On the Semi-Simplicity of Galois Actions.

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Let $K$ be a finitely generated field and $X$ be a smooth projective variety over $K$; let $G_K$ denote the absolute Galois group of $K$ and $l$ a prime number different from char $K$. Then we have

Conjecture 1 (Grothendieck-Serre). The action of $G_K$ on the $l$-adic cohomology groups $H^*(X, \mathbb{Q}_l)$ is semi-simple.

There is a weaker version of this conjecture:

Conjecture 2 ($S^n(X)$). For all $n \geq 0$, the action of $G_K$ on the $l$-adic cohomology groups $H^{2n}(X, \mathbb{Q}_l(n))$ is «semi-simple at the eigenvalue 1», i.e. the composite map

$$H^{2n}(X, \mathbb{Q}_l(n))^{G_K} \hookrightarrow H^{2n}(X, \mathbb{Q}_l(n)) \twoheadrightarrow H^{2n}(X, \mathbb{Q}_l(n))^{G_K}$$

is bijective.

If $K$ is a finite field, then Conjecture 2 implies Conjecture 1. This is well-known and was written-up in [8] and [4], manuscript notes distributed at the 1991 Seattle conference on motives. Strangely, this is the only result of op. cit. that was not reproduced in [10]. We propose here a simpler proof than those in [8] and [4], which does not involve Jordan blocks, representations of $SL_2$ or the Lefschetz trace formula.

We also show that Conjecture 2 for $K$ finite implies Conjecture 1 for any $K$ of positive characteristic. The proof is exactly similar to that in [3, pp. 212-213], except that it relies on Deligne’s geometric semi-simplicity theorem [2, cor. 3.4.13]; I am grateful to Yves André for explaining it to me. This gives a rather simple proof of Zarhin’s semi-simplicity theorem.

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for abelian varieties (see Remark 8.1). There is also a small result for $K$
of characteristic 0 (see Remark 8.2). Besides, this paper does not claim
much originality.

In order to justify later arguments we start with a well-known elementary lemma:

**Lemma 3.** Let $E$ be a topological field of characteristic 0 and $G$ a
topological group acting continuously on some finite-dimensional $E$-
vector space $V$. Suppose that the action of some open subgroup of finite
index $H$ is semi-simple. Then the action of $G$ is semi-simple.

**Proof.** Let $W \subseteq V$ be a $G$-invariant subspace. By assumption, there is
an $H$-invariant projector $e \in \text{End}(V)$ with image $W$. Then

$$e' = \frac{1}{(G : H)} \sum_{g \in G \setminus H} geg^{-1}$$

is a $G$-invariant projector with image $W$. ■

**Lemma 4.** Let $K$ be a field of characteristic 0, $A$ a finite-dimensiona-
semi-simple $K$-algebra and $M$ an $A$-bimodule. Let $\mathfrak{g}$ be the Lie al-
gebra associated to $A$, and let $\mathfrak{N}$ be the $\mathfrak{g}$-module associated to $M$
$(\text{ad}(a)m = am - ma)$. Then $\mathfrak{N}$ is semi-simple.

**Proof.** Since $K$ has characteristic 0, $A \otimes_K A^{\text{op}}$ is semi-simple.
We may reduce to the case where $K$ is algebraically closed by a trace
argument, and then to $M$ simple (as a left $A \otimes_K A^{\text{op}}$-module). Write

$$A = \prod_i \text{End}_K(V_i); \quad A \otimes_K A^{\text{op}} = \prod_{i,j} \text{End}_K(V_i \otimes V_j^*)$$

and $M$ is isomorphic to one of the $V_i \otimes V_j^*$. We distinguish two cases:

a) $i = j$. We may assume $A = \text{End}(V) \ (V = V_i)$. Then $\mathfrak{g} = \mathfrak{gl}(V) =
= \mathfrak{sl}(V) \times K$, and $\mathfrak{sl}(V)$ is simple. By [9, th. 5.1], to see that $\mathfrak{N} = V \otimes V^*$ is
semi-simple, it suffices to check that the action of $K = \text{Cent}(\mathfrak{g})$ can be
diagonalised. But $a \in \mathfrak{g}$ acts by

$$\text{ad}(a)(v \otimes w) = a(v) \otimes w - v \otimes a(w)$$

and if $a$ is a scalar, then $\text{ad}(a) = 0$.

b) $i \neq j$. We may assume $A = \text{End}(V) \times \text{End}(W) \ (V = V_i, W = V_j)$.
This time, $\mathfrak{g} = \mathfrak{gl}(V) \times \mathfrak{gl}(W) = \mathfrak{sl}(V) \times \mathfrak{sl}(W) \times K \times K$. The action of $\mathfrak{g}$

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This time, $\mathfrak{g} = \mathfrak{gl}(V) \times \mathfrak{gl}(W) = \mathfrak{sl}(V) \times \mathfrak{sl}(W) \times K \times K$. The action of $\mathfrak{g}$
onto \( \mathfrak{g} = V \otimes W^* \) is given by the formula

\[
\text{ad}(a, b)(v \otimes w) = a(v) \otimes w - v \otimes b(w).
\]

Hence the centre acts by \( \text{ad}(\lambda, \mu) = \lambda - \mu \) and the conditions of [9, th. 5.1] are again verified.

**Proposition 5.** Let \( V \) be a finite-dimensional vector space over a field \( K \) of characteristic 0. For \( u \) an endomorphism of \( V \), denote by \( \text{ad}(u) \) the endomorphism \( v \mapsto uv - vu \) of \( \text{End}_K(V) \). Let \( A \) be a \( K \)-subalgebra of \( \text{End}_K(V) \) and \( B \) its commutant. Consider the following conditions:

(i) \( A \) is semi-simple.

(ii) \( \text{End}_K(V) = B \oplus \sum_{a \in A} \text{ad}(a) \text{End}_K(V) \).

(iii) \( B \) is semi-simple.

Then (i) \( \Rightarrow \) (ii) \( \Rightarrow \) (iii), and (iii) \( \Rightarrow \) (i) if \( A \) is commutative.

**Proof.** (i) \( \Rightarrow \) (ii): let \( \mathfrak{g} \) be the Lie algebra associated to \( A \). By lemma 4, \( \text{End}_K(V) \) is semi-simple for the adjoint action of \( \mathfrak{g} \). Then (ii) follows from [1, §3, prop. 6].

(ii) \( \Rightarrow \) (iii): let us show that the radical \( J \) of \( B \) is 0. Let \( x \in J \). For \( y \in B \), we have \( xy \in J \); in particular, \( xy \) is nilpotent, hence \( \text{Tr}(xy) = 0 \). For \( z \in \text{End}_K(V) \), and \( u \in A \), we have

\[
\text{Tr}(xz - zu)) = \text{Tr}(xuz - xzu) = \text{Tr}(uxz - xzu) = 0.
\]

Hence \( \text{Tr}(xy) = 0 \) for all \( y \in \text{End}_K(V) \), and \( x = 0 \).

(iii) \( \Rightarrow \) (i) supposing \( A \) commutative: let us show this time that the radical \( R \) of \( A \) is 0. Suppose the contrary, and let \( r > 1 \) be minimal such that \( R' = 0 \); let \( I = R^{r-1} \). Then \( I^2 = 0 \). Let \( W = IV \); then \( W \) is \( B \)-invariant, hence \( B \) acts on \( V/W \). Let

\[
N = \{ v \in B | v(V) \subseteq W \}
\]

be the kernel of this action: then \( N \) is a two-sided ideal of \( B \) and obviously \( NI = IN = 0 \). Let \( v, v' \in N \) and \( x \in V \). Then there exist \( y \in V \) and \( w \in I \) such that \( v(x) = w(y) \). Hence

\[
v' v(x) = v' w(y) = 0
\]

and \( N^2 = 0 \). Since \( B \) is semi-simple, this implies \( N = 0 \). But, since \( A \) is commutative, \( I \subseteq N \), a contradiction.
THEOREM 6. Let $X$ be a smooth, projective variety of dimension $d$ over a field $k$ of characteristic $\neq 1$. Let $k_s$ be a separable closure of $k$, $G = \text{Gal}(k_s/k)$, and $\overline{X} = X \times_k k_s$. Consider the following conditions:

(i) For all $i \geq 0$, the action of $G$ on $H^i(X, \mathbb{Q}_l)$ is semi-simple.

(ii) $S^d(X \times X)$ holds.

(iii) The algebra $H^{2d}(\overline{X} \times_k \overline{X}, \mathbb{Q}_l(d))^G$ is semi-simple.

Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii), and (iii) $\Rightarrow$ (i) if $k$ is contained in the algebraic closure of a finite field.

PROOF. By the Künneth formula and Poincaré duality, we have the well-known Galois-equivariant isomorphism of $\mathbb{Q}_l$-algebras

$$H^{2d}(\overline{X} \times \overline{X}, \mathbb{Q}_l(d)) = \mathbb{Q}_l \otimes_{\mathbb{Q}_l} H^{2d}_{\text{cont}}(\overline{X} \times \overline{X}, \mathbb{Q}_l(d)).$$

For $q \in [0, 2d]$, let $A_q$ be the image of $\mathbb{Q}_l[G]$ in $\text{End}_{\mathbb{Q}_l}(H^{2d}_{\text{cont}}(\overline{X} \times \overline{X}, \mathbb{Q}_l(d)))$. Then condition (i) (resp. (ii), (iii)) of theorem 6 is equivalent to condition (i) (resp. (ii), (iii)) of proposition 5 for all $A_q$. The conclusion follows by remarking that the $A_q$ are commutative if $k$ is contained in the algebraic closure of a finite field. $\blacksquare$

I don’t know how to prove (iii) $\Rightarrow$ (i) in general in theorem 6, but in fact there is something better:

THEOREM 7. Let $F$ be a finitely generated field over $\mathbb{F}_p$ and let $X$ be a smooth, projective variety of dimension $d$ over $F$. Let $\mathcal{O}$ be a valuation ring of $F$ with finite residue field, such that $X$ has good reduction at $\mathcal{O}$. Let $Y$ be the special fibre of a smooth projective model $\overline{X}$ of $X$ over $\mathcal{O}$. Assume that $S^d(Y \times Y)$ holds. Then the Galois action on the $\mathbb{Q}_l$-adic cohomology of $X$ is semi-simple.

PROOF. For the proof we may assume that $X$ is geometrically irreducible. By Lemma 3 we may also enlarge $F$ by a finite extension and hence, by de Jong [5, Th. 4.1], assume that it admits a smooth projective model $T$ over $\mathbb{F}_p$. By the valuative criterion for properness, $\mathcal{O}$ has a centre $u$ on $T$ with finite residue field $k$. Up to extending the field of constants of $T$ to $k$, we may also assume that $u$ is a rational point. Now
spread $X$ to a smooth, projective morphism

$$f : \mathcal{X} \to U$$

over an appropriate open neighbourhood $U$ of $u$ (in a way compatible to $\mathcal{X}$).

The action of $G_F$ on $H^*(\mathcal{X}, \mathbb{Q}_l)$ factors through $\pi_1(U)$. Moreover $u$ yields a section $\sigma$ of the homomorphism $\pi_1(U) \to \pi_1(\text{Spec} k)$; in other terms, we have a split exact sequence of profinite groups

$$1 \to \pi_1(U) \to \pi_1(U) \to \pi_1(\text{Spec} k) \to 1.$$  

Let $i \geq 0$, $V = H^i(\mathcal{X}, \mathbb{Q}_l)$, $\Gamma = GL(V)$ and $\rho : \pi_1(U) \to \Gamma$ the monodromy representation. Denote respectively by $A$, $B$, $C$ the Zariski closures of the images of $\pi_1(U)$ in $\pi_1(U)$ and $\sigma(\pi_1(\text{Spec} k))$. Then $A$ is closed and normal in $B$, and $B = AC$.

By [2, cor. 3.4.13], $\pi_1(U)$ acts semi-simply on $V$; this is also true for $\sigma(\pi_1(\text{Spec} k))$ by the smooth and proper base change theorem and Theorem 6 applied to $Y$. It follows that $A$ and $C$ act semi-simply on $V$; in particular they are reductive. But then $B$ is reductive, hence its representation on $V$ is semi-simple and so is that of $\pi_1(U)$.

**REMARKS 8.** 1. If $X$ is an abelian variety, we recover a result of Zarhin [11, 12]. Theorem 7 applies more generally by just assuming that $Y$ is of abelian type in the sense of [6], for example is an abelian variety or a Fermat hypersurface [7]. (Recall, e.g. [6, Lemma 1.9], that the proof of semi-simplicity for an abelian variety $X$ over a finite field boils down to the fact that Frobenius is central in the semi-simple algebra $\text{End}(X) \otimes \mathbb{Q})$).

2. If $F$ is finitely generated over $\mathbb{Q}$, this argument gives the following (keeping the notation of Theorem 7). Let $F_0$ be the field of constants of $F$. Assume that $S^d(Y \times Y)$ holds and that, moreover, the action of $\text{Gal}(\overline{K}/K^{ab})$ on the $\mathbb{Q}_l$-adic cohomology of $Z$ is semi-simple, where $Z$ is the special fibre of $\mathcal{X} \otimes_\mathbb{Z} F_0$ and $K$ is the residue field of $F_0$. Then the conclusion of theorem 7 still holds.

To see this, enlarge $F$ as before so that it has a regular projective model $g : T \to \text{Spec} A$ (where $A$ is the ring of integers of $F_0$), this time by [5, Th. 8.2]. Let $u$ be the centre of $\mathcal{O}$ on $T$ and $U$ an open neighbourhood of $u$, small enough so that $X$ spreads to a smooth projective morphism $f : \mathcal{X} \to U$. Let $S = g(U)$ and $s = g(u)$. Up to extending $F_0$ and then shrinking $S$, we may assume that $g : U \to S$ has a section $\sigma$ such that $u = \sigma(s)$, that $\mu_{2 \ell} \subset \Gamma(S, \mathcal{O}_S^\times)$ and that $\mu_{1 \ell}(\kappa(s)) = \mu_{1 \ell}(S)$.

Let $S_\infty$ be a connected component of $S \otimes_\mathbb{Z} \mathbb{Z}[\mu_{1 \ell}]$ and $U_\infty = U \times_S S_\infty$. 
We then have two short exact sequences

\[ 1 \to \pi_1(U) \to \pi_1(U_\infty) \to \pi_1(S_\infty) \to 1 \]

\[ 1 \to \pi_1(U_\infty) \to \pi_1(U) \xrightarrow{\chi} \mathbb{Z}/p \]

where \( \chi \) is the cyclotomic character. The first sequence is split by \( s \); the second one is almost split in the sense that \( \chi(\pi_1(u)) = \chi(\pi_1(U)) \). By assumption, \( \pi_1(S_\infty) \) acts semi-simply on the cohomology of the generic geometric fibre of \( Z \) and (using theorem 6) \( \pi_1(u) \) acts semi-simply on the cohomology of \( Y \). Arguing as in the proof of theorem 7, we then get that \( \pi_1(U_\infty) \), and then \( \pi_1(U) \), act semi-simply on \( H^*_\mathrm{cont}(X, \mathbb{Q}) \). (To justify applying the smooth and proper base change theorem to \( Y \), note that \( g| : X \to S \) is smooth at \( s \) by the good reduction assumption.)

REFERENCES


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