialization \((t, x) : M \to T \times X \), determined by the integral lines of any vector field \( M \to T^x M \); the family of these lines can be identified with the fibre type \( X \) of \( t \). It should be noted that, in general (differently from the flat case), the manifolds \( T^x \) and \( X^x \) do not inherit distinguished metric structures. One may choose adapted coordinate charts \((x^i) \) on \( M \), determined by a chart \((x^i) \) on \( T^x \) and a chart \((x^4) \) on \( X^x \). Obviously, one has \( g_{i4} = 0, \ i = 1, 2, 3 \).

Besides adapted charts, it is also convenient to work with a tetrad, which is defined to be an orthonormal frame \((U^l) \) such that \( U^0 : M \to T^x M \) and \( U^i : M \to T^+_x M \), \( i = 1, 2, 3 \). One also sets \( \tilde{x}_a = \Theta_a^l U^l \), with \( \Theta_a^l : M \to \mathbb{R} \).

The given time and spacetime orientations of \( M \) yield a space orientation, namely an orientation of each \( M^t \); one has the positive semi-vector bundle \( \nabla^+ := (\wedge^3 T^\perp M)^+ \subset \wedge^3 TM \to M \), and the spacetime volume form can be decomposed as \( \eta = \Theta^0 \wedge \eta_0, \eta_0 : M \to \nabla^+ \). It is not difficult to see that the spacetime connection determines connections on \( T^0 M \to M \) and \( T^\perp M \to M \) by the rules

\[
\nabla^+_u v := (\nabla_u^a)^+, \quad u : M \to T^0 M, \quad v : M \to T^\perp M,
\]

and that \( \nabla^+ \Theta_0 = 0, \nabla^+ \eta_0 = 0 \).

Next, consider a \( 4 \)-spinor bundle (see also \[CJ, C00\] for details); this is defined to be a complex vector bundle \( W \to M \) with 4-dimensional fibres, endowed with a fibred Hermitian metric \( k \) with signature \((+ - - -)\), a Clifford map \( \gamma : TM \to \text{End}(W) \) over \( M \) fulfilling \( k(\gamma(v) \psi', \psi) = k(\psi', \gamma(v) \psi) \forall (v, \psi', \psi) \in TM \times W \times W \), and a \( k \)-preserving linear connection \( \Gamma : W \to JW \) such that \( \nabla[\Gamma \otimes I] \gamma = 0 \). Then, in suitable linear fibre coordinates, \( \Gamma \) is related to the spacetime connection \( \Gamma \) by the expression

\[
F^a_{\alpha \beta} = i A_{\alpha} \delta^a_{\beta} + \frac{1}{4} F^{\alpha \beta}_{\gamma \delta} (\gamma^a_{\gamma} \gamma^b_{\delta})^a_{\beta}, \quad \gamma_{\gamma} = \gamma(\Theta_{\gamma}), \quad \alpha, \beta = 1, 2, 3, 4,
\]

where the functions \( A_{\alpha} : M \to \mathbb{R} \) can be seen as the components of the
connection induced on $\bigwedge^2 S \to M$, $S \subset W$ being a maximal $k$-isotropic subbundle (2-dimensional fibres). The time fibration yields a further Hermitian structure $h$ in the fibres of $W$, given by

$$h(\psi', \psi) := k(\gamma^0 \psi', \psi) = k(\psi', \gamma^0 \psi),$$

which turns out to have positive signature.

The Dirac equation for a (generalized) section $\psi : M \to W$,

$$i\gamma^\lambda \nabla_\lambda \psi - \mu \psi + i T_\lambda \gamma^\lambda \psi = 0, \quad \mu \in \mathbb{R}^+$$

(here $\gamma^\lambda := g^{\lambda\nu} \gamma_\nu$ and $T_\lambda := T_\lambda^\nu$, $T$ being the torsion of the spacetime connection), can be rewritten, after composition by $\gamma^0$ on the left, as (5)

$$\partial_t \psi - \Gamma^\alpha_\beta \psi + \Theta^\alpha_\beta(\Theta^{-1})^\beta_\gamma \gamma^\gamma (\partial_\gamma \psi - \Gamma_\gamma \psi) + \Theta^\alpha_\beta \left( i\mu \gamma^0 \psi + \frac{1}{2} T_\lambda \gamma^\lambda \gamma^\lambda \psi \right) = 0.$$

Let now $\mathcal{W} := \mathcal{O}_T(M, W) \to T$ be the distributional bundle whose fibre over any $t \in T$ is the space of all generalized sections of the classical bundle $W_M \to M$. This is called the bundle of 1-electron states, and a section $\psi : T \to \mathcal{W}$ is called a 1-electron quantum history. It is clear, from the latter way of writing it, that the Dirac equation can be seen as an equation for quantum histories of the form $\nabla(\mathcal{C}) \psi = 0$, relatively to a linear connection $\mathcal{C} : \mathcal{W} \to J \mathcal{W}$ which I call the quantum Dirac connection. It should be noted that $\mathcal{C}$ does not derive from a connection on the underlying classical bundle (§ 7).

The adjoint bundle of $\mathcal{W} \to T$ is

$$\mathcal{W}^* = \mathcal{O}_T(M, \bigwedge^* M \otimes W^*) \to T,$$

its fibres being constituted by $W^*$-valued generalized densities on the spacelike fibres of $t$. Because the Hermitian metric $k$ determines an anti-isomorphism $W \leftrightarrow W^*$, the conjugate Dirac equation is a field equation for (generalized) sections $\phi : M \to W^*$, namely

$$i\nabla_\lambda \phi \gamma^\lambda + \mu \phi + \frac{1}{2} T_\lambda \phi \gamma^\lambda = 0.$$

As one has a connection on $\bigwedge^* \to M$, determined by the spacetime con-

(5) As customary, here spinor indices are not explicitly shown.
nection, and since $\nabla \eta_0 = 0$, one can equivalently write the above equation for $\phi$ as a formally identical equation for $\tilde{\phi} \equiv \eta_0 \otimes \phi : M \to \mathbb{V}_{\chi}^* \otimes \mathbb{W}^*$ (coordinates expressions, however, are not exactly the same). One can rewrite the equation for $\tilde{\phi}$ using the same procedure used for $\psi$ above, getting

$$0 = \partial_4 \tilde{\phi} - (\partial_4 \log \det \Theta^1) \tilde{\phi} + \partial_4 I^4_4 + \Theta^0_4(\Theta^{-1})^i_j [\partial_4 \tilde{\phi} - (\partial_4 \log \det \Theta^1) \tilde{\phi} + \partial_4 I^4_4] \gamma^j \gamma^0 + \Theta^0_4(-\ddot{u} \tilde{\phi} \gamma^0 + \frac{1}{2} T_1 \tilde{\phi} \gamma^1 \gamma^0),$$

where $(\Theta^1)$ denotes the «spacelike» matrix $(\Theta^1)$, $k, i = 1, 2, 3$. Then, one sees that the equation for $\phi$ can be also written in the form $\nabla(\tilde{\phi}) \phi = 0$, relatively to a connection $(\tilde{\phi}^b) : \mathcal{W}^a \to \mathbb{J} \mathcal{W}^a$. Naturally, one wishes to compare this connection with the distributional adjoint of $\partial_4$. It turns out that $\partial_4$ is not $\partial_4^\ast$, but rather it is the adjoint of $\tilde{\phi}$ relatively to a contraction mediated by the observer through $\gamma_0$ (thus related to the positive Hermitian metric $h$). In fact:

**Proposition 11.1.** Whenever all contractions are defined, one has

$$\partial_4(\tilde{\phi}, \gamma_0 \psi) = \langle \nabla_4[\tilde{\phi}^b] \tilde{\phi}, \gamma_0 \psi \rangle + \langle \tilde{\phi}, \gamma_0 \nabla_4[\tilde{\phi}] \psi \rangle.$$  

**Proof.** By an argument similar to the proof of proposition 6.1 there is a connection $(\tilde{\phi}^b) : \mathcal{W}^a \to \mathbb{J} \mathcal{W}^a$ determined by the requirement $\partial_4(\tilde{\phi}, \gamma_0 \psi) = \langle \nabla_4[\tilde{\phi}^b] \tilde{\phi}, \gamma_0 \psi \rangle + \langle \tilde{\phi}, \gamma_0 \nabla_4[\tilde{\phi}] \psi \rangle$. The operator $\nabla_4[\tilde{\phi}^b]$ can be calculated by assuming that $\tilde{\phi}$ and $\psi$ are represented in each fibre by ordinary sections, and $\phi$ in particular by a test section. Then contractions can be written as integrals, and integration by parts gives

$$\nabla_4[\tilde{\phi}^b] \tilde{\phi} = \partial_4 \tilde{\phi} + \partial_4 I^4_4 + \tilde{T}_4^0 \tilde{\phi} \gamma^j \gamma^0 + \Theta^0_4(\Theta^{-1})^i_j [\partial_4 \tilde{\phi} + \partial_4 I^4_4] \gamma^j \gamma^0 + \Theta^0_4(\Theta^{-1})^i_j [\partial_4 \gamma^j \gamma^0 + \Theta^0_4(\Theta^{-1})^i_j \tilde{T}_4 \tilde{\phi} \gamma^j \gamma^0 + -i \Theta^0_4 \mu \tilde{\phi} \gamma^0 - \frac{1}{2} \Theta^0_4 \tilde{T}_4 \tilde{\phi} \gamma^j \gamma^0.$$

The comparison between $\mathcal{C}^b$ and $\mathcal{C}'$ now involves some coordinate calculations by which one relates the derivatives of the tetrad components to the torsion; eventually, these two distribitional connections are seen to coincide.

By similar arguments, one can show that $\mathcal{C}^\ast$ is related to the field equation obeyed by $\psi'$, the adjoint of $\psi$ through the positive Hermitian metric $h$.


A convenient way of describing quantum states consists in viewing them as distributions on the phase bundle of the particle under consideration. Let $\mu \in \{0\} \cup \mathbb{R}^+$ be the particle’s mass and consider the subbundle $K^\mu_c \subset \mathcal{T}M$ over $M$ constituted by all future-pointing vectors $v \in \mathcal{T}M$ such that $g(v, v) = \mu^2$ (using spacetime metric signature $(+, - , - , -)$); the fibres are 3-hyperboloids for $\mu > 0$, null half-cones for $\mu = 0$.

Let $(y^0, y^i)$ be (not necessarily orthonormal) coordinates in the fibres of $\mathcal{T}M \rightarrow M$ such that $g_{00} > 0$ (namely $y^0$ is timelike) and $g_{0i} = 0$, $i = 1, 2, 3$. Then the restrictions of $(y^i)$ are coordinates in the fibres of $K^\mu_c \rightarrow M$.

The following is a generalization of a result by Janyška and Modugno [JM96].

**Proposition 12.1.** The spacetime connection $\Gamma$ is reducible to a (non-linear) connection $\Gamma_\mu$ in $K^\mu_c \rightarrow M$; in orthonormal fibred coordinates $(y^0, y^i)$, its expression is

$$(\Gamma_\mu)^a_i = \Gamma_a^b \delta^0_0 (\mu^2 + \delta_{kk} y^0 y^k)^{1/2} + \Gamma_a^i y^i.$$ 

**Proof.** The subbundle $K_c \subset \mathcal{T}M$ over $M$, constituted by all $v \in \mathcal{T}M$ (of any time orientation) such that $g(v, v) = \mu^2$, is characterized in coordi-

(5) For a precise physical setting, physical constants should be described as elements of certain «unit spaces», namely 1-dimensional vector spaces or semi-vector spaces [3, 5, 7, 12]. Accordingly, some geometric structures and fields, such as the spacetime metric, the Dirac map $\gamma$ and a quantum history $\psi$ have unit spaces attached to them as tensor products. The metric, in particular, is valued into $L^2 \equiv L \otimes L$ where $L$ is the unit space of lengths. For the purpose of this paper, however, one can simply work with (arbitrarily) chosen units.
mates by the condition $g_{ij}y^i y^j = \mu^2$; hence, $TK_\mu$ is the submanifold of $T^2TM$ characterized by

$$g_{ij}y^i y^j = \mu^2, \quad \hat{x}^a \partial_a g_{ij}y^i y^j + 2g_{ij}y^j \hat{y}^i = 0,$$

and $\nabla K_\mu$ is the submanifold of $K^\mu \times TM$ characterized by $g_{ij}y^i \hat{y}^j = 0$.

The vertical-valued form $\Omega : T^2TM \to VTM \equiv TM \times TM$, associated with the spacetime connection restricts to a form $\Omega_\mu : TK_\mu \to K_\mu \times TM$; using the above coordinates identities, and $\Omega = \left(\hat{y}^i - \hat{x}^a \Gamma^i_{ab} y^b\right) \partial_a$, it is immediate to check that $\Omega_\mu$ is actually valued onto $\nabla K_\mu$, namely it is the vertical-valued form associated with a connection on $K_\mu \to M$. On turn, this is obviously reducible to the subbundle $K_\mu \subset K_0$ of future-pointing vectors. In orthonormal fibre coordinates, on $TK_\mu$ one has

$$y^0 = \sqrt{\mu^2 + \delta_{ik}y^k y^k}, \quad g_{ij}y^i \hat{y}^j = 0, \quad \Gamma_{ij} y^i \hat{y}^j = 0,$$

$$\Rightarrow \hat{y}^i \cdot \Omega_\mu = \hat{y}^i - \hat{x}^a \left(\Gamma^i_{ab} y^b + \Gamma^i_{aj} y^j\right), \quad y^0 = \sqrt{\mu^2 + \delta_{ik}y^k y^k}.$$

Let $W \to M$ be the spinor bundle introduced in § 11 and $V \equiv K^+_\mu \times W$. The couple $(\Gamma_\mu, \ell)$ is a classical connection on the 2-fibred bundle $V \to K^+_\mu \to M$, linear projectable over $\Gamma_\mu$; thus one gets (§ 7) a linear connection $\mathcal{C}$ on the distributional bundle $\mathcal{V} := \Omega_M(K^+\mu, V) \to M$ (which is related to the quantum description of electrons and other massive $1/2$-spin particles: here $K^+_\mu$ is the particle’s phase bundle). Its coordinate expression is

$$(\mathcal{C} y^i)_\beta = \Gamma^i_{\alpha\beta} - \delta^i_{\alpha\beta}\left[\Gamma^i_{ab} (\mu^2 + \delta_{ik}y^k y^k)^{1/2} + \Gamma^i_{aj} y^j\right] \partial_a.$$

For massless particles, the phase bundle is not $K^+ \equiv K^+_0$ but rather its projective bundle over $M$

$$P \equiv PK^+ := K^+ / R^+.$$

That is, $P$ is the quotient of $K^+$ by the action of the multiplicative group $R^+$: its fibres are the sets of generatrices of the future null cone, namely 2-spheres (the so-called celestial spheres).
PROPOSITION 12.2. There exists a unique connection $\Gamma_P : P \rightarrow JP$ such that the diagram

$$
\begin{array}{ccc}
K^+ & \xrightarrow{\Gamma_0} & JK^+ \\
P & \Downarrow & JP \\
\Psi_P & \xrightarrow{\Gamma_0} & JP
\end{array}
$$

commutes, where $P : K^+ \rightarrow P$ is the natural projection.

PROOF. Let $k \in K^+$, $r \in R^+$; then, by means of coordinate expressions, it is not difficult to see that $\Gamma_0(k)$, $\Gamma_0(rk) \in JK^+$ are in the same orbit of the prolonged $R^+$-action.

In order to write down a coordinate expression for $\Gamma_P$, one may take spherical fibre coordinates $(r, \theta, \phi)$ associated with orthonormal fibre coordinates $(y^i)$. Then $(\theta, \phi)$ are fibre coordinates for $P$, and after some calculations one finds

$$(\Gamma_P)^{a}_{0} = \cos \theta \cos \phi \Gamma_{a 0}^{1} + \cos \theta \sin \phi \Gamma_{a 0}^{2} - \sin \theta \Gamma_{a 0}^{3} + \cos \phi \Gamma_{a 1}^{1} + \sin \phi \Gamma_{a 1}^{2},$$

$$(\Gamma_P)^{a}_{2} = -\Gamma_{a 2}^{1} + \frac{1}{\sin \theta} (-\sin \phi \Gamma_{a 0}^{1} + \cos \phi \Gamma_{a 2}^{1} - \cos \theta \sin \phi \Gamma_{a 3}^{1} + \cos \theta \cos \phi \Gamma_{a 3}^{2}).$$

A classical photon field can be described as a section $\Phi : M \rightarrow VP$ (see [C00b] for details). Accordingly, in view of its quantum description one is lead to consider the distributional bundle $\mathcal{E} := \mathcal{O}(P, VP)$. The vertical prolongation of $\Gamma_P$ is a connection (§ 10) $VP \rightarrow JVP$ which is linear projectable over $\Gamma_P$, thus one obtains a linear connection $\mathcal{E} \rightarrow J\mathcal{E}$.

Applications of these constructions to quantum field theory will be expounded in a forthcoming paper.

REFERENCES


Connections on distributional bundles


Manoscritto pervenuto in redazione il 3 febbraio 2003.