

Elementary Pseudoconcavity and fields of CR meromorphic functions.

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Let M be a smooth CR manifold of CR dimension n and CR codimension k , which is not compact, but has the local extension property E . We introduce the notion of «elementary pseudoconcavity» for M , which extends to CR manifolds the concept of a «pseudoconcave» complex manifold. This notion is then used to obtain generalizations, to the noncompact case, of the results of our previous paper about algebraic dependence, transcendence degree and related matters for the field $\mathcal{K}(M)$ of CR meromorphic functions on M .

It was A. Andreotti who, in a series of papers ([A], [AG], [ASi], [ASt], [AT]), realized that a number of fundamental theorems about compact complex manifolds (or compact analytic spaces) could be carried over to the non-compact case. This involved the introduction of an «elementary notion» of pseudoconcavity: *Let X be a connected non-compact complex manifold. Then X is called elementary pseudoconcave if one can find a non-empty open subset $Y \subset X$ with the following properties:*

- i) Y is relatively compact in X , and ∂Y is smooth.*
- ii) The Levi form of ∂Y restricted to the analytic tangent space has at least one negative eigenvalue at each point of ∂Y .*

In particular, for any point $z_0 \in \partial Y$ there is an analytic disc of complex dimension ≥ 1 which is tangent at z_0 to ∂Y , and is contained in Y except for the point z_0 . A compact X may be considered elementary pseudoconcave, since one may take $Y = X$ and $\partial Y = \emptyset$, so that (i) and (ii) are trivially valid. With this condition it became possible to carry over to X the basic results of

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Siegel [Si] about the field $\mathcal{K}(X)$ of meromorphic functions on X . Thus one was able to discuss the transcendence degree of $\mathcal{K}(X)$ over \mathbb{C} , algebraic versus analytic dependence, the fact that $\mathcal{K}(X)$ is a simple algebraic extension of the field $\mathbb{C}(f_1, \dots, f_d)$ of rational functions of a given transcendence basis f_1, \dots, f_d , as well as an extension of Chow's theorem.

The present work is a sequel to our previous paper [HN12]. There we managed to replace a compact X by a *compact CR* manifold M of *CR* dimension n and *CR* codimension k , which was assumed to have a certain local extension property E . This property E was taken as an axiom. In particular this axiom is satisfied in the important case where M is pseudoconcave. [Note that here the word «pseudoconcave» is being used in a completely different context, and has a completely different meaning than the «elementary pseudoconcave» mentioned above. M being pseudoconcave is a local property of M , having to do with an arbitrarily small neighborhood of each point of M : every complex manifold, compact or not, is pseudoconcave in this sense.] The purpose of the present paper is to find an analogue of the notion of «elementary pseudoconcavity», for a *non-compact CR* manifold M , and to use it to obtain generalizations of most of the results of [HN12]. Thus we study the algebraic dependence, transcendence degree and related matters for the field $\mathcal{K}(M)$ of *CR* meromorphic functions on M .

In order to find a suitable replacement for the notion of «elementary pseudoconcavity», in the context of a *CR* manifold M satisfying property E , we first discuss the complex Hessian $i(\partial\bar{\partial})_M$ acting on real transversal 1-jets on M (see [MN]). This enables us to consider weakly and strongly pseudoconcave domains $Y \subset M$ with smooth boundaries ∂Y . We prove two lemmas which serve to replace the analytic disc, tangent to $z_0 \in \partial Y$, and otherwise contained in Y .

For more information about *CR* manifolds, we refer the reader to [HN1], [HN2], ..., [HN12] and to a number of interesting papers of C. Laurent-Thiébaud and J. Leiterer.

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1. The complex Hessian on *CR* manifolds.

In this paper M will be a smooth (C^∞) paracompact manifold, whose $\dim_{\mathbb{R}} M = 2n + k$, which has a smooth *CR* structure of type (n, k) ; i.e. n is

the $CR - \dim_{\mathbb{C}}$ and k the $CR - \text{codim}_{\mathbb{R}}$. We recall what this means: as an abstract CR manifold M is really a triple $M = (M, HM, J)$, where HM is a smooth real vector subbundle of rank $2n$ of the real tangent bundle TM , and where $J : HM \rightarrow HM$ is a smooth fiber preserving isomorphism such that $J^2 = -I$. It is also required that the *formal integrability conditions* $[\Gamma(M, T^{0,1}M), \Gamma(M, T^{0,1}M)] \subset \Gamma(M, T^{0,1}M)$ be satisfied (Γ means smooth sections). Here $T^{0,1}M = \{X + iJX \mid X \in HM\}$ is the complex subbundle of the complexification $\mathbb{C}HM$ of HM corresponding to the eigenvalue $-i$ of J ; we have $T^{1,0}M \cap T^{0,1}M = 0$ and $T^{1,0}M \oplus \oplus T^{0,1}M = \mathbb{C}HM$, where $T^{1,0}M = \overline{T^{0,1}M}$. When $k = 0$, we recover the abstract definition of a *complex* manifold, via the Newlander-Nirenberg theorem.

We denote by $H^0M = \{\zeta \in T^*M \mid \langle X, \zeta \rangle = 0 \quad \forall X \in H_{\pi(\zeta)}M\}$ the *characteristic bundle* of M . To each $\zeta \in H_x^0M$, we associate the Levi form at ζ :

$$(1.1) \quad L(\zeta, X) = \zeta([\tilde{J}X, \tilde{X}]) = d\tilde{\zeta}(X, JX) \quad \text{for } X \in H_xM$$

which is Hermitian for the complex structure of H_xM defined by J . It is of interest to carry over to smoothly bounded domains Y of an abstract CR manifold M the notions of pseudoconvexity and pseudoconcavity, expressed in terms of locally defining functions, as one has for complex manifolds X , such as in the classical work of Andreotti-Grauert [AG]. Classically these notions are expressed in terms of the complex Hessian of a locally defining function; however in the case of an abstract CR manifold the complex Hessian of a real function cannot be defined intrinsically as a Hermitian form on HM . In order to obtain an invariant notion, one must consider real transversal 1-jets on M .

We denote by $\mathcal{E}^j(M)$ the space of smooth complex valued exterior forms homogeneous of degree j and by $\mathcal{E}^*(M) = \bigoplus_{j=0}^{2n+k} \mathcal{E}^j(M)$ the algebra of complex valued smooth exterior forms on M . Then we consider the ideal \mathcal{J} of $\mathcal{E}^*(M)$ generated by the one-forms vanishing on $T^{0,1}M$ and by $\overline{\mathcal{J}}$ its conjugate, which is the ideal of $\mathcal{E}^*(M)$ generated by the one-forms vanishing on $T^{1,0}M$.

The notion of transversal 1-jet on M is best explained in terms of a choice of a splitting:

$$(1.2) \quad \lambda : \mathcal{E}^1(M) \rightarrow [\mathcal{J} \cap \mathcal{E}^1(M)] \oplus [\overline{\mathcal{J}} \cap \mathcal{E}^1(M)]$$

for the exact sequence:

$$(1.3) \quad 0 \rightarrow \mathcal{J} \cap \overline{\mathcal{J}} \cap \mathcal{E}^1(M) \rightarrow [\mathcal{J} \cap \mathcal{E}^1(M)] \oplus [\overline{\mathcal{J}} \cap \mathcal{E}^1(M)] \rightarrow \mathcal{E}^1(M) \rightarrow 0.$$

Note that such splittings always exist because $\mathcal{J}, \bar{\mathcal{J}}, \mathcal{J} \cap \bar{\mathcal{J}}$ and $\mathcal{E}^*(M)$ are all locally free graded $\mathcal{E}(M)$ -modules. The splitting $\lambda = (\lambda_1, \lambda_2)$ was called a *CR gauge* in [MN]. A *CR gauge* is characterized by

$$(1.4) \quad a = \lambda_1(a) - \lambda_2(a), \quad \lambda_1(a) \in \mathcal{J} \cap \mathcal{E}^1(M), \quad \lambda_2(a) \in \bar{\mathcal{J}} \cap \mathcal{E}^1(M)$$

for all $a \in \mathcal{E}^1(M)$.

In a *CR gauge* λ a *real transversal 1-jet* ψ is represented by the pair $(\psi_0, \psi_1)_\lambda$, where ψ_0 is a smooth real valued function on M , and ψ_1 is a smooth section of H^0M . In a different *CR gauge* λ' , ψ is represented by the pair $(\psi'_0, \psi'_1)_{\lambda'}$, where

$$(1.5) \quad \begin{cases} \psi'_0 = \psi_0 \\ \psi'_1 = \psi_1 + i\lambda'_1(d\psi_0) - i\lambda_1(d\psi_0) \\ \quad = \psi_1 + i\lambda'_2(d\psi_0) - i\lambda_2(d\psi_0). \end{cases}$$

The *complex Hessian* $i(\partial\bar{\partial})_M\psi$ of a real transversal one jet ψ on M is then the Hermitian form on HM defined by

$$(1.6) \quad \begin{aligned} i(\partial\bar{\partial})_M\psi(X, JX) &= d[\psi_1 - i\lambda_1(d\psi_0)](X, JX) \\ &= d[\psi_1 - i\lambda_2(d\psi_0)](X, JX), \end{aligned}$$

for all $X \in HM$. By (1.5) the right hand side does not depend on the choice of the *CR gauge* λ ; hence $i(\partial\bar{\partial})_M\psi$ is an invariant notion on the abstract *CR manifold* M . By our definition of the Levi form on M , we have that

$$(1.7) \quad L(\psi_1, X) = L(\psi_1, JX) = d\psi_1(X, JX).$$

Notice that if M were a complex manifold ($k = 0$), then we would have $\mathcal{J} \cap \bar{\mathcal{J}} = 0$, and (1.2) would be an isomorphism, and so there would be a natural unique choice of the *CR gauge*; moreover $H^0M = 0$, so there is no ψ_1 -term, and hence (1.6) reduces to the usual complex Hessian of a function on the complex manifold. Thus for an open subset Y of a *CR manifold* M with a defining function ψ_0 , there are many ways to extend ψ_0 to a real transversal 1-jet ψ ; however the various ways lead to complex Hessians (1.6) which differ by a Levi form (1.7) of M .

Let us now consider the situation where M is a generic *CR submanifold* of a complex manifold X . If f is a smooth real valued function defined on X , we shall associate to it the real transversal 1-jet ψ on M defined by

$$(1.8) \quad \psi = (f|_M, i\lambda_1(df|_M) - i(\partial f)|_M)|_\lambda,$$

where λ is some fixed *CR gauge* on M . Then the complex Hessian $i(\partial\bar{\partial})_M\psi$

of the associated transversal 1-jet is the restriction to HM of the pullback to M of the complex Hessian $i\partial\bar{\partial}f$ on X :

$$(1.9) \quad i(\partial\bar{\partial})_M\psi = i(\partial\bar{\partial}f)|_M \quad \text{on } HM.$$

On the other hand, given a smooth transversal 1-jet ψ on M , by the Whitney extension theorem, we can find a smooth real valued function f on X such that (1.9) holds.

2. The E -property.

As in [HN12], also in this paper we shall consider CR manifolds M of type (n, k) which have property E (E is for *extension*). We recall that M is said to have property E iff there is an E -pair (M, X) . By an E -pair we mean that

- i) M is a generic CR submanifold of the complex manifold X , and
- ii) for each $a \in M$, the restriction map induces an isomorphism $\mathcal{O}_{X,a} \rightarrow \mathcal{CR}_{M,a}$.

We use the notation: $\mathcal{O}(X)$ and $\mathcal{O}_{X,a}$ are the spaces of holomorphic functions on X and of germs of holomorphic functions defined on a neighborhood of a point $a \in X$, respectively; likewise $\mathcal{CR}(M)$ and $\mathcal{CR}_{M,a}$ are the spaces of smooth CR functions on M and the space of germs of smooth CR functions defined on a neighborhood of a point $a \in M$, respectively.

REMARK. *If M is a pseudoconcave CR manifold, then M has property E .* (see [HN12]).

When $k = 0$, so M is of type $(n, 0)$, then M is an n -dimensional complex manifold, and we obtain an E pair by choosing $X = M$. Hence we adopt the convention that *any complex manifold has property E* .

When $n = 0$, so M is of type $(0, k)$, then M is a smooth totally real k -dimensional manifold, and we can never obtain an E -pair, (unless $M = X = \text{a point}$), because then any smooth function belongs to $\mathcal{CR}(M)$.

In [HN12] it was proved:

THEOREM 2.1. *Let (M, X) be an E -pair. Then for any open set $\omega \subset M$ there is a corresponding open set $\Omega \subset X$ such that*

- i) $\Omega \cap M = \omega$, and
- ii) $r : \mathcal{O}(\Omega) \rightarrow \mathcal{CR}(\omega)$ is an isomorphism.

iii) If $f \in CR(\omega)$, and f vanishes of infinite order at $a \in \omega$, then $f \equiv 0$ in the connected component of a in ω .

iv) $(r^*f)(\Omega) = f(\omega)$.

v) If $|f|$ has a local maximum at a point $a \in \omega$, then f is constant on the connected component of a in ω .

COROLLARY 2.2. *Let (M, X) and (N, Y) be E -pairs, and let $f : M \rightarrow N$ be a smooth CR isomorphism. Then there are E -pairs (M, X') and (N, Y') , with $X' \subset X$ and $Y' \subset Y$, such that f extends to a biholomorphic diffeomorphism $\hat{f} : X' \rightarrow Y'$.*

THEOREM 2.3. *M has property E if and only if for each $a \in M$, there is an open neighborhood ω_a of a in M such that ω_a has property E .*

3. Weakly and strongly pseudoconcave domains.

In this section we consider domains $Y \subset M$ with smooth boundaries. We say that Y is *weakly pseudoconcave* at $a \in \partial Y$ if there is a smooth transversal 1-jet $\phi = (\phi_0, \phi_1)_i$, defined on an open neighborhood U of a in M such that:

1) $\phi_0 : U \rightarrow \mathbb{R}$ is a smooth locally defining function for Y at a : this means that $d\phi_0(a) \neq 0$ and $Y \cap U = \{p \in U \mid \phi_0(p) < 0\}$;

2) $i(\partial\bar{\partial})_M \phi(p) + L(\xi, \cdot)$ has at least 2 negative eigenvalues on $H_p M$ for every $p \in U$ and every $\xi \in H_p^0 M$.

LEMMA 3.1. *Assume that M has property E . Let Y be an open subset of M with a smooth boundary and let U be a connected relatively compact open subset of M . If Y is weakly pseudoconcave at every point of $\partial Y \cap \bar{U}$, then for every function u , which is defined and CR on a neighborhood of \bar{U} , we have:*

$$(3.1) \quad \sup_{\partial Y \cap U} |u| \leq \sup_{\partial(Y \cap U) \setminus \partial Y \cap U} |u|.$$

PROOF. Using a partition of unity, we may assume that $\phi = (\phi_0, \phi_1)_i$ is globally defined on an open neighborhood U' of \bar{U} , that $d\phi_0 \neq 0$ on an open neighborhood W of $\partial Y \cap \bar{U}$, and that condition (2) of the definition is valid at each point $p \in W$.

We argue by contradiction. If the statement were false, one could find a CR function u , that we can assume to be defined on U' , and a point

$p_0 \in \partial Y \cap U$ such that

$$(3.2) \quad \mu = |u(p_0)| = \max_{\overline{Y \cap U}} |u| > \sup_{\partial(Y \cap U) \setminus \partial Y \cap U} |u| = \mu'.$$

Set $F_\delta = \{p \in \overline{U} \mid \phi_0(p) \leq -\delta\}$. Then for small $\delta > 0$ the function $\mu(\delta) = \max_{p \in F_\delta} |u(p)|$ is continuous and decreasing. Choose a small $\delta \geq 0$ such that $\mu \geq \mu(\delta) > \mu'$, $\partial F_\delta \setminus (\partial(Y \cap U) \setminus \partial Y \cap U) \subset W$, and $\mu^2(\delta)$ is not a critical value for the smooth real valued function $|u|^2$, which we may do by Sard's theorem. Then $\mu(\delta) = |u(p_\delta)|$ for some $p_\delta \in \partial Y_\delta \cap U$, because $|u(p)| \leq \mu' < \mu(\delta)$ on $\partial U \cap Y_\delta \setminus (U \cap \partial Y_\delta) \subset \partial(U \cap Y) \setminus (U \cap \partial Y_\delta)$. Then we have $d|u|^2(p_\delta) = kd\phi_0(p_\delta)$ with some $k > 0$, that we can arrange to be 1.

Let X be a complex manifold such that (M, X) is an E pair and consider the open $\tilde{U} \subset X$ corresponding to U given by Theorem 2.1. We denote by $\tilde{\phi}$ a smooth real valued function on \tilde{U} defining on U the transversal 1-jet ϕ . Let ρ_1, \dots, ρ_k be smooth real valued functions on a neighborhood Ω of p_δ in X such that $\partial\rho_1 \wedge \dots \wedge \partial\rho_k \neq 0$ in Ω and $M \cap \Omega = \{z \in \Omega \mid \rho_1 = 0, \dots, \rho_k = 0\}$. Then, denoting by d_X the differential on X , for the holomorphic extension \tilde{u} of u to \tilde{U} we have $d_X|\tilde{u}|^2(p_\delta) = d_X\tilde{\phi}(p_\delta) + \sum_{i=1}^k \xi_i d_X\rho_i(p_\delta)$, for suitable $\xi_1, \dots, \xi_k \in \mathbb{R}$. Then from the inclusion:

$$(3.3) \quad F_\delta \subset \{z \in \tilde{U} \mid |\tilde{u}|^2 \leq \mu^2(\delta)\}$$

it follows that we can find a large constant $C > 0$ such that

$$(3.4) \quad |\tilde{u}|^2 - \mu^2(\delta) \leq \tilde{\phi} - \delta + \sum_{i=1}^k \xi_i \rho_i + C \sum_{i=1}^k \rho_i^2$$

on a neighborhood of p_δ .

By the E.E. Levi theorem [L], all holomorphic functions defined in $\Omega = \{z \in \tilde{U} \mid \tilde{\phi} - \delta + \sum_{i=1}^k \xi_i \rho_i + C \sum_{i=1}^k \rho_i^2 < 0\}$ have a holomorphic extension to an open neighborhood of the point $p_\delta \in \partial\Omega \cap \tilde{U}$. However, the function $z \rightarrow (\tilde{u}(z) - u(p_\delta))^{-1}$ is holomorphic in Ω and cannot be holomorphically extended to an open neighborhood of p_δ in \tilde{U} . This gives a contradiction, completing the proof of the lemma.

Next we fix a Hermitian metric h on the complex vector bundle $HM \rightarrow M$. If $\phi = (\phi_0, \phi_1)_\lambda$ is a transversal 1-jet defined on an open subset U of M , we can consider for each $p \in U$ and $\xi \in H_p^0 M$ the eigenvalues

$$(3.5) \quad \lambda_1(\phi; \xi) \leq \lambda_2(\phi; \xi) \leq \dots \leq \lambda_n(\phi; \xi)$$

of the Hermitian form $i(\partial\bar{\partial})_M \phi(p) + L(\xi, \cdot)$ with respect to h .

Let Y be an open subset of M with smooth boundary. We say that Y is *strongly pseudoconcave* at $a \in \partial Y$ if there is a smooth transversal 1-jet $\phi = (\phi_0, \phi_1)_\lambda$, defined on an open neighborhood U of a in M such that:

- 1) $\phi_0 : U \rightarrow \mathbb{R}$ is a smooth locally defining function for Y at a ;
- 2) there exists a positive constant $c_0 > 0$ and an open neighborhood ω of a in U such that

$$\lambda_2(\phi; \xi) \leq -c_0 < 0 \quad \forall p \in \omega, \forall \xi \in H_p^0 M.$$

We obtain the following

LEMMA 3.2. *Assume that M has property E. Let Y be an open subset of M with a smooth boundary. If Y is strongly pseudoconcave at a point $a \in \partial Y$, then for every open neighborhood U of a in M there is an open neighborhood ω of a in U such that*

$$(3.6) \quad \sup_{\omega} |u| \leq \sup_{Y \cap U} |u| \quad \forall u \in \mathcal{CR}(U).$$

PROOF. We can assume that the open set U is so chosen that there is a 1-jet $\phi = (\phi_0, \phi_1)_\lambda$, defined on a neighborhood of \bar{U} , with ϕ_0 being a locally defining function for Y at all points of $\partial Y \cap \bar{U}$, and $\lambda_2(\phi; \xi) \leq -c_0 < 0$ for all $p \in \bar{U}$ and $\xi \in H_p^0 M$. Let ψ_0 be a non negative smooth real valued function, with compact support in U , and with $\psi_0(a) > 0$. Set $\phi_t = (\phi_0 - t\psi, \phi_1)_\lambda$. Then there exists $\delta > 0$ such that $\lambda_2(\phi_t; \xi) \leq -c_0/2 < 0$ for all $t \in \mathbb{R}$ with $|t| \leq \delta$. Using the previous lemma we obtain (3.6) with $\omega = \{p \in U \mid \phi_0(p) - \delta\psi(p) < 0\}$.

EXAMPLE 1. Let $q \geq 3$ and consider in $\mathbb{C}P^{2q-1}$ homogeneous coordinates (w, z) with $z, w \in \mathbb{C}^q$. Let K denote the union of the two disjoint projective $q - 1$ -planes:

$$K = \{w = iz\} \cup \{w = -iz\}.$$

Let M be the CR submanifold of type $(2q - 3, 2)$ of points in $\mathbb{C}P^{2q-1} \setminus K$ satisfying the homogeneous equations

$$\begin{cases} \sum_{j=1}^q |w_j|^2 = \sum_{j=1}^q |z_j|^2 \\ \sum_{j=1}^q w_j \bar{z}_j + z_j \bar{w}_j = 0. \end{cases}$$

Then, for every $1/2 < \varepsilon < 1$ the relatively compact open subset

$$Y_\varepsilon = \{\min\{|z + iw|^2, |z - iw|^2\} > \varepsilon(|z|^2 + |w|^2)\} \cap M$$

has a smooth boundary and is strongly pseudoconcave. To see this we observe that the boundary consists of two disjoint pieces and that we can take the functions $\phi_\pm = \varepsilon(|z|^2 + |w|^2) - |z \pm iw|^2$ as locally defining functions for ∂Y_ε . Note also that M is $(q - 2)$ -pseudoconcave at each point and hence has property E .

EXAMPLE 2. Denote by $w_0, w_1, w_2, z_0, z_1, z_2$ the homogeneous coordinates of $\mathbb{C}P^5$. Let Q be the ruled quadric described by the homogeneous equations

$$w_0 = 0, z_0 = 0, w_1 z_2 = w_2 z_1.$$

Then consider the noncompact CR manifold M consisting of the points of $\mathbb{C}P^5 \setminus Q$ whose homogeneous coordinates satisfy:

$$\begin{cases} w_0 \bar{w}_1 + w_1 \bar{w}_0 = z_0 \bar{z}_1 + z_1 \bar{z}_0 \\ w_0 \bar{w}_2 + w_2 \bar{w}_0 = z_0 \bar{z}_2 + z_2 \bar{z}_0 \end{cases}$$

Then M is 1-pseudoconcave at all points which do not belong to the 3-plane $\Sigma = \{w_0 = 0, z_0 = 0\}$. The Levi form is identically zero at the points of $\Sigma \cap M$, because $\Sigma \setminus Q \subset M$. Let d be the distance in the Fubini-Study metric of $\mathbb{C}P^5$ and denote by Y_ε the set of points of M having a distance $> \varepsilon$ from Q . Since Q is smooth, for small $\varepsilon > 0$ the boundary of Y_ε is smooth and one can verify, by using the function $\phi(p) = \varepsilon - d(p, Q)$, that for small $\varepsilon > 0$ the domain Y_ε is strongly pseudoconcave at all points $a \in \partial Y_\varepsilon$. Note that M does not have the property E , because it is not minimal at the points of $\Sigma \setminus Q$.

4. CR meromorphic functions on elementary pseudoconcave CR manifolds.

Let M be a connected smooth CR manifold of type (n, k) . We say that M is *elementary pseudoconcave* if it contains a relatively compact non empty open subset Y , with a smooth boundary which is strongly pseudoconcave at every point $a \in \partial Y$.

Note that for a compact M we can take $Y = M$, so that $\partial Y = \emptyset$ and the condition above is trivially satisfied: hence compact CR manifolds are tri-

vially elementary pseudoconcave. The *CR* manifolds M described in the Examples 1,2 at the end of the previous section provide examples of non-compact elementary pseudoconcave *CR* manifolds, the first having and the second not having property E .

For the notion of *CR* meromorphic functions we refer to our previous article [HN12].

THEOREM 4.1. *Let M be a connected smooth *CR* manifold of type (n, k) . If M has property E and is elementary pseudoconcave, then the field $\mathcal{K}(M)$ of *CR* meromorphic functions on M has transcendence degree over \mathbb{C} less or equal to $n + k$.*

Setting $k = 0$ above, we recover Andreotti’s generalization [A] of Satz 1 of Siegel [Si].

PROOF. The statement means: Given $n + k + 1$ *CR* meromorphic functions

$$f_0, f_1, \dots, f_{n+k} \quad \text{on } M,$$

there exists a non zero polynomial with complex coefficients $F(x_0, x_1, \dots, x_{n+k})$ such that

$$(4.1) \quad F(f_0, f_1, \dots, f_{n+k}) \equiv 0 \quad \text{on } M.$$

Because of property (E) we may regard M as a generic *CR* submanifold of an $n + k$ dimensional complex manifold X .

For each point $a \in M$ there is a connected open coordinate neighborhood Ω_a , in which the holomorphic coordinate z_a is centered at a . We choose Ω_a in such a way that $\omega_a = \Omega_a \cap M$ is a connected neighborhood of a in M . Moreover we can arrange that, for $j = 0, 1, \dots, n + k$, each f_j has a representation

$$(4.2) \quad f_j = \frac{p_{ja}}{q_{ja}} \quad \text{on } \omega_a$$

with p_{ja} and q_{ja} being smooth *CR* functions in ω_a . According to Theorem 2.1 we may also assume that the restriction map $\mathcal{O}(\Omega_a) \rightarrow \mathcal{CR}(\omega_a)$ is an isomorphism. For each *CR* function g on ω_a , we denote its unique holomorphic extension to Ω_a by \tilde{g} . By a careful choice of the p_{ja} and q_{ja} , and an additional shrinking of ω_a, Ω_a , we can also arrange that

$$(4.3) \quad \tilde{f}_j = \frac{\tilde{p}_{ja}}{\tilde{q}_{ja}} \quad \text{on } \Omega_a,$$

with the functions \tilde{p}_{ja} and \tilde{q}_{ja} being holomorphic and having no nontrivial common factor at each point in a neighborhood of $\overline{\Omega}_a$. For each pair of points a, b on M we have the transition functions

$$(4.4) \quad \tilde{q}_{ja} = g_{jab} \tilde{q}_{jb},$$

which are holomorphic and non vanishing on a neighborhood of $\overline{\Omega}_a \cap \overline{\Omega}_b$.

Let Y be a relatively compact open subset of M with ∂Y smooth and strongly pseudoconcave at each point. For each $a \in \overline{Y}$ we can choose the polydiscs:

$$(4.5) \quad K_a = \{|z_a| \leq r_a\} \quad \text{and} \quad L_a = \{|z_a| < e^{-1}r_a\},$$

where $|z_a|$ denotes the max norm in \mathbb{C}^{n+k} , and $r_a > 0$, so that $K_a \subseteq \Omega_a$ and for all $u \in \mathcal{CR}(\omega_a)$ we have

$$(4.6) \quad \sup_{K_a} |\tilde{u}| \leq \sup_{Y \cap \omega_a} |u|.$$

This is trivial when $a \in Y$, as we can take $K_a \subset \widetilde{Y \cap \omega_a}$ in this case. When $a \in \partial Y$, we apply Lemma 3.2 to find an open neighborhood ω of a in ω_a so that (3.6) is valid with $U = \omega_a$, and next we choose $K_a \subset \Omega$, where Ω is the open set in X of Theorem 2.1.

By the compactness of \overline{Y} , we may fix a finite number of points a_1, a_2, \dots, a_m on M , such that the $L_{a_1}, L_{a_2}, \dots, L_{a_m}$ provide an open covering of \overline{Y} . Then we choose positive real numbers μ and ν to provide the bounds:

$$(4.7) \quad |g_{0ab}| < e^\mu \quad \text{and} \quad \left| \prod_{j=1}^{n+k} g_{jab} \right| < e^\nu$$

on $\overline{\Omega}_a \cap \overline{\Omega}_b$ for $a, b = a_1, a_2, \dots, a_m$.

Consider a polynomial with complex coefficients to be determined later, $F(x_0, x_1, \dots, x_{n+k})$ of degree s with respect to x_0 and of degree t with respect to each x_i for $i = 1, 2, \dots, n+k$. The number of coefficients to be determined is

$$(4.8) \quad A = (s+1) \cdot (t+1)^{n+k}.$$

Now, letting a stand for any one of the a_1, a_2, \dots, a_m , we introduce the functions

$$(4.9) \quad Q_a = \tilde{q}_{0a}^s \prod_{j=1}^{n+k} \tilde{q}_{ja}^t, \quad P_a = Q_a F(\tilde{f}_0, \tilde{f}_1, \dots, \tilde{f}_{n+k})$$

which are holomorphic on a neighborhood of $\overline{\Omega}_a$. For a positive integer h , to be made precise later, we wish to impose the condition, for $a = a_1, a_2, \dots, a_m$, that P_a vanishes to order h at a . In terms of our local coordinates z_a , this means that all partial derivatives of order $\leq h - 1$ must vanish at $z_a = 0$. This imposes a certain number of linear homogeneous conditions on the unknown coefficients of the polynomial F . The number of such conditions is

$$(4.10) \quad B = m \binom{n+k+h-1}{n+k} \leq m h^{n+k}.$$

If we can arrange that $B < A$, then this system of linear homogeneous equations has a non trivial solution.

However, in order to apply the Schwarz lemma later, we need also to arrange that s , t and h satisfy

$$(4.11) \quad \mu s + \nu t < h.$$

To this end we fix s to be an integer with $s > m\nu^{n+k}$. Thus, for each positive h , we denote by t_h the largest positive integer satisfying $st_h^{n+k} < mh^{n+k}$. In this way we obtain that

$$(4.12) \quad B \leq mh^{n+k} \leq s(t_h + 1)^{n+k} < (s+1)(t_h + 1)^{n+k} = A.$$

On the other hand, since $t_h \rightarrow \infty$ as $h \rightarrow \infty$, by choosing h sufficiently large we have

$$(4.13) \quad m \left(\frac{\mu s}{t_h} + \nu \right)^{n+k} < s,$$

which implies (4.11) for $t = t_h$. Set

$$(4.14) \quad Y = \max_{1 \leq i \leq m} \max_{K_{a_i}} |P_{a_i}|.$$

This maximum is obtained at some point z^* belonging to some K_{a^*} , for a^* equal to some one of a_1, a_2, \dots, a_m . Since $z^* \in K_{a^*} \subset \Omega_{a^*}$, because of our choices of the ω_a , Ω_a , according to (iv) in Theorem 2.1 and (4.6), there is another point $z^{**} \in \omega_{a^*} \cap \overline{Y}$ such that

$$(4.15) \quad P_{a^*}(z^*) = P_{a^*}(z^{**}).$$

But the point z^{**} belongs to some $L_{a^{**}} \subset K_{a^{**}}$, where a^{**} is one of the a_1, a_2, \dots, a_m . Hence by the Schwartz lemma of Siegel [Si] we obtain

$$(4.16) \quad |P_{a^{**}}(z^{**})| \leq Y e^{-h}.$$

However

$$(4.17) \quad P_{a^*}(z^{**}) = P_{a^{**}}(z^{**}) \left[g_{0a^*a^{**}}^s(z^{**}) \prod_{j=1}^{n+k} g_{ja^*a^{**}}^t(z^{**}) \right].$$

Hence from (4.7), (4.15), (4.16) we obtain

$$(4.18) \quad Y = |P_{a^*}(z^{**})| \leq Y e^{\mu s + \nu t - h}.$$

By (4.11) this implies that $Y = 0$. Hence each $P_{\alpha_j} \equiv 0$, which in turn yields $F(\tilde{f}_0, \tilde{f}_1, \dots, \tilde{f}_{n+k}) \equiv 0$. Therefore restricting to M we get (4.1). This completes the proof.

We note that our proof follows closely that of the corresponding result (Theorem 2.1) of [HN12], the only change consisting in the restriction of the covering by polycylinders to the points of the closure of the subdomain Y of M , and the use of Lemma 3.2 to reduce again the discussion to polycylinders centered at points of \bar{Y} .

In a completely similar way we can extend also the other results of [HN12] to the present situation. We shall therefore refer, for the proofs of the following results, to [HN12], as only small changes, similar to those explained in detail in the proof of Theorem 4.1 are needed.

Let $f_0, f_1, \dots, f_\ell \in \mathcal{K}(M)$. We recall that they are *analytically dependent* if

$$(4.19) \quad df_0 \wedge df_1 \wedge \dots \wedge df_\ell = 0 \quad \text{where it is defined.}$$

THEOREM 4.2. *Let M be a connected smooth elementary pseudoconcave CR manifold of type (n, k) , having property E. Let $f_0, f_1, \dots, f_\ell \in \mathcal{K}(M)$. Then they are algebraically dependent over \mathbb{C} if and only if they are analytically dependent.*

THEOREM 4.3. *Let M be a connected smooth elementary pseudoconcave CR manifold of type (n, k) , having property E. Let d be the transcendence degree of $\mathcal{K}(M)$ over \mathbb{C} , and let f_1, f_2, \dots, f_d be a maximal set of algebraically independent CR meromorphic functions in $\mathcal{K}(M)$. Then $\mathcal{K}(M)$ is a simple finite algebraic extension of the field $\mathbb{C}(f_1, f_2, \dots, f_d)$ of rational functions of f_1, f_2, \dots, f_d .*

Setting $k = 0$ above, and taking the special case where $d = n$, we recover Andreotti's generalization [A] of Satz 2 of Siegel [Si].

As in [HN12], this theorem can be derived from the following:

PROPOSITION 4.4. *Let f_1, f_2, \dots, f_ℓ be CR meromorphic functions in $\mathcal{K}(M)$. Then there exists a positive integer $\kappa = \kappa(f_1, f_2, \dots, f_\ell)$ such that every $f_0 \in \mathcal{K}(M)$, which is algebraically dependent on f_1, f_2, \dots, f_ℓ , satisfies a nontrivial polynomial equation of degree $\leq \kappa$ whose coefficients are rational functions of f_1, f_2, \dots, f_ℓ .*

In particular, fix a maximal set f_1, f_2, \dots, f_d of algebraically independent CR meromorphic functions on M , where d is the transcendence degree of $\mathcal{K}(M)$. Consider an $f \in \mathcal{K}(M)$. Then f is algebraically dependent on f_1, f_2, \dots, f_d ; i.e. it satisfies an equation

$$(4.20) \quad f^\lambda + g_1 f^{\lambda-1} + \dots + g_\lambda = 0,$$

where $g_1, g_2, \dots, g_\lambda \in \mathbb{C}(f_1, f_2, \dots, f_d)$. The minimal λ for which such an equation holds is called the *degree* of f over $\mathbb{C}(f_1, f_2, \dots, f_d)$. By Proposition 4.4 this degree is bounded from above by $\kappa = \kappa(f_1, f_2, \dots, f_d)$. Now choose an element $\theta \in \mathcal{K}(M)$ so that its degree a is maximal. For any $f \in \mathcal{K}(M)$ consider the algebraic extension field $\mathbb{C}(f_1, f_2, \dots, f_d, \theta, f)$. By the primitive element theorem this extension is simple; i.e. there exists an element $h \in \mathbb{C}(f_1, f_2, \dots, f_d, \theta, f)$ such that $\mathbb{C}(f_1, f_2, \dots, f_d, \theta, f) = \mathbb{C}(f_1, f_2, \dots, f_d, h)$. Then

$$(4.21) \quad \begin{aligned} a &\geq [\mathbb{C}(f_1, f_2, \dots, f_d, h) : \mathbb{C}(f_1, f_2, \dots, f_d)] \\ &= [\mathbb{C}(f_1, f_2, \dots, f_d, \theta, f) : \mathbb{C}(f_1, f_2, \dots, f_d, \theta)] \\ &\quad \times [\mathbb{C}(f_1, f_2, \dots, f_d, \theta) : \mathbb{C}(f_1, f_2, \dots, f_d)] \\ &\geq a. \end{aligned}$$

Hence the first factor on the right must be one; therefore $f \in \mathbb{C}(f_1, f_2, \dots, f_d, \theta)$. The conclusion is that

$$(4.22) \quad \mathcal{K}(M) = \mathbb{C}(f_1, f_2, \dots, f_d, \theta) = \mathbb{C}(f_1, f_2, \dots, f_d)[\theta],$$

and any $f \in \mathcal{K}(M)$ can be written as a polynomial of degree $< a$ having coefficients that are rational functions of f_1, f_2, \dots, f_d .

From the above remark we derive the

PROPOSITION 4.5. *There is an open neighborhood U of M in X such that the restriction map*

$$(4.23) \quad \mathcal{K}(U) \rightarrow \mathcal{K}(M)$$

is an isomorphism.

Here $\mathcal{K}(U)$ denotes the field of meromorphic functions on U .

Let M be a connected smooth abstract CR manifold of type (n, k) . Consider a complex CR line bundle $F \xrightarrow{\pi} M$ over M . Introduce the graded ring

$$(4.24) \quad \mathcal{A}(M, F) = \bigcup_{\ell=0}^{\infty} \mathcal{CR}(M, F^{\ell}),$$

where $\mathcal{CR}(M, F^{\ell})$ are the smooth global CR sections of the ℓ -th tensor power of F . Note that if $\sigma_1 \in \mathcal{CR}(M, F^{\ell_1})$ and $\sigma_2 \in \mathcal{CR}(M, F^{\ell_2})$, then $\sigma_1 \sigma_2 \in \mathcal{CR}(M, F^{\ell_1 + \ell_2})$.

Assume that we are in a situation where smooth sections of F have the weak unique continuation property; e.g. we could take M to be essentially pseudoconcave (see [HN8]). Then $\mathcal{A}(M, F)$ is an integral domain because M is connected. Let

$$(4.25) \quad \mathcal{Q}(M, F) = \left\{ \frac{\sigma_1}{\sigma_2} \mid \sigma_1, \sigma_2 \in \mathcal{CR}(M, F^{\ell}) \text{ for some } \ell, \text{ and } \sigma_2 \neq 0 \right\}$$

denote the field of quotients.

Then $(^1) \mathcal{Q}(M, F) \subset \hat{\mathcal{K}}(M)$, and $\mathcal{CR}(M) = \mathcal{A}(M, \text{trivial bundle})$.

PROPOSITION 4.6. *Assume that M is elementary pseudoconcave and has property E.*

1) *If F is locally CR trivializable, then $\mathcal{Q}(M, F)$ is an algebraically closed subfield of $\mathcal{K}(M)$.*

2) *There exists a choice of a locally CR trivializable F such that $\mathcal{Q}(M, F) = \mathcal{K}(M)$.*

Assume moreover that M is essentially pseudoconcave $(^2)$. Then

$(^1)$ See [HN12]: We can associate a CR meromorphic function f to any pair (p, q) , where p and q are smooth global CR sections of a smooth complex CR line bundle $F \xrightarrow{\pi} M$, with $q \neq 0$. Another pair (p', q') , which are smooth CR global sections of another such $F' \xrightarrow{\pi} M$, with $q' \neq 0$, define the same f iff $p q' = p' q$ as sections of $F \otimes F'$. Note that $f = p/q$ is a well defined smooth CR function where $q \neq 0$. With this more general definition, we get a new collection $\hat{\mathcal{K}}(M)$ of objects called CR meromorphic functions on M . Observe that $\hat{\mathcal{K}}(M)$ is a field. For an essentially pseudoconcave M , which has property E, $\mathcal{K}(M)$ is a subfield of $\hat{\mathcal{K}}(M)$. If in addition M is 2-pseudoconcave, then all smooth complex CR line bundles over M are locally CR trivializable, and then $\mathcal{K}(M) = \hat{\mathcal{K}}(M)$.

$(^2)$ See [HN8]: M is *essentially pseudoconcave* iff it is *minimal*, i.e. does not contain germs of CR manifolds with the same CR dimension and a smaller CR codimension, and admits a Hermitian metric on HM for which the traces of the Levi forms are zero at each point.

3) $\mathcal{Q}(M, F)$ is algebraically closed in $\hat{\mathcal{K}}(M)$.

In case M is compact and satisfies both hypothesis, then

4) $\mathcal{K}(M)$ is algebraically closed in $\hat{\mathcal{K}}(M)$.

Let M be a connected smooth elementary pseudoconcave CR manifold of type (n, k) , having property E . Then

THEOREM 4.7. *Let $F \xrightarrow{\pi} M$ be a locally CR trivializable smooth complex CR line bundle over M . Then*

$$(4.26) \quad \dim_{\mathbb{C}} \mathcal{CR}(M, F) < \infty.$$

THEOREM 4.8. *Let M be a paracompact connected elementary pseudoconcave CR manifold of type (n, k) having property E . Let $\tau : M \rightarrow \mathbb{C}P^N$ be a smooth CR map. Suppose that τ has maximal rank $2n + k$ at one point of M . Then $\tau(M)$ is contained in an irreducible algebraic subvariety of complex dimension $n + k$, and the transcendence degree of $\mathcal{K}(M)$ over \mathbb{C} is $n + k$.*

THEOREM 4.9. *The following are equivalent:*

1) M has a smooth CR embedding as a locally closed CR submanifold of some $\mathbb{C}P^N$.

2) There exists over M a smooth complex CR line bundle F such that the graded ring $\mathcal{A}(M, F) = \bigcup_{\ell=0}^{\infty} \mathcal{CR}(M, F^{\ell})$ separates points and gives «local coordinates» at each point of M .

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