On the Semi-Simplicity of Galois Actions.

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Let $K$ be a finitely generated field and $X$ be a smooth projective variety over $K$; let $G_K$ denote the absolute Galois group of $K$ and $l$ a prime number different from char $K$. Then we have

**Conjecture 1** (Grothendieck-Serre). The action of $G_K$ on the $l$-adic cohomology groups $H^*(X, \mathbb{Q}_l)$ is semi-simple.

There is a weaker version of this conjecture:

**Conjecture 2** ($S^n(X)$). For all $n \geq 0$, the action of $G_K$ on the $l$-adic cohomology groups $H^{2n}(X, \mathbb{Q}_l(n))$ is «semi-simple at the eigenvalue 1», i.e. the composite map

$$H^{2n}(X, \mathbb{Q}_l(n))^{G_K} \hookrightarrow H^{2n}(X, \mathbb{Q}_l(n)) \to H^{2n}(X, \mathbb{Q}_l(n))^{G_K}$$

is bijective.

If $K$ is a finite field, then Conjecture 2 implies Conjecture 1. This is well-known and was written-up in [8] and [4], manuscript notes distributed at the 1991 Seattle conference on motives. Strangely, this is the only result of op. cit. that was not reproduced in [10]. We propose here a simpler proof than those in [8] and [4], which does not involve Jordan blocks, representations of $SL_2$ or the Lefschetz trace formula.

We also show that Conjecture 2 for $K$ finite implies Conjecture 1 for any $K$ of positive characteristic. The proof is exactly similar to that in [3, pp. 212-213], except that it relies on Deligne’s geometric semi-simplicity theorem [2, cor. 3.4.13]; I am grateful to Yves André for explaining it to me. This gives a rather simple proof of Zarhin’s semi-simplicity theorem

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for abelian varieties (see Remark 8.1). There is also a small result for \( K \) of characteristic 0 (see Remark 8.2). Besides, this paper does not claim much originality.

In order to justify later arguments we start with a well-known elementary lemma:

**Lemma 3.** Let \( E \) be a topological field of characteristic 0 and \( G \) a topological group acting continuously on some finite-dimensional \( E \)-vector space \( V \). Suppose that the action of some open subgroup of finite index \( H \) is semi-simple. Then the action of \( G \) is semi-simple.

**Proof.** Let \( W \subseteq V \) be a \( G \)-invariant subspace. By assumption, there is an \( H \)-invariant projector \( e \in \text{End}(V) \) with image \( W \). Then

\[
e' = \frac{1}{(G : H)} \sum_{g \in G/H} geg^{-1}
\]

is a \( G \)-invariant projector with image \( W \). \( \square \)

**Lemma 4.** Let \( K \) be a field of characteristic 0, \( A \) a finite-dimensional semi-simple \( K \)-algebra and \( M \) an \( A \)-bimodule. Let \( \mathfrak{g} \) be the Lie algebra associated to \( A \), and let \( \mathfrak{m} \) be the \( \mathfrak{g} \)-module associated to \( M \) (\( \text{ad}(a)m = am - ma \)). Then \( \mathfrak{m} \) is semi-simple.

**Proof.** Since \( K \) has characteristic 0, \( A \otimes_K A^{\text{op}} \) is semi-simple. We may reduce to the case where \( K \) is algebraically closed by a trace argument, and then to \( M \) simple (as a left \( A \otimes_K A^{\text{op}} \)-module). Write \( A = \prod_i \text{End}_K(V_i) \); then \( A \otimes_K A^{\text{op}} = \prod_i \text{End}_K(V_i \otimes V_i^*) \) and \( M \) is isomorphic to one of the \( V_i \otimes V_i^* \). We distinguish two cases:

a) \( i = j \). We may assume \( A = \text{End}(V) \) (\( V = V_i \)). Then \( \mathfrak{g} = \mathfrak{gl}(V) = \mathfrak{sl}(V) \times K \), and \( \mathfrak{sl}(V) \) is simple. By [9, th. 5.1], to see that \( \mathfrak{m} = V \otimes V^* \) is semi-simple, it suffices to check that the action of \( K = \text{Cent}(\mathfrak{g}) \) can be diagonalised. But \( a \in \mathfrak{g} \) acts by

\[
\text{ad}(a)(v \otimes w) = a(v) \otimes w - v \otimes a(w)
\]

and if \( a \) is a scalar, then \( \text{ad}(a) = 0 \).

b) \( i \neq j \). We may assume \( A = \text{End}(V) \times \text{End}(W) \) (\( V = V_i \), \( W = V_j \)). This time, \( \mathfrak{g} = \mathfrak{gl}(V) \times \mathfrak{gl}(W) = \mathfrak{sl}(V) \times \mathfrak{sl}(W) \times K \times K \). The action of \( \mathfrak{g} \)
onto $\mathcal{W} = V \otimes W^*$ is given by the formula
\[
ad(a, b)(v \otimes w) = a(v) \otimes w - v \otimes b(w).
\]

Hence the centre acts by $\ad(l, m) \otimes \zeta = l \otimes m$ and the conditions of [9, th. 5.1] are again verified.

**Proposition 5.** Let $V$ be a finite-dimensional vector space over a field $K$ of characteristic 0. For $u$ an endomorphism of $V$, denote by $\ad(u)$ the endomorphism $v \mapsto vu - uv$ of $\text{End}_K(V)$. Let $A$ be a $K$-subalgebra of $\text{End}_K(V)$ and $B$ its commutant. Consider the following conditions:

(i) $A$ is semi-simple.

(ii) $\text{End}_K(V) = B \oplus \sum_{a \in A} \ad(a) \text{End}_K(V)$.

(iii) $B$ is semi-simple.

Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii), and (iii) $\Rightarrow$ (i) if $A$ is commutative.

**Proof.** (i) $\Rightarrow$ (ii): let $\mathfrak{g}$ be the Lie algebra associated to $A$. By lemma 4, $\text{End}_K(V)$ is semi-simple for the adjoint action of $\mathfrak{g}$. Then (ii) follows from [1, §3, prop. 6].

(ii) $\Rightarrow$ (iii): let us show that the radical $J$ of $B$ is 0. Let $x \in J$. For $y \in B$, we have $xy \in J$; in particular, $xy$ is nilpotent, hence $\text{Tr}(xy) = 0$. For $z \in \text{End}_K(V)$, and $u \in A$, we have
\[
\text{Tr}(uxz - zu) = \text{Tr}(xuz - zxu) = \text{Tr}(uxz - zxu) = 0.
\]

Hence $\text{Tr}(xy) = 0$ for all $y \in \text{End}_K(V)$, and $x = 0$.

(iii) $\Rightarrow$ (i) supposing $A$ commutative: let us show this time that the radical $R$ of $A$ is 0. Suppose the contrary, and let $x > 1$ be minimal such that $R^x = 0$; let $I = R^{x-1}$. Then $I^2 = 0$. Let $W = IV$: then $W$ is $B$-invariant, hence $B$ acts on $V/W$. Let
\[
N = \{ v \in B | v(V) \subseteq W \}
\]

be the kernel of this action: then $N$ is a two-sided ideal of $B$ and obviously $NI = IN = 0$. Let $v, v' \in N$ and $x \in V$. Then there exist $y \in V$ and $w \in I$ such that $v(x) = w(y)$. Hence
\[
v'v(x) = v'w(y) = 0
\]

and $N^2 = 0$. Since $B$ is semi-simple, this implies $N = 0$. But, since $A$ is commutative, $I \subseteq N$, a contradiction. □
THEOREM 6. Let $X$ be a smooth, projective variety of dimension $d$ over a field $k$ of characteristic $\neq 1$. Let $k_s$ be a separable closure of $k$, $G = \text{Gal}(k_s/k)$ and $\overline{X} = X \times_k k_s$. Consider the following conditions:

(i) For all $i \geq 0$, the action of $G$ on $H^i(X, \mathbb{Q}_l)$ is semi-simple.

(ii) $S^d(X \times X)$ holds.

(iii) The algebra $H^{2d}(\overline{X} \times_k \overline{X}, \mathbb{Q}_l(d))^G$ is semi-simple.

Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii), and (iii) $\Rightarrow$ (i) if $k$ is contained in the algebraic closure of a finite field.

PROOF. By the Künneth formula and Poincaré duality, we have the well-known Galois-equivariant isomorphism of $\mathbb{Q}_l$-algebras

$$H^{2d}(\overline{X} \times \overline{X}, \mathbb{Q}_l(d)) = \prod_{q=0}^{2d} \text{End}_{\mathbb{Q}_l}(H^{q}_\text{cont}(\overline{X}, \mathbb{Q}_l)).$$

For $q \in [0, 2d]$, let $A_q$ be the image of $\mathbb{Q}_l[G]$ in $\text{End}_{\mathbb{Q}_l}(H^{q}_\text{cont}(\overline{X}, \mathbb{Q}_l))$. Then condition (i) (resp. (ii), (iii)) of theorem 6 is equivalent to condition (i) (resp. (ii), (iii)) of proposition 5 for all $A_q$. The conclusion follows by remarking that the $A_q$ are commutative if $k$ is contained in the algebraic closure of a finite field.

I don’t know how to prove (iii) $\Rightarrow$ (i) in general in theorem 6, but in fact there is something better:

THEOREM 7. Let $F$ be a finitely generated field over $\mathbb{F}_p$ and let $X$ be a smooth, projective variety of dimension $d$ over $F$. Let $\mathcal{O}$ be a valuation ring of $F$ with finite residue field, such that $X$ has good reduction at $\mathcal{O}$. Let $Y$ be the special fibre of a smooth projective model $\overline{X}$ of $X$ over $\mathcal{O}$. Assume that $S^d(Y \times Y)$ holds. Then the Galois action on the $\mathbb{Q}_l$-adic cohomology of $X$ is semi-simple.

PROOF. For the proof we may assume that $X$ is geometrically irreducible. By Lemma 3 we may also enlarge $F$ by a finite extension and hence, by de Jong [5, Th. 4.1], assume that it admits a smooth projective model $T$ over $\mathbb{F}_p$. By the valuative criterion for properness, $\mathcal{O}$ has a centre $u$ on $T$ with finite residue field $k$. Up to extending the field of constants of $T$ to $k$, we may also assume that $u$ is a rational point. Now
spread $X$ to a smooth, projective morphism

$$f : \mathcal{X} \to U$$

over an appropriate open neighbourhood $U$ of $u$ (in a way compatible to $\overline{X}$).

The action of $G_{F}$ on $H^{*}(\mathcal{X}, \mathbb{Q}_{l})$ factors through $\pi_{1}(U)$. Moreover $u$ yields a section $\sigma$ of the homomorphism $\pi_{1}(U) \to \pi_{1}(\text{Spec } k)$; in other terms, we have a split exact sequence of profinite groups

$$1 \to \pi_{1}(U) \to \pi_{1}(U) \to \pi_{1}(\text{Spec } k) \to 1.$$

Let $i \geq 0$, $V = H^{i}(\mathcal{X}, \mathbb{Q}_{l})$, $\Gamma = G L(V)$ and $\rho : \pi_{1}(U) \to \Gamma$ the monodromy representation. Denote respectively by $A$, $B$, $C$ the Zariski closures of the images of $\pi_{1}(U)$, $\pi_{1}(U)$ and $\sigma(\pi_{1}(\text{Spec } k))$. Then $A$ is closed and normal in $B$, and $B = A C$.

By [2, cor. 3.4.13], $\pi_{1}(U)$ acts semi-simply on $V$; this is also true for $\sigma(\pi_{1}(\text{Spec } k))$ by the smooth and proper base change theorem and Theorem 6 applied to $Y$. It follows that $A$ and $C$ act semi-simply on $V$; in particular they are reductive. But then $B$ is reductive, hence its representation on $V$ is semi-simple and so is that of $\pi_{1}(U)$.

REMARKS 8. 1. If $X$ is an abelian variety, we recover a result of Zarhin [11, 12]. Theorem 7 applies more generally by just assuming that $Y$ is of abelian type in the sense of [6], for example is an abelian variety or a Fermat hypersurface [7]. (Recall, e.g. [6, Lemma 1.9], that the proof of semi-simplicity for an abelian variety $X$ over a finite field boils down to the fact that Frobenius is central in the semi-simple algebra $\text{End}(X) \otimes \mathbb{Q}_{l}$.)

2. If $F$ is finitely generated over $\mathbb{Q}$, this argument gives the following (keeping the notation of Theorem 7). Let $F_{0}$ be the field of constants of $F$. Assume that $S^{d}(Y \times Y)$ holds and that, moreover, the action of $\text{Gal}(\overline{K}/K^{ab})$ on the $\mathbb{Q}_{l}$-adic cohomology of $Z$ is semi-simple, where $Z$ is the special fibre of $\overline{X} \otimes_{F_{0}} \mathcal{O}$ and $K$ is the residue field of $F_{0}$. Then the conclusion of theorem 7 still holds.

To see this, enlarge $F$ as before so that it has a regular projective model $g : T \to \text{Spec } A$ (where $A$ is the ring of integers of $F_{0}$), this time by [5, Th. 8.2]. Let $u$ be the centre of $\mathfrak{O}$ on $T$ and $U$ an open neighbourhood of $u$, small enough so that $X$ spreads to a smooth projective morphism $f : \mathcal{X} \to U$. Let $S = g(U)$ and $s = g(u)$. Up to extending $F_{0}$ and then shrinking $S$, we may assume that $g : U \to S$ has a section $\sigma$ such that $u = \sigma(s)$, that $\mu_{2} \subset \Gamma(S, \mathcal{O}^{\underline{s}})$ and that $\mu_{1}^{-1}(\sigma(s)) = \mu_{1}^{-1}(S)$.

Let $S_{\infty}$ be a connected component of $S \otimes_{F} \mathbb{Z}[\mu_{1}]$ and $U_{\infty} = U \times_{S} S_{\infty}$.
We then have two short exact sequences

\[ 1 \to \pi_1(\mathcal{U}) \to \pi_1(U_\mathbb{Q}) \to \pi_1(S_\mathbb{Q}) \to 1 \]

\[ 1 \to \pi_1(U_\mathbb{Q}) \to \pi_1(U) \xrightarrow{\chi} \mathbb{Z}^p \]

where \( \chi \) is the cyclotomic character. The first sequence is split by \( \sigma \); the second one is almost split in the sense that \( \chi(\pi_1(u)) = \chi(\pi_1(U)) \). By assumption, \( \pi_1(S_\mathbb{Q}) \) acts semi-simply on the cohomology of the generic geometric fibre of \( Z \) and (using theorem 6) \( \pi_1(u) \) acts semi-simply on the cohomology of \( Y \). Arguing as in the proof of theorem 7, we then get that \( \pi_1(U_\mathbb{Q}) \), and then \( \pi_1(U) \), act semi-simply on \( H^*_{\text{cont}}(X, \mathbb{Q}) \). (To justify applying the smooth and proper base change theorem to \( Y \), note that \( gf : X \to S \) is smooth at \( s \) by the good reduction assumption.)

REFERENCES


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