Generic 2-Coverings of Finite Groups of Lie Type (*).

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Dedicated to G. Zappa for his 90th birthday

Abstract - In [10] it was shown by R.H. Dye that in a symplectic group

\[ G := \text{Sp}_{2n}(2^f) = \text{Iso}(V, \langle \cdot, \cdot \rangle) \]

defined over a finite field of characteristic 2 every
element in \( G \) stabilizes a quadratic form of maximal or non-maximal Witt index
inducing the bilinear form \( \langle \cdot, \cdot \rangle \). Thus \( G \) is the union of the two \( G \)-conjugacy
classes of subgroups isomorphic to \( O^+_2(2^f) \) and \( O^-_2(2^f) \) embedded naturally. In
this paper we classify all finite groups of Lie type \( (G,F) \) with this generic 2-
covering property (Thm. A). In particular, we will show that there exists also an
interesting example in characteristic 3, i.e., in the finite group of Lie type

\[ G := \text{F}_4(3^f) \]

every element in \( G \) is conjugate to an element of the subgroup

\[ B_4(3^f) \leq F_4(3^f) \]
or of the subgroup \( 3.\text{D}_4(3^f) \leq F_4(3^f) \).

1. Introduction.

Let \( G \) be a finite group and let \( H, K \leq G \) be proper subgroups of \( G \). The
pair of subgroups \( \{H, K\} \) is called a 2-covering of \( G \), if every element in \( G \) is
conjugate to an element in \( H \) or \( K \), i.e.,

\[ G = \bigcup_{g \in G} H^g \cup \bigcup_{g \in G} K^g. \tag{1.1} \]

If \( G \) is a Frobenius group with complement \( H \) and kernel \( K \) then \( \{H, K\} \) is
obviously a 2-covering of \( G \). Another example of a finite group with a 2-

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covering is the group $G := SL_2(q)$, $q = p^f$. It can be 2-covered by the pair \( \{ H, K \} \), where $H$ is the normalizer of a $p$-Sylow subgroup, and $K$ is the normalizer of a maximal non-split torus. A more sophisticated class of examples is the following: Let $\mathbb{F}_{2^f}$ be a finite field of characteristic 2 and order $2^f$, let $V$ be an $\mathbb{F}_{2^f}$-vector space of dimension $2\ell$ and let $\langle ., . \rangle : V \otimes V \rightarrow \mathbb{F}_{2^f}$ be a non-degenerate symmetric bilinear form on $V$. In particular, $G := \text{Iso}(V, \langle ., . \rangle) = Sp_{2\ell}(2^f)$ is the symplectic group of degree $2\ell$ defined over the finite field $\mathbb{F}_{2^f}$. Let $Q : V \rightarrow V$ be a quadratic form of maximal Witt index inducing $\langle ., . \rangle$, and let $Q' : V \rightarrow V$ be a quadratic form of non-maximal Witt index inducing $\langle ., . \rangle$. Then $M_1 := \text{Iso}(V, Q) \simeq O_{2\ell}^+(2^f)$ and $M_2 := \text{Iso}(V, Q') \simeq O_{2\ell}^-(2^f)$ are maximal subgroups of $G$, and it was shown by R.H. Dye ([10]) that every element of $G$ is contained in a $G$-conjugate of $M_1$ or $M_2$, i.e., $\{ M_1, M_2 \}$ is a 2-covering of $G = Sp_{2\ell}(2^f)$.

The study of covering properties of finite groups was originally motivated by number theoretic questions. In [15], W. Jehne discovered that Kronecker equivalence of number fields is related to the existence of a certain type of covering of a finite group (cf. [17, §1C]). Therefore, one calls the proper subgroup $U$ of the finite group $G$ a covering subgroup of $G$, if $G = \bigcup_{a \in \text{Aut}(G)} U^a$. The study of finite groups with a covering subgroup was initiated by R. Brandl (cf. [2]), who also conjectured that a finite group with solvable covering subgroup must itself be solvable. Finally, J. Saxl proved this conjecture in showing that a non-abelian finite simple group does not have a covering subgroup (cf. [19]).

The examples mentioned above show that in contrast to the non-existence of covering subgroups there are non-abelian finite simple groups possessing 2-coverings. Special classes of finite simple groups with this covering property have been investigated in [3] and [4]. For finite groups of Lie type there exists a more particular type of covering – a generic 2-covering – which allows to analyze the class of groups with this covering property systematically. The main purpose of this paper will be to classify all generic 2-coverings of finite groups of Lie type.

Let $G$ be a simple algebraic group defined over the algebraically closed field $\bar{\mathbb{F}}$. A closed subgroup $H$ of $G$ is called of maximal rank, if it contains a maximal torus $T$ of $G$. By a finite group of Lie type $(^1)(G, F)$ we understand a simple algebraic group $G$ defined over the algebraic closure $\bar{\mathbb{F}_p}$ of a finite field of characteristic $p$ together with a Frobenius morphism $F : G \rightarrow G$.

$(^1)$ The finite group $G_F$ of elements fixed under $F$ is usually called the finite group of Lie type.
For such a group a pair of proper closed $F$-stable subgroups $\{H, K\}$ will be called a \textit{generic 2-covering} of $(G, F)$, if

(i) every $F$-stable maximal torus $T$ is $G_F$-conjugate to a subgroup of $H$ or $K$,

(ii) $\{H_F, K_F\}$ is a 2-covering of $G_F$,

where $-_F$ denotes the subgroup of elements fixed under $F$. In [25, §1] it was shown that there is a canonical one-to-one correspondence between $G_F$-conjugacy classes of $F$-stable reductive subgroups $H$ of maximal rank of $G$ and $W$-orbits of certain cosets of Weyl subgroups of $W$, where $W$ denotes the Weyl group of $G$. We refer to the $W$-orbit of this coset as the \textit{type} of $H$. This correspondence extends the well-known correspondence between $G_F$-conjugacy classes of $F$-stable maximal tori, and $F$-conjugacy classes in the Weyl group of $G$ (cf. [8, Prop. 3.3.3]). The main purpose of this article is to establish the following theorem (Thm. 4.5, § 5).

**Theorem A.** Let $(G, F)$ be a finite simple group of Lie type, and let $\{H, K\}$ be a generic 2-covering of $(G, F)$.

(a) Assume that for one of the subgroups - say $H$ - the connected component $H^\circ$ is not reductive. Then $(G, F)$ is of type $A_1, A_2, A_3$ or $2A_3$, $H$ is maximal parabolic, and $K^\circ$ is reductive. Moreover, the types of $H/R_n(H)$ and $K^\circ$ are as listed in Table 1.1. Every pair of $F$-stable subgroups $\{H, K\}$, $H$ parabolic, $K := N_G(K^\circ)$, satisfying the conditions of Table 1.1 defines a generic 2-covering of $(G, F)$.

<table>
<thead>
<tr>
<th>$\text{type } (G, F)$</th>
<th>$\text{type } H/R_n(H)$</th>
<th>$\text{type } K^\circ$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_\ell, \ell \geq 1$</td>
<td>$A_1$</td>
<td>$1^W$</td>
</tr>
<tr>
<td></td>
<td>$A_2$</td>
<td>$W(A_1)^W$</td>
</tr>
<tr>
<td></td>
<td>$A_3$</td>
<td>$W(A_2)^W$</td>
</tr>
<tr>
<td></td>
<td>$2A_3$</td>
<td>$2.W(A_2)^W$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$2.W(A_1)^W$</td>
</tr>
</tbody>
</table>

(b) Assume that $H^\circ$ and $K^\circ$ are reductive subgroups. Then the types of $(G, F)$, $H$ and $K$ are as listed in Table 1.2 modulo interchanging the role of $H$ and $K$. Moreover, the characteristic of the algebraically closed field $\mathbb{F}$ must be as indicated in the 5th-column. Every pair of $F$-stable subgroups
\{H, K\}, H := N_G(H^o), K := N_G(K^o), satisfying the conditions of Table 1.2, defines a generic 2-covering of (G, F).

**Table 1.2. – Reductive-type generic 2-coverings**

<table>
<thead>
<tr>
<th>type ((G, F))</th>
<th>type (H^o)</th>
<th>type (K^o)</th>
<th>conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>(A_\ell, \ell \geq 1)</td>
<td>(A_1)</td>
<td>(1^W)</td>
<td>(2^W)</td>
</tr>
<tr>
<td>(A_2)</td>
<td>(W(A_1)^W)</td>
<td>(3^W)</td>
<td>(p = 3)</td>
</tr>
<tr>
<td>(2A_{\ell}, \ell \geq 2)</td>
<td>(2A_2)</td>
<td>(2.W(A_1)^W)</td>
<td>(6^W)</td>
</tr>
<tr>
<td>(C_{\ell}, \ell \geq 2)</td>
<td>(C_2)</td>
<td>(W(A_1^2)^W)</td>
<td>(2.W(A_1^2)^W)</td>
</tr>
<tr>
<td>(C_3)</td>
<td>(W(C_2 \times A_1)^W)</td>
<td>(3.W(A_1^2)^W)</td>
<td>(p = 3)</td>
</tr>
<tr>
<td>(C_{\ell})</td>
<td>(W(\tilde{D}_\ell)^W)</td>
<td>(2.W(\tilde{D}_\ell)^W)</td>
<td>(p = 2)</td>
</tr>
<tr>
<td>(F_4)</td>
<td>(W(B_4)^W)</td>
<td>(3.W(D_4)^W)</td>
<td>(p = 3)</td>
</tr>
<tr>
<td>(G_2)</td>
<td>(W(A_2)^W)</td>
<td>(2.W(A_2)^W)</td>
<td>(p = 2)</td>
</tr>
</tbody>
</table>

Here the connected component of an algebraic group \(X\) is denoted by \(X^o\). In Table 1.2 we used the convention that a root system with a \(\sim\) consists of short roots. Moreover, for a subroot system \(\Psi\) of the root system \(\Phi\) of \(G\) and \(n \in \mathbb{N}\), \(n.W(\Psi)\) will stand for a coset \(n_\Psi.W(\Psi)\), \(n_\Psi \in N_W(W(\Psi))\), with \(n_\Psi.W(\Psi) \in N_W(W(\Psi)).W(\Psi)\) of order \(n\). While this notations is not in-ambiguous, it will suffice for our needs, and from the context it will always be clear which particular coset is meant to be considered.

It should be remarked that Theorem A is independent of the isogeny class of \(G\). The proof of Theorem A is organized in several steps. Using the results in [25, § 1] one concludes that a pair of reductive subgroups \(\{H, K\}\) satisfying property (i) corresponds to a certain type of covering of the Weyl group (coset) which will be called an admissible 2-coset covering (cf. Thm. 2.2). All such coverings can be classified (cf. Thm. 3.4, Table 3.1).

In [20] closed subgroups of a simple algebraic group of positive dimension containing a regular unipotent element were classified. Fortunately, there are few examples of these groups which occur also in Table 3.1. Apart from 3 exceptions this exclusion principle will lead precisely to the examples listed in Table 1.1 and 1.2 (cf. Thm. 4.5).

In order to obtain a full classification of all generic 2-coverings, we have to show that a pair of subgroups \(\{H, K\}\) with the properties described in
Table 1.1 and 1.2 is indeed a generic 2-covering. This will be done in section 5, 6, 7 by a case-by-case analysis. Unfortunately, for some examples we could not find a geometric argument, and hence in those cases the only way in proving the 2-covering property is by brute force (cf. § 6, § 7).

In a subsequent paper [5] we will determine all 2-coverings of classical groups of Lie type using the classification of finite simple group. It turns out that there are also 2-coverings of classical groups of Lie type which are non-generic.

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2. F-stable reductive subgroups of maximal rank

2.1 – Weyl groups and admissible cosets

By $W(\Phi) \leq \text{Iso}(E, (., .))$ we denote the Weyl group of the finite crystallographic rootsystem $\Phi \subset E$ in the euclidean space $(E, (., .)).$ Let $\Delta \subset \Phi$ be a basis of $\Phi,$ and let $\gamma \in \text{Aut}(W(\Phi))$ be a graph automorphism of $W(\Phi)$ leaving $\Delta$ invariant. In particular,

\begin{equation}
\tilde{W} := \langle \gamma \rangle . W(\Phi) \leq GL(R(E)).
\end{equation}

The coset $\gamma . W(\Phi)$ carries a natural right $W(\Phi)$-action given by

\begin{equation}
(\gamma . w)^x := \gamma . (\gamma (x^{-1}) w x), \quad x, w \in W,
\end{equation}

where $\gamma (x) = \gamma ^{-1} . x \gamma.$ We call the coset $\gamma . W(\Phi) \subseteq \tilde{W}$ an admissible $W(\Phi)$-coset.

Let $(G, F)$ be a finite group of Lie type, i.e., $G$ is a simple algebraic group defined over $\overline{F}_p,$ the algebraic closure the finite field $F_p,$ and $F': G \to G$ is a Frobenius morphism. Let $B \leq G$ be an $F$-stable Borel subgroup of the algebraic group $G$ containing the $F$-stable maximal torus $T$. Then $W := N_G(T)/T$ is the Weyl group of a finite irreducible crystallographic rootsystem $\Phi,$ i.e., $W = W(\Phi).$ Let $\Delta \subset \Phi$ be a basis of $\Phi$ corresponding to the Borel subgroup $B.$ The Frobenius morphism $F': G \to G$ is acting on $\Phi$ leaving $\Delta$ invariant, and thus corresponds to a graph automorphism $\gamma \in \text{Aut}(W).$ On the contrary, every pair $(\gamma, \Phi),$ where $\Phi$ is a finite irreducible crystallographic rootsystem and $\gamma$ is a graph automorphism of $W(\Phi),$ defines a finite group of Lie type $(G, F_{\gamma})$ for every standard Frobenius morphism $F_0$ (cf. [8, § 1.1.18]).
We use the same symbols which are used in the classification of finite
groups of Lie type for the definition of the type of an admissible coset
\( \gamma W(\Phi) \), i.e., we say that \( \gamma W(\Phi) \) is of type \( ^2A_\ell \), if \( \Phi \) is a rootsystem of type
\( A_\ell \) and if \( \gamma \) is the graph automorphism of order 2, etc.

2.2 – Subrootsystems and saturated subrootsystems

Every subrootsystem \( \Psi \subseteq \Phi \) of the finite crystallographic root system
\( \Phi \) is again crystallographic. Moreover, there exists an easy algorithm
described in [6, p. 8] to determine all maximal subrootsystems of \( \Phi \).

For two roots \( \alpha, \beta \in \Phi \) one defines the root cone through \( \alpha \) and \( \beta \) by

\[
C(\alpha, \beta) := \{ k_1 \alpha + k_2 \beta \mid k_1, k_2 \in \mathbb{N} \} \cap \Phi.
\]

A subrootsystem \( \Psi \) is called saturated, if for all \( \alpha, \beta \in \Psi \), \( C(\alpha, \beta) \subseteq \Psi \).

Obviously, if in the finite irreducible crystallographic root system \( \Phi \) all
roots are of the same length, every subrootsystem \( \Psi \subseteq \Phi \) is saturated
(cf. [12, § 9.4]). But for example, the subrootsystem \( \tilde{A}_1^\ell \) consisting of all
short roots in \( \Phi(B_\ell) \) is not saturated. It is an easy exercise to show that a
subrootsystem \( \Psi \) is saturated, if and only if for every pair of roots \( \alpha, \beta \in \Psi \),
\( \Psi \) contains also the \( a \)-string through \( \beta \).

Let \( \mathfrak{L} = \mathfrak{L}(\Phi) \) be the simple finite-dimensional \( \mathbb{C} \)-Lie algebra with
rootsystem \( \Phi \) and Cartan decomposition

\[
\mathfrak{L} = \mathfrak{C} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{L}_\alpha.
\]

Then \( \mathfrak{L}(\Psi) := \mathfrak{C} \oplus \bigoplus_{\alpha \in \Psi} \mathfrak{L}_\alpha \) is a Lie-subalgebra, if and only if \( \Psi \subseteq \Phi \) is saturated
(cf. [7, § 4.2], [12, Prop. 8.4(e)]).

Let \( G \) be the simple simply-connected algebraic group defined over \( \mathbb{C} \)
with rootsystem \( \Phi \), and let \( T \leq G \) be a maximal torus. From Chevalley’s commutator formula
(cf. [7, Thm. 5.2.2]) one concludes that
\( H := \langle T, U_\alpha \mid \alpha \in \Psi \rangle \) is a reductive subgroup with rootsystem of type \( \Psi \),
if and only if \( \Psi \subseteq \Phi \) is saturated.

Let \( \tilde{G} \) be the simple simply-connected algebraic group defined over \( \tilde{\mathbb{F}}_p \)
with rootsystem \( \Phi \), and let \( T \leq \tilde{G} \) be a maximal torus. We call the subrootsystem
\( \Psi \subseteq \Phi \) \( p \)-saturated, if \( H := \langle T, U_\alpha \mid \alpha \in \Psi \rangle \) is a reductive sub-
group with rootsystem of type \( \Psi \). One has the following:

**Proposition 2.1.** Let \( \Phi \) be a finite irreducible crystallographic root-
system and let \( \Psi \subseteq \Phi \) be a subrootsystem.
(a) If \( \Psi \subseteq \Phi_{\text{long}} \) consists only of long roots, \( \Psi \) is saturated.

(b) If \( \Psi \) is saturated, it is \( p \)-saturated for all primes \( p \).

(c) Assume \( \Phi \) has two root lengths \( r_l, r_s \) and that \( p \neq r_l/r_s \). Then, if \( \Psi \) is \( p \)-saturated, it is also saturated.

(d) If \( \Phi \) has two root lengths \( r_l, r_s \) and \( p = r_l/r_s \), every subrootsystem is \( p \)-saturated.

**Proof.** Let \( a, \beta \in \Psi \) be linearly independent roots, and let \( \Gamma \subseteq \Phi \) be the subrootsystem generated by \( C(a, \beta) \). If \( \Gamma \) is of type \( A_2 \) or \( A_1 \times A_1 \), \( \Gamma \) coincides with the rootsystem spanned by \( a \) and \( \beta \). The same is true if \( \Gamma \) is of type \( B_2 \) or \( G_2 \) and \( a \) and \( \beta \) are long roots (cf. [7, p. 46, fig. 1]). This yields (a). (b) is a direct consequence of Chevalley’s commutator formula. Assume that \( \Psi \) is \( p \)-saturated. The rootsystem \( \Gamma \) coincides with the rootsystem spanned by \( a \) and \( \beta \) unless \( \Gamma \) is of type \( B_2 \) or \( G_2 \) and \( a \) and \( \beta \) are short roots. If \( p = r_l/r_s \), the constants \( C_{i,j,a,\beta} \) (cf. [7, p. 62, p. 77]) vanish whenever \( ia + j\beta \) is a long root. This yields (d). However, if \( p \neq r_l/r_s \), they do not vanish. Thus, if \( \Psi \) is \( p \)-saturated, \( C(a, \beta) \subseteq \Psi \) must hold which implies (c).

\( \square \)

### 2.3 \(-\gamma\)-Subrootsystems

Let \( \gamma \) be a graph automorphism of the finite irreducible crystallographic rootsystem \( \Phi \) and let \( \Psi \subseteq \Phi \) be a subrootsystem of \( \Phi \). Then \( \Psi \) is called a \( \gamma \)-subrootsystem, if there exists an element \( w \in W(\Phi) \) such that \( \gamma \Psi = w\Psi \).

Let \( G \) a simple algebraic group with rootsystem of type \( \Phi \) defined over \( \bar{F}_p \), and let \( F: G \rightarrow G \) be a Frobenius morphism acting on \( \Phi \) through \( \gamma \). Let \( T \leq G \) be a maximal torus and let \( \Psi \subseteq \Phi \) be a subrootsystem of \( \Phi \). In [25, Prop. 2] it was shown that \( G \) contains an \( F \)-stable reductive group \( H = \langle T, U_a \mid a \in \Psi \rangle^g, g \in G \), if and only if \( \Psi \) is \( p \)-saturated and if \( \Psi \) is a \( \gamma \)-subrootsystem. Additionally, for all twisted types the \( \gamma \)-subrootsystems were determined.

### 2.4 Admissible 2-coset coverings for Weyl groups

Let \( \Phi \) be a finite irreducible crystallographic rootsystem, let \( W = W(\Phi) \) be its Weyl group, and let \( \gamma: W \rightarrow W \) be as in § 2.1. Let \( G_F \) be a finite group of Lie type, where \( G \) is a simple algebraic group defined over \( \bar{F}_p \) with rootsystem \( \Phi \) and assume that the Frobenius morphism \( F: G \rightarrow G \)
is acting on $\Phi$ through $\gamma$. Let $B$ be an $F$-stable Borel subgroup of $G$ containing the maximal torus $T$ and let $U_a, a \in \Phi$, denote the root groups for $T$ in $B$ and $B^{-}$.

For a $p$-saturated $\gamma$-subrootsystem $\Psi \subseteq \Phi$ and $n_{\Psi} \in N_w(W(\Psi)) \cap \cap \gamma.W(\Phi)$, the $W$-orbit $(n_{\Psi}.W(\Psi))^{W}$ of the coset $n_{\Psi}.W(\Psi)$ corresponds to a $G_F$-conjugacy class of finite reductive groups $H_F$, where $H := \langle T, U_a | a \in \Psi \rangle^g$ is $F$-stable, $F(g)g^{-1} \in N_G(T)$, $\pi(F(g)g^{-1}) = n_{\Psi}$, $\pi : N_G(T) \to W$ the canonical projection. This follows from [25, Prop. 4] bearing in mind that left translation with $\gamma, \gamma^{-1} : W \to \gamma.W$, is a bijection of right $W$-sets, where $W$ carries the right $W$-action given by $wx = \gamma x^{-1}wx$, $x, w \in W$. In order to analyze the problem further we make the following definition:

**Definition 2.1.** Let $\gamma.W$ be an admissible $W$-coset of the finite Weyl group $W = W(\Phi)$. Let $\Psi, \Xi \subseteq \Phi$ be proper $\gamma$-subrootsystems of $\Phi$, and let $n_{\Psi}, n_{\Xi} \in \gamma.W$ be elements with the following properties:

(i) $n_{\Psi} \in N_w^{-}(W(\Psi))$, $n_{\Xi} \in N_w^{-}(W(\Xi))$,

(ii) $(\gamma.W)^W = (n_{\Psi}.W(\Psi))^W \cup (n_{\Xi}.W(\Xi))^W$.

Then we call $(n_{\Psi}.W(\Psi), n_{\Xi}.W(\Xi))$ an admissible 2-coset covering of $\gamma.W$.

From the definition and [25, §1] one concludes the following:

**Theorem 2.2.** Let $(G, F)$ be a finite group of Lie type with root system $\Phi$ whose $F$-action is given by $\gamma \in \text{Aut}(\Phi)$. Put $W := W(\Phi)$. Let $H$ and $K$ be $F$-stable reductive groups of maximal rank with $G_F$-conjugacy class corresponding to the $W$-conjugacy class $(n_{\Psi}.W(\Psi))^W$ and $(n_{\Xi}.W(\Xi))^W$, respectively, $n_{\Psi}, n_{\Xi} \in \gamma.W$ (cf. [25, Prop. 4]).

(a) $(n_{\Psi}.W(\Psi), n_{\Xi}.W(\Xi))$ is an admissible 2-coset covering of $\gamma.W$, if and only if every $F$-stable maximal torus $T$ is $G_F$-conjugate to a torus in $H$ or $K$.

(b) If $(n_{\Psi}.W(\Psi), n_{\Xi}.W(\Xi))$ is an admissible 2-coset covering of $\gamma.W$, every semisimple element $s \in G_F$ is $G_F$-conjugate to an element in $H_F$ or $K_F$.

**Proof.** For (a) see [25, Prop. 4, Thm. 5]. Since every semisimple element $s \in G_F$ is contained in a maximal $F$-stable torus, (b) is a direct consequence of (a).
3. Admissible 2-coset coverings for Weyl groups

3.1 – The Coxeter class

Let $\Phi$ be a finite irreducible crystallographic root system and let $W = W(\Phi)$ denote its Weyl group. There exists a very particular $W$-conjugacy class $w^W$, the class of Coxeter elements (cf. [7, §10.4]). Elements of this class have the following well-known property:

**Proposition 3.1.** Let $\Phi$ be a finite irreducible crystallographic root system and let $W = W(\Phi)$ denote its Weyl group. Let $w$ be a Coxeter element of $W$ and let $\Psi \subseteq \Phi$ be a subsystem of $\Phi$, such that $w \in W(\Psi)$. Then $\Psi = \Phi$.

**Proof.** We denote by $E_C := E \otimes \mathbb{C}$ the complexification of $E$ and denote by $(\cdot, \cdot)_C$ the standard $W$-invariant hermitian form induced by $(\cdot, \cdot)$. Let $\Psi \subseteq \Phi$ be a subsystem such that $w \in W(\Psi)$.

The Coxeter element $w \in W$ is an element of order $k := |\Phi|/\ell$, where $\ell = \text{rk}(\Phi)$ is the rank of $\Phi$. It is also regular, i.e., there exists a primitive $k^{th}$ root of unity $\xi \in \mathbb{C}^*$ such that the eigenspace $V_\xi(w)$ of $w$ on $E_C$ for the eigenvalue $\xi$ is one-dimensional, and all eigenvectors $v \in V_\xi(w) \setminus \{0\}$ are regular, i.e., they are not invariant under any non-trivial element of $W$ (cf. [23, Thm. 4.2]).

Let $\Psi := \bigcup_{i=1}^r \Psi_i$ be the decomposition of $\Psi$ in irreducibility components, and let $V_i := \text{span}_C (\Psi_i)$, $V_0 := (V_1 \oplus \cdots \oplus V_r)^\perp$, where $\perp$ denotes the orthogonal complement with respect to the hermitian form $(\cdot, \cdot)_C$. The subspaces $V_i$, $i = 0, \ldots, r$, are $W(\Psi)$-invariant subspaces, and hence in particular $w$-invariant. Since $w$ does not have the eigenvalue 1 (cf. [7, Prop. 10.5.6]), $V_0 = 0$ and $\Psi$ must be of rank $\ell = \text{rk}(\Phi)$.

Assume $r \geq 2$. As $V_\xi(w)$ coincides with the $\xi$-Fitting component of $w$ on $E_C$, it must be contained in one of the $V_i$, $i \in \{1, \ldots, r\}$. Hence $W(\Psi_j), j \neq i$, fixes $V_\xi(w)$, a contradiction. Hence $\Psi$ must be an irreducible subsystem of maximal rank.

The element $w \in W(\Psi)$ is also regular in $W(\Psi)$, and hence its order $o(w)$ is a regular number of $\Psi$. In particular, $o(w) \leq |\Psi|/\ell$ (cf. [23, §5]). This yields $\Psi = \Phi$. □

3.2 – Conjugacy classes in $W(A_\ell)$

The Weyl group $W = W(A_\ell)$ of type $A_\ell$ is isomorphic to $S_{\ell+1}$. The conjugacy classes of $W$ can be parametrized by the set of partitions of
The mapping is given by assigning each element \( w \in W \) its cycle type of the action on the set \( \{1, \ldots, \ell + 1 \} \). The Coxeter elements are the elements of cycle type \((\ell + 1)\). We will make use of the following property:

**Proposition 3.2.** (a) Let \( U \leq S_n, \ n \geq 2, \) be a subgroup containing elements \( \rho, \sigma, \tau \) of cycle type \((2,1,\ldots,1),(n)\) and \((1,n-1)\). Then \( U = S_n.\)

(b) Let \( U \leq S_n, \ n \geq 4, \) be a subgroup containing elements \( \rho, \sigma, \tau \) of cycle type \((2,1,\ldots,1),(1,n-1)\) and \((2,n-2)\). Then \( U = S_n.\)

**Proof.** (a) The subgroup \( U \) is 2-transitive on \( \{1, \ldots, n\} \), and thus in particular primitive. Thus the claim follows from Jordan’s theorem (cf. [14, Satz 4.5(b)]).

(b) From the hypothesis \( n \geq 4 \), one concludes that \( U \) is transitive, and the cycle type of \( \sigma \) yields that \( U \) is 2-transitive. The claim then follows by the same argument as used in (a).

\[ \square \]

### 3.3 – Conjugacy classes in \( W(B_\ell) \)

Let \( W = W(B_\ell) \) be the Weyl group associated to the root system of type \( B_\ell \). In this case \( W \) is acting faithfully on the set \( \mathcal{E}_\ell := \{ \pm e_i \mid 1 \leq i \leq \ell \} \) of positive and negative vectors of an orthonormal basis \( e_1, \ldots, e_\ell \) of the euclidean space \((\mathbb{E},(,))\). Assigning every element \( w \in W \) the cycle type of this action yields a parametrization of \( W \)-conjugacy classes by partitions \(( , )\) with positive and negative parts. The \( W \)-conjugacy class containing the unit element \( 1_W \in W \) corresponds to the \( \pm \)-partition \((1,\ldots,1)\); the Coxeter class corresponds to the \( \pm \)-partition \((\ell)\).

The Weyl group \( W(B_\ell) \) contains the element \( w_0 \in W(B_\ell) \) acting as \(-id_\mathbb{E}\) on \( \mathbb{E} \). Multiplication with \( w_0 \) yields an automorphism \( w_0 \cdot (,): W^W \rightarrow W^W \) of order 2. It corresponds to the automorphism of the set of \( \pm \)-partitions of \( \ell \) fixing blocks of even length and changing the sign of blocks of odd length. The conjugacy class consisting of the element \( w_0 \) corresponds to the \( \pm \)-partition \((1,\ldots,1)\).

The subrootsystem \( \Phi_{long} \subset \Phi \) consisting of long roots is of type \( D_\ell \). A conjugacy class is contained in the Weyl subgroup \( W(\Phi_{long}) \), if and only if the number of negative blocks is even. The conjugacy class of long reflections, i.e., reflections corresponding to long roots, corresponds to the \( \pm \)-partition \((2,1,\ldots,1)\). The conjugacy class of short reflections, i.e., reflections corresponding to short roots, corresponds to the \( \pm \)-partition \((1,\ldots,1,1)\).
There exists a unique $W$-orbit of subroot systems $\Psi$ of type $A_{\ell-1}$ consisting of long roots. The $W$-conjugacy classes of elements being contained in a $W$-conjugate of $W(\Psi)$ correspond to the $\pm$-partitions with trivial negative part (\(\_\_\_\_\_\)).

Let $E \triangleleft W$ denote the normal subgroup generated by all reflections corresponding to short roots. The canonical projection $\pi: W \to W/E \cong S_\ell$ is mapping reflections to reflections, and thus in particular Weyl subgroups to Weyl subgroups. Omitting the “$|$” in the notation describes its effect on the conjugacy classes. Let $\Psi \subseteq \Phi$ be a subroot system such that $\pi(W(\Psi)) = W/E$. Then $\Psi$ is of type $A_{\ell-1}, D_\ell$ or $B_\ell$. This is a consequence of the following more general property:

**Proposition 3.3.** Let $\Phi$ be a crystallographic finite root system with at most two root lengths. Let $\tau: W(\Phi) \to S_m$, $m \geq 2$, be a surjective morphism mapping long reflections to long reflections. Then $\Phi$ contains a subrootsystem of type $A_{m-1}$ consisting of long roots.

**Proof.** Let $R(S_m)$ denote the set of reflections in $S_m$, and let $R_\ell$ denote the set of long reflections in $W(\Phi)$. Since $R(S_m)$ is a single $S_m$-conjugacy class, the hypothesis implies that $\tau$ induces a surjective map $\tau_R: R_\ell \to R(S_m)$. Let $\Sigma' = \{\sigma_i \mid 1 \leq i \leq m - 1\} \subset R(S_m)$ be a simple generation system of $S_m$, and let $\Sigma = \{\sigma_i \mid 1 \leq i \leq m - 1\}$ be a preimage of $\Sigma'$ under $\tau_R$. For any pair $(\sigma_i, \sigma_j)$ of long reflections the order of its product satisfies $o(\sigma_i \sigma_j) \in \{1, 2, 3\}$, and either two of these numbers are coprime. This implies that the Coxeter matrices of $\Sigma$ and $\Sigma'$ coincide. This yields the claim.

\[\square\]

### 3.4 – Conjugacy classes in $W(D_\ell)$

Conjugacy classes in the Weyl group $W = W(D_\ell)$ can be best understood to think of $W(D_\ell)$ as the Weyl subgroup of $W(B_\ell)$ being generated by long reflections. For $w \in W(D_\ell)$ let $t(w)$ denote the $\pm$-partition associated to $w$ being considered as an element in $W(B_\ell)$. We call $t(w)$ the type of $w$. Obviously, $W(D_\ell)$-conjugate elements have the same type, and a $\pm$-partition is the type of an element $w \in W(D_\ell)$, if and only if the number of negative blocks is even. If $\ell$ is odd, one has $W(B_\ell) = \langle w_0 \rangle W(D_\ell)$. Hence for every $\pm$-partition $\lambda$ with an even number of negative blocks, $t^{-1}(\{\lambda\})$ consists of a single $W(D_\ell)$-conjugacy class. When $\ell$ is even, $t^{-1}(\{\lambda\})$ consists of either 2 or 1 $W(D_\ell)$-conjugacy classes.
Restriction of the mapping \( \pi: W(B_\ell) \to S_\ell \) of section 3.3 yields a surjective mapping \( \pi': W(D_\ell) \to S_\ell \) with kernel \( \ker(\pi') = W(A^{\ell-1}_1) \). Note that the only proper rootsubsystems \( \Phi \) of \( \Phi \) for which \( \pi'(W(\Phi)) = S_\ell \) are of type \( A^{\ell-1}_1 \) (cf. Prop. 3.3). There is a unique \( W(D_\ell) \)-orbit of such subrootsystems in case that \( \ell \) is odd, and there are two such \( W(D_\ell) \)-orbits for \( \ell \) even.

3.5 – The classification of admissible 2-coset coverings for Weyl groups

The information we have collected in the previous subsection can now be used to classify all admissible 2-coset coverings:

**Theorem 3.4.** Let \( \Phi \) be a finite irreducible crystallographic rootsystem, let \( W = W(\Phi) \) be its Weyl group and let \( \gamma.W \) be an admissible 2-coset (cf. 2.1). Assume that \((n_\Phi,W(\Phi),n_\Xi,W(\Xi))\) is an admissible 2-coset covering of \( \gamma.W \). Then the types of \((n_\Phi,W(\Phi),n_\Xi,W(\Xi))\) - or \((n_\Xi,W(\Xi),n_\Phi,W(\Phi))\) - are as listed in Table 3.1. Moreover, every tuple \((n_\Xi,W(\Xi),n_\Phi,W(\Phi))\) given as in Table 3.1 is an admissible 2-coset covering of \( \gamma.W \).

**Table 3.1. – Admissible 2-coset coverings**

<table>
<thead>
<tr>
<th>type ( \gamma.W )</th>
<th>( n_\Phi.W(\Phi) )</th>
<th>( n_\Xi.W(\Xi) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A_\ell, \ell \geq 1 )</td>
<td>( A_1 )</td>
<td>1</td>
</tr>
<tr>
<td>( A_2 )</td>
<td>( W(A_1) )</td>
<td>3</td>
</tr>
<tr>
<td>( A_3 )</td>
<td>( W(A_2) )</td>
<td>2.W(( A_1^2 ))</td>
</tr>
<tr>
<td>( A_\ell, \ell \geq 2 )</td>
<td>( 2.A_2 )</td>
<td>2.W(( A_1 ))</td>
</tr>
<tr>
<td>( 2.A_3 )</td>
<td>2.W(( A_2 ))</td>
<td>2.W(( A_1^2 ))</td>
</tr>
<tr>
<td>( B_\ell, \ell \geq 2 )</td>
<td>( B_2 )</td>
<td>( W(\tilde{A}_1^2) )</td>
</tr>
<tr>
<td>( B_3 )</td>
<td>( W(B_2 \times \tilde{A}_1) )</td>
<td>3.W(( \tilde{A}_1^2 ))</td>
</tr>
<tr>
<td>( B_4 )</td>
<td>( W(B_3 \times \tilde{A}_1) )</td>
<td>2.W(( B_2^2 ))</td>
</tr>
<tr>
<td>( B_\ell )</td>
<td>( W(D_\ell) )</td>
<td>2.W(( D_\ell ))</td>
</tr>
<tr>
<td>( F_4 )</td>
<td>( W(B_4) )</td>
<td>3.W(( D_4 ))</td>
</tr>
<tr>
<td>( G_2 )</td>
<td>( W(A_2) )</td>
<td>2.W(( A_2 ))</td>
</tr>
<tr>
<td>( W(C_4) )</td>
<td>3.W(( D_4 ))</td>
<td></td>
</tr>
<tr>
<td>( W(\tilde{A}_2) )</td>
<td>2.W(( A_2 ))</td>
<td></td>
</tr>
</tbody>
</table>
The case $\Phi$ of type $C_\ell$ leads to the same examples as for $\Phi$ of type $B_\ell$
with short and long roots exchanged. Therefore, we have not listed these
cases in Table 3.1 explicitly.

**Proof.** Assume that $(n_\Psi, W(\Psi), n_\Xi, W(\Xi))$ is an admissible 2-coset
covering of $\gamma_1 W$. We denote by $\bar{n}_\Psi$ and $\bar{n}_\Xi$
the canonical homomorphic images of $n_\Psi$ and $n_\Xi$ in $N_W(W(\Psi))/W(\Psi)$
and $N_W(W(\Xi))/W(\Xi)$, respectively. We may assume that $\gamma \in n_\Psi, W(\Psi)$.
Thus in case that $\gamma = 1, n_\Xi, W(\Xi)$
contains a Coxeter element and thus $\bar{n}_\Xi \neq 1$ (cf. Prop. 3.1). The proof
proceeds by a case-by-case analysis depending on the type of $\gamma_1 W$.

### 3.5.1 — $\gamma_1 W$ of type $A_\ell$

For $\ell \leq 3$ an easy exercise shows that the cases listed in Table 3.1 are
the only examples one has in this case. Since for $\ell \geq 4$ no proper Weyl
subgroup of $W$ is normal in $W$, $(W(\Psi), N_W(W(\Xi)))$ is a 2-covering of $W$.
From [3, Thm. 4.2] one concludes that this implies $\ell \leq 5$. It remains to
analyze the cases $\ell = 4, 5$ explicitly.

Assume $\ell = 4$, i.e. $W = S_5$. For any proper non-trivial subrootsystem $\Xi$
of $\Phi$, $N_W(W(\Xi))/W(\Xi)$ is a 2-group. Hence $\Xi = \emptyset$ must be trivial, and $n_\Xi$
is an element of order 5. The maximal subrootsystems of $\Phi$ are of type $A_3$ and
$A_2 \times A_1$, and thus $W(\Psi)$ does not contain elements of cycle type (2, 3)
in the first case and does not contain elements of cycle type (1, 4) in the latter,
respectively. Hence admissible 2-coset coverings do not exist for $\ell = 4$.

Assume $\ell = 5$, i.e., $W = S_6$. There is one $W$-conjugacy class of Weyl
subgroups $W(\Psi)^W$ containing a 5-cycle corresponding to the subrootsystems of type $A_4$.
The group $W(\Psi)$ does not contain elements of cycle
type (2, 2, 2), (3, 3) and (6). The only proper non-trivial subrootsystems $\Xi$
for which $N_W(W(\Xi))$ contains elements of cycle type (6) are of type $A_1^3$ and
$A_2^2$. However, for $\Xi$ of type $A_1^3$ and $\bar{n}_\Xi$ of order 3, $n_\Xi, W(\Xi)$
does not contain elements of cycle type (2, 2, 2); for $\Xi$ of type $A_2^2$ and $\bar{n}_\Xi$ of order 2, $n_\Xi, W(\Xi)$
does not contain elements of cycle type (3, 3). This implies that $n_\Xi, W(\Xi)$
must contain elements of cycle type (1, 5). As it already contains a Coxeter
element, one has $\Xi \neq \emptyset$. This implies that $N_W(W(\Xi)) = S_6$ (cf. Prop. 3.2(a)),
and thus $W(\Xi) = W$, a contradiction. Hence admissible 2-coset coverings
do not exist for $\ell = 5$.

### 3.5.2 — $\gamma_1 W$ of type $^2A_\ell$

In this case $\gamma$ is an inner automorphism, and $\widetilde{W} = \langle \gamma \rangle_1 W = \langle -id_E \rangle_1 W$,
where $-id_E$ denotes the endomorphism acting as $-1$ on $E$. Thus if
(n_\gamma \cdot W(\Psi), n_\Xi \cdot W(\Xi)) is an admissible 2-coset covering of \gamma \cdot W, the pair of admissible cosets \((- id_E)n_\gamma \cdot W(\Psi), (- id_E)n_\Xi \cdot W(\Xi)) is an admissible 2-coset covering of W. Thus the claim can be deduced from 3.5.1.

3.5.3 — \gamma \cdot W of type B_\ell (or C_\ell)

By 3.5.1 and the previous discussion (cf. §3.3), we have to consider three cases:

1. \pi(W(\Psi)) = W/E,
2. \pi(n_\Xi \cdot W(\Xi)) = W/E,
3. \ell \leq 4 and (\pi(W(\Psi)), \pi(n_\Xi \cdot W(\Xi))) is an admissible 2-coset covering of S_\ell.

Note that \pi(n_\Xi \cdot W(\Xi)) is a group, if and only if n_\Xi \cdot W(\Xi) \cap E \neq \emptyset. Hence \pi(n_\Xi \cdot W(\Xi)) = W/E \implies \pi(W(\Xi)) = W/E.

Cases (1) and (2): The only proper subroot systems \Psi of \Phi satisfying (1) are of type D_\ell and A_{\ell-1} (cf. Prop. 3.3). A similar argument yields that (2) implies that \Xi is either of type D_\ell or A_{\ell-1}. Hence in order to prove the claim it suffices to show that \Psi of type D_\ell implies \Xi of type D_\ell, and that \Xi of type D_\ell implies \Psi of type D_\ell.

Assume that \Psi = \Phi_{long} is of type D_\ell. Then n_\Xi \cdot W(\Xi) contains a Coxeter element and elements of the W-conjugacy classes (2, 1, \ldots, 1|1) and (1|\ell - 1). Hence \pi(n_\Xi \cdot W(\Xi)) = W/E (cf. Prop. 3.2(a)). As \pi(n_\Xi \cdot W(\Xi)) = \pi(W(\Xi)), the root system \Xi must be of type A_{\ell-1} or D_\ell. Assume \Xi is of type A_{\ell-1}. As w_0 \in N_W(W(\Xi)) and \left| N_W(W(A_{\ell-1}))/W(A_{\ell-1}) \right| = 2, one has n_\Xi \cdot W(\Xi) = \ldots = w_0 \cdot W(\Xi). However, in this case \Psi \cdot W(\Xi) does not contain short reflections corresponding to the ±-partition (1, \ldots, 1|1) (cf. §3.3). Thus \Psi = \Phi_{long} implies \Xi = \Phi_{long}.

Assume that \Xi = \Phi_{long} and that n_\Xi is inducing the graph automorphism of order 2. Then n_\Xi \cdot W(\Xi) does not contain any W-conjugacy class being contained in \Phi_{long}. In particular, W(\Psi) contains elements of type (2, 1, \ldots, 1|1, 1), (1, \ell - 1), (\ell). Thus \pi(W(\Psi)) = W/E (cf. Prop. 3.2(a)), and \Psi is either of type A_{\ell-1} or D_\ell. However, since W(\Psi) contains elements of type (2, 1, \ldots, 1|1, 1) the first case is impossible. This yields the claim.

Case (3): The 3 examples for \Phi of type A_\ell, \ell = 1, 2, 3 yield the 3 additional cases for \Phi of type B_\ell, \ell = 2, 3, 4. A straightforward calculation shows that these are the only exceptions.

3.5.4 — \gamma \cdot W of type D_\ell

By §3.5.1 one has to consider three cases:

1. \pi(W(\Psi)) \simeq W/E',
(2) \( \pi(n_\Xi.W(\Xi)) \simeq W/E' \),
(3) \( \ell = 4, \pi(W(\Psi)) \simeq S_3, \pi(n_\Xi.W(\Xi)) = 2.W(A_1^2) \subseteq S_4. \)

Case (1): The hypothesis implies that \( W(\Psi) \) is of type \( A_{\ell-1} \) (cf. Prop. 3.3). Hence \( n_\Xi.W(\Xi) \) must contain an element of type \((1, \ldots, 1|1,1)\), and thus \( \pi(n_\Xi.W(\Xi)) \) is a group. Furthermore, \( n_\Xi.W(\Xi) \) must contain elements of type \((1,\ell-1), (2,\ell-2) \) and \((2,1,\ldots,1|1,1)\), and this implies that \( \pi(n_\Xi.W(\Xi)) = W/E' \) (cf. Prop. 3.2(b)). So also \( W(\Xi) \) must be of type \( A_{\ell-1} \). But this yields that \( \ell \) must be odd and that \( n_\Xi.W(\Xi) = w_0.W(\Xi) \). However, in this case neither \( W(\Psi) \) nor \( n_\Xi.W(\Xi) \) contains elements of type \((1,\ldots,1|1,1)\). So an admissible 2-coset covering cannot exist in this case.

Case (3) can be excluded, since in this case \( N_W(W(\Xi)) \) is a 2-group, but must also contain a Coxeter element which is of order 6, a contradiction.

3.5.5 \( \gamma.W \) of type \( 2D_\ell \)

If \( \ell \) is odd, \( \gamma.W(\Phi) = w_0.W(\Phi) \subseteq W(B_\ell) \), and with the same argument as used in §3.5.2 one concludes that there are no admissible 2-coset coverings in this case. So \( \ell \) must be even. We may assume that \( n_\Psi = \gamma \). Note that \( \gamma \) is of type \((1,\ldots,1|1)\) considered as an element in \( W(B_\ell) \). Hence \( \pi(W(\Psi)) \) is a group. By §3.5.1, we have to consider 2 cases:

(1) \( \pi(W(\Psi)) \simeq W/E' \),
(2) \( \pi(n_\Xi.W(\Xi)) \simeq W/E' \),
(3) \( \ell = 4, \pi(W(\Psi)) \simeq S_3, \pi(n_\Xi.W(\Xi)) = 2.W(A_1^2) \subseteq S_4. \)

Case (1): The hypothesis implies that \( \Psi \) is of type \( A_{\ell-1} \). However, for \( \ell \) even there does not exist a \( \gamma \)-subroot system of type \( A_{\ell-1} \) (cf. [25, §2, (C)])

Case (2): The hypothesis implies that \( \pi(W(\Xi)) \simeq S_\ell \), and thus \( \Xi \) must be of type \( A_{\ell-1} \). The argument which was used in case (1) yields that this is impossible.

Case (3): By hypothesis, \( \pi(n_\Xi.W(\Xi)) \) is not a group. Thus \( n_\Xi.W(\Xi) \cap \cap E = \emptyset \). The hypothesis \( \pi(\gamma.W(\Psi)) \simeq S_3 \) yields that \( \Psi \) is of type \( A_2 \) and that \( \gamma \) is centralizing \( W(\Psi) \). So the elements of \( \gamma.W(\Xi) \) are of type \((1,1,1|1),(1,2|1),(3|1)\). Hence neither \( \gamma.W(\Psi) \) nor \( n_\Xi.W(\Xi) \) contains elements of type \((1|1,1,1)\), and thus an admissible 2-coset covering cannot exist in this case.
3.5.6 — $\gamma W$ of type $^3D_4$

If one identifies the Weyl group $W$ with the Weyl subgroup $W(D_4)$ ≤ $W(F_4)$ associated to the subrootsystem of long roots in the rootsystem of type $F_4$, $\gamma W = 3W(D_4) \subset W(F_4)$ is a $W(D_4)$-coset in the Weyl group $W(F_4)$ of type $F_4$. Since $W(F_4)$ has an admissible 2-covering ($W(B_4), 3W(D_4)$) (cf. § 3.5.11), one concludes that $3W(D_4)$ contains Coxeter elements of the Weyl group $W(F_4)$ which are of order 12 and which are the twisted Coxeter elements in $3W(D_4)$ (cf. [23, § 7]). Let $w \in \gamma W$ be such an element being contained in $n_{\Xi}W(\Xi)$. From [25, Prop. 7] one concludes that $\Xi = \emptyset$ must be trivial.

The proper $\gamma$-invariant subrootsystems of the rootsystem $\Phi$ are of type $A_1^1, A_1^3, A_2, A_1, \emptyset$ (cf. [25, § 2, (B)]). The coset $n_{\Psi}W(\Psi)$ must intersect the two $W$-conjugacy classes of elements of order 3 being contained in $3W(D_4)$ non-trivially. Hence $W(\Psi)$ cannot be a 2-group which leaves only the case $\Psi$ of type $A_2$.

In particular, one may assume that $n_{\Psi} = \gamma$ and that $\gamma$ is centralizing $W(\Psi)$. It is straightforward, that $n_{\Psi}W(\Psi)$ is a subset of a Weyl subgroup $W(A_2 \times \tilde{A}_2) \leq W(F_4)$ of type $A_2 \times \tilde{A}_2$ in $W(F_4)$. Hence $n_{\Psi}W(\Psi)$ has trivial intersection with the $W(F_4)$-conjugacy class containing $\gamma(-id_{\Xi})$, a contradiction.

3.5.7 — $\gamma W$ of type $E_6$

We may assume that $\Psi \subset \Phi$ is a maximal subrootsystem of $\Phi$. The maximal subrootsystems of the rootsystem of type $E_6$ are of type $D_5, A_5 \times A_1$ and $A_2^2$. From Table 3.2 one concludes that $W(\Psi)$ has trivial intersection with a $W$-conjugacy class which is not the Coxeter class. This implies that $\Xi$ is not empty.

Since proper Weyl subgroups of $W = W(\Phi)$ are not normal in $W$, $N_W(W(\Xi)) \neq W$. Hence it suffices to show that $W(\Phi)$ cannot be covered by the $W$-conjugates of $W(\Psi)$, $\Psi$ a maximal subrootsystem of $\Phi$, and a maximal subgroup $M \subset W$ containing $N_W(W(\Xi))$. The maximal subgroups $M \subset W$ of $W \simeq O_6^-(2)$ are known (cf. [9, p. 26]), and from their permutation characters $\chi_M$ corresponding to the left $\mathbb{C}[W]$ module $\text{Ind}_M^W_{\mathbb{C}}$ one can determine the $W$-conjugacy classes they do not contain.

In Table 3.2 we have listed some $W$-conjugacy classes which have trivial intersection with some of the maximal subgroups, and also the decomposition of the permutation character $\chi_M$ in terms of the characters occurring in the character table in [9]. We have used the symbol $\times$ to express that the conjugacy class has non-trivial intersection with the maximal subgroup and the symbol $\emptyset$ in case the intersection is trivial. By $\text{sgn}: W \to \{\pm 1\}$ we denote the sign character.
Table 3.2. – Maximal subgroups and conjugacy classes of $O_6^-(2)$

<table>
<thead>
<tr>
<th>maximal subgroup</th>
<th>$\chi_M$</th>
<th>9A</th>
<th>12A</th>
<th>8A</th>
<th>10A</th>
<th>12C</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_1 = W(D_5)$</td>
<td>$\chi_1 + \chi_4 + \chi_9$</td>
<td>∅</td>
<td>∅</td>
<td>×</td>
<td>∅</td>
<td>×</td>
</tr>
<tr>
<td>$M_2 = W(A_5 \times A_1)$</td>
<td>$\chi_1 + \chi_8 + \chi_9$</td>
<td>∅</td>
<td>∅</td>
<td>∅</td>
<td>×</td>
<td>∅</td>
</tr>
<tr>
<td>$M_3 = 3^{1+2}: 2S_4$</td>
<td>$\chi_1 + \text{sgn} \cdot \chi_7 + \chi_{10}$</td>
<td>×</td>
<td>×</td>
<td>×</td>
<td>∅</td>
<td>∅</td>
</tr>
<tr>
<td>$M_4 = N_W(W(A_2^3))$</td>
<td>$\chi_1 + \chi_8 + \chi_{10}$</td>
<td>×</td>
<td>∅</td>
<td>∅</td>
<td>∅</td>
<td>×</td>
</tr>
<tr>
<td>$M_5 = C_W(2A)$</td>
<td>$\chi_1 + \chi_9 + \chi_{10}$</td>
<td>∅</td>
<td>×</td>
<td>×</td>
<td>∅</td>
<td>∅</td>
</tr>
</tbody>
</table>

Since $M$ has to contain Coxeter elements (cf. Prop. 3.1) which correspond to the conjugacy class 12A, one concludes from Table 3.2 that $M$ has to be conjugate to either $M_3$ or $M_5$. Since $M_1$ and $M_4$ have trivial intersection with the conjugacy class 10A, $\Psi$ has to be of type $A_5 \times A_1$. Hence $M$ cannot be conjugate to $M_5$, as this group has trivial intersection with the conjugacy class of type 9A. But $M$ cannot be of type $M_3$ either, since this group has trivial intersection with the conjugacy class of type 12C.

3.5.8 — $\gamma.W$ of type $^2E_6$.

Since $\gamma.W = (−id_E).W$, the same argument which was used in §3.5.2 and the non-existence of admissible 2-coset covers for $W = W(E_6)$ (cf. §3.5.7) show that admissible 2-coset covers do not exist in this case.

3.5.9 — $\gamma.W$ of type $E_7$

We may assume that $\Psi$ is a maximal subrootsystem of $\Phi$. Thus $\Psi$ is of type $E_6$, $A_7$, $D_6 \times A_1$ or $A_5 \times A_2$. Let $\overline{\gamma} : W \rightarrow Sp_6(2)$ denote the canonical projection on the symplectic group of degree 6 over the field with 2 elements. Some information on maximal subgroups and conjugacy classes of the group $Sp_6(2)$ has been collected in Table 3.3 (cf. [9, p. 46]).

From Table 3.3 one concludes that there exists a $W$-conjugacy class which is not the Coxeter class having trivial intersection with $W(\Psi)$. Hence $\Xi$ is non-trivial. Since proper Weyl subgroups of $W$ are not normal, $N_W(W(\Xi))$ is contained in a maximal subgroup $M$ of $W$ containing $w_0 = −id_E$. The same technique which was used in §3.5.7 and the information listed in Table 3.3 show that either $\Psi$ is of type $E_6$ and $M$ is $W$-conjugate to $\langle w_0 \rangle.W(A_7)$, or $\Psi$ is of type $A_7$ and $M$ is $W$-conjugate to
Table 3.3. – Maximal subgroups and conjugacy classes of $Sp_6(2)$

<table>
<thead>
<tr>
<th>maximal subgroup</th>
<th>3B</th>
<th>7A</th>
<th>8A</th>
<th>9A</th>
<th>15A</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{M}_1 = \mathcal{W}(E_6)$</td>
<td>$\times$</td>
<td>$\emptyset$</td>
<td>$\times$</td>
<td>$\times$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>$\mathcal{M}_2 = \mathcal{W}(A_7)$</td>
<td>$\emptyset$</td>
<td>$\times$</td>
<td>$\emptyset$</td>
<td>$\emptyset$</td>
<td>$\times$</td>
</tr>
<tr>
<td>$\mathcal{M}_3 = \mathcal{W}(D_6 \times A_1)$</td>
<td>$\emptyset$</td>
<td>$\emptyset$</td>
<td>$\times$</td>
<td>$\times$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>$\mathcal{M}_4 = U_3(3) : 2$</td>
<td>$\times$</td>
<td>$\emptyset$</td>
<td>$\times$</td>
<td>$\emptyset$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>$\mathcal{M}_5 = 2^6 : L_3(2)$</td>
<td>$\times$</td>
<td>$\times$</td>
<td>$\times$</td>
<td>$\emptyset$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>$\mathcal{M}_6 = 2.[2^6] : (S_3 \times S_3)$</td>
<td>$\times$</td>
<td>$\emptyset$</td>
<td>$\times$</td>
<td>$\emptyset$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>$\mathcal{M}_7 = \mathcal{W}(A_5 \times A_2)$</td>
<td>$\times$</td>
<td>$\emptyset$</td>
<td>$\emptyset$</td>
<td>$\emptyset$</td>
<td>$\times$</td>
</tr>
<tr>
<td>$\mathcal{M}_8 = S_2(8)$</td>
<td>$\times$</td>
<td>$\times$</td>
<td>$\emptyset$</td>
<td>$\times$</td>
<td>$\emptyset$</td>
</tr>
</tbody>
</table>

$\langle w_0 \rangle . \mathcal{W}(E_6)$. Since $w_0$ is not contained in Weyl subgroups of type $A_7$ or $E_6$, $w_0 \in n_2.\mathcal{W}(\mathcal{Z})$, and we may assume that $\mathcal{Z}$ is of type $A_7$ or $E_6$, respectively. However, in the first case, neither $\mathcal{W}(E_6)$ nor $w_0.\mathcal{W}(A_7)$ contain Coxeter elements which are of order 18; in the second case neither $\mathcal{W}(A_7)$ nor $w_0.\mathcal{W}(E_6)$ contain elements of order 9. Hence admissible 2-coset coverings cannot exist.

3.5.10 — $\gamma \cdot \mathcal{W}$ of type $E_8$

Let $\chi: \mathcal{W} \to \{\pm 1\}$ denote the sign character of the reflection group $\mathcal{W} = \mathcal{W}(E_8)$, and let $\gamma: \mathcal{W} \to O^+_8(2)$ denote the canonical projection on the orthogonal group of maximal Witt index of degree 8 over the field with 2 elements. As in the previous case we may assume that $\Psi$ is a maximal subrootsystem of $\Phi$. Hence $\Psi$ is of type $E_7 \times A_1, D_8, A_8, E_6 \times A_2$ or $A^2_1$ (cf. [6]), and thus $\mathcal{W}(\Psi)$ is $\mathcal{W}$-conjugate to a subgroup of one of the maximal subgroups $\mathcal{M}_1, \mathcal{M}_4, \mathcal{M}_7, \mathcal{M}_{10}, \mathcal{M}_{15}$ of $\mathcal{W} = O^+_8(2)$ (cf. Table 3.4). We have chosen the enumeration of the $\mathcal{W}$-conjugacy classes of maximal subgroups in such a way that it coincides with its appearance in the list of maximal subgroups in [9, p. 85].

For each of the maximal subgroups $\mathcal{M}_1, \mathcal{M}_4, \mathcal{M}_7, \mathcal{M}_{10}, \mathcal{M}_{15}$, there exists one $\mathcal{W}$-conjugacy class which has trivial intersection with the maximal subgroup and which is not the homomorphic image of the Coxeter class (cf. Table 3.4). As $\chi(w) = 1$ for a Coxeter element $w$, this implies that $\mathcal{Z}$ is not empty.
Since $W$ does not contain proper normal Weyl subgroups, $N_W(W(\Xi)) \neq W$ and $N_W(W(\Xi))$ is contained in a maximal subgroup $M < W$. Let $\overline{M}$ denote its homomorphic image in $\overline{W}$. From the previous remark one concludes that $\chi(M) = \{ \pm 1 \}$. Hence $\overline{M}$ is $\overline{W}$-conjugate to one of the maximal subgroups $\overline{M}_1, \overline{M}_4, \overline{M}_7, \overline{M}_{10}, \overline{M}_{13} = C_{\overline{W}}(2A), \overline{M}_{14}, \overline{M}_{15}$. As $\overline{M}_{13}$ and $\overline{M}_{14}$ do not have elements of order 5, they cannot contain the homomorphic image of a Coxeter element whose order is divisible by 15. Hence $\overline{M}$ is $\overline{W}$-conjugate to one of the maximal subgroups listed in Table 3.4.

Assume that $\Psi$ is of type $E_7 \times A_1$. Then $\overline{W}(\Psi)$ has trivial intersection with the $\overline{W}$-conjugacy classes 9B and 15B. The first fact implies that $\overline{M}$ cannot be $\overline{W}$-conjugate to $\overline{M}_{15}$, while the latter implies that $\overline{M}$ cannot be $\overline{W}$-conjugate to $\overline{M}_1, \overline{M}_4, \overline{M}_7$ or $\overline{M}_{10}$ (cf. Table 3.4). The cases $\Psi$ of type $D_8, A_8, E_6 \times A_2$ and $A_2^2$ can be ruled out by a similar argument and the information listed in Table 3.4.

**Table 3.4. – Maximal subgroups and conjugacy classes of $O_5^+(2)$**

<table>
<thead>
<tr>
<th>maximal subgroup</th>
<th>7A</th>
<th>9A</th>
<th>9B</th>
<th>15B</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\overline{M}_1 = \overline{W}(E_7 \times A_1)$</td>
<td>$\times$</td>
<td>$\times$</td>
<td>$\emptyset$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>$\overline{M}_4 = \overline{W}(D_8)$</td>
<td></td>
<td></td>
<td>$\emptyset$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>$\overline{M}_7 = \overline{W}(A_8)$</td>
<td>$\times$</td>
<td>$\emptyset$</td>
<td>$\times$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>$\overline{M}_{10} = \overline{W}(E_6 \times A_2)$</td>
<td></td>
<td></td>
<td>$\emptyset$</td>
<td>$\times$</td>
</tr>
<tr>
<td>$\overline{M}<em>{15} = N</em>{\overline{W}}(\overline{W}(A_2^2))$</td>
<td></td>
<td></td>
<td>$\emptyset$</td>
<td>$\emptyset$</td>
</tr>
</tbody>
</table>

3.5.11 — $\gamma W$ of type $F_4$

The maximal subrootsystems of $\Phi$ are of type $B_4, C_4, C_3 \times A_1, B_3 \times \tilde{A}_1, \tilde{A}_3 \times A_1, A_3 \times \tilde{A}_1$ and $A_2 \times \tilde{A}_2$. Note that since there exists a surjective morphism $\beta : W(F_4) \to S_3^2$ with $\ker(\beta)$ a 2-group, $W = W(F_4)$ has precisely 3-conjugacy classes $\sigma_i^W$, $i = 1, 2, 3$, of elements of order 3. The graph automorphism of $F_4$ permutes 2 of these conjugacy classes and leaves $\sigma_3^W$ fixed. We may assume that $\sigma_1^W$ has non-trivial intersection with Weyl subgroups of type $C_4$, and $\sigma_2^W$ has non-trivial intersection with Weyl subgroups of type $B_4$. Since no maximal Weyl subgroup contains Coxeter elements of the Weyl subgroup of type $B_4$ - which have order 8 - and also elements from the $W$-conjugacy class $\sigma_3^W$, $\Xi$ cannot be trivial.
Since $n_{\Xi}.W(\Xi)$ contains Coxeter elements, one concludes from [25, Prop. 7] that $N_W(W(\Xi)) = N_W(W(D_4))$ or $N_W(W(\Xi)) = N_W(W(\tilde{D}_4))$. Hence $\Xi$ is either of type $D_4$ or $\tilde{D}_4$.

Assume $\Xi$ is of type $D_4$. Since $(W(B_4), 3.W(D_4))$ is an admissible 2-coset covering, $W(\Psi)$ must contain elements from the $W$-conjugacy class $\sigma^W_2$ and also Coxeter elements from the Weyl group $W(B_4)$. Thus from the knowledge of maximal Weyl subgroups of $W$ one concludes that $W(\Psi) \leq W(B_4)$ is a subgroup of a Weyl group of type $B_4$. Proposition 3.1 then implies that $W(\Psi)$ is of type $B_4$. Applying the graph automorphism yields the case $\Xi$ of type $D_4$.

3.5.12 — $\gamma.W$ of type $G_2$

The subrootsystems of $\Phi$ are of type $A_2$, $\tilde{A}_2$, $A_1 \times \tilde{A}_1$, $A_1$, $\tilde{A}_1$ or $\emptyset$. The $W$-conjugacy classes are $1$, $\omega_0 = -id_\Xi$, $\rho_1^W$, $\rho_2^W$, $\sigma^W$, $\tau^W$, where $\rho_i$ is a long root, $\rho_i$ a short root, $\sigma$ of order 3 and $\tau$ of order 6. Straightforward arguments show that for $\Psi = A_2$ or $\tilde{A}_2$, $n_{\Xi}.W(\Xi)$ must be as described in Table 3.1, and that there does not exist an admissible 2-coset covering for $\Psi$ of type $A_1 \times \tilde{A}_1$.

3.5.13 — $\gamma.W$ of type $^2B_2$

Non-trivial $\gamma$-invariant proper subrootsystems do not exist in this case (cf. [25, § 2, (D)]). Hence $\Psi = \Xi = \emptyset$, and this yields that admissible 2-coset coverings do not exist in this case.

3.5.14 — $\gamma.W$ of type $^2G_2$

Every non-trivial proper $\gamma$-subrootsystem is of type $\tilde{A}_1 \times A_1$ (cf. [25, § 2, (D)]). Note that $\tilde{W} \simeq D(24)$ is isomorphic to a dihedral group of order 24. In particular, $\gamma.W$ contains two $W$-conjugacy classes of elements of order 12 and one $W$-conjugacy class of elements of order 4. As $N_W(W(\tilde{A}_1 \times A_1))$ is a 2-group of order 8, an admissible 2-coset covering $(n_{\Psi}.W(\Psi), n_{\Xi}.W(\Xi))$ must satisfy $\Psi = \Phi = \emptyset$ and $n_{\Psi}$ and $n_{\Phi}$ must be of order 12. However, for this choice neither $n_{\Psi}.W(\Psi)$ nor $n_{\Xi}.W(\Xi)$ contains elements of order 4, and thus admissible 2-coset coverings cannot exist in this case.

3.5.15 — $\gamma.W$ of type $^2F_4$

Every non-trivial proper $\gamma$-subrootsystem is of type $B_2^2$, $\tilde{A}_2 \times A_2$ or $\tilde{A}_1 \times A_1$. (cf. [25, § 2, (D)]). Let $\rho_i$, $i = 1, 2, 3, 4$, be the simple reflection associated to a basis $\Delta$ of $\Phi$ ordered in the natural way. Then $\sigma := \gamma \rho_1 \rho_2 \in \gamma.W$ is an element of order 24. We may assume that $n_{\Xi}.W(\Xi)$ contains an element $W$-conjugate to $\sigma$. Since $N_W(W(\tilde{A}_1 \times A_1))$ and
$N_W(W(B_2^3))$ are 2-groups and as $N_W(W(A_2 \times A_2)) = (-id_{U}).W(\tilde{A}_2 \times A_2)$ (cf. [6, Tab. 8]), $\Xi$ must be trivial.

Moreover, $\gamma W$ contains elements of order 6 and 8. Hence the same argument yields that $\Psi$ cannot be of type $\tilde{A}_1 \times A_1, B_2^2$ or $\tilde{A}_2 \times A_2$. But as $\gamma W$ has more than two $W$-orbits, also $\Psi = \emptyset$ is impossible showing that admissible 2-coset coverings do not exist in this case. \hfill \Box

4. The proof of Theorem A - part I

In this section we will prove that a generic 2-covering $\{H, K\}$ of the finite group of Lie type $(G, F)$ must satisfy the conditions of Theorem A (cf. Thm. 4.5). In the subsequent 3 sections it will be proved that every pair of subgroup $\{H, K\}$ as described in Theorem A is indeed a generic 2-covering. We start with the following property - the easy proof is left to the reader.

**Proposition 4.1.** Let $(\tilde{G}, F)$ be a finite group of Lie type, and let $\iota: \tilde{G} \to G$ be a finite $F$-invariant central isogeny. Let $\{H, K\}$ be a pair of $F$-stable closed subgroups of maximal rank, and let $H$ and $K$ denote their canonical images under $\iota$. Then, if $\{H, K\}$ is a generic 2-covering of $(G, F)$, $\{\tilde{H}, \tilde{K}\}$ is a generic 2-covering of $(\tilde{G}, F)$. In particular, one has the following:

(a) Let $(G, F)$ be a finite group of Lie type of adjoint type. Assume that the pair of $F$-stable subgroups $\{H, K\}$ of maximal rank is a generic 2-covering. Then for every group of Lie type $G$ isogeneous to $\tilde{G}$ and $H$ and $K$ defined as above, $\{\tilde{H}, \tilde{K}\}$ is a generic 2-covering of $(\tilde{G}, F)$.

(b) Let $(\tilde{G}, F)$ be a simply-connected finite group of Lie type, and let $\tilde{H}, \tilde{K}$ be $F$-stable subgroups of maximal rank. Assume that $\{H_F,K_F\}$ is not a 2-covering of $\tilde{G}_F$. Then $\{H_F,K_F\}$ - with $H$ and $K$ as defined above - is not a 2-covering of $\tilde{G}_F$ for all groups of Lie type $(G, F)$ isogeneous to $(\tilde{G}, F)$.

4.1 – Three exceptions

It seems remarkable to us that almost all the admissible 2-coset coverings listed in Table 3.1 give rise to a generic 2-covering - sometimes with a further characteristic restriction. However, there are three cases in Table 3.1 which will not correspond to a generic 2-covering:

1. $(G, F)$ of type $C_4$, $H$ reductive of type $W(C_3) \times W(C_1)$, $K$ reductive of type $2.W(C_2^3)$,
(2) \((G, F)\) of type \(A_3\), \(H\) reductive of type \(W(A_2)\), \(K^o\) reductive of type \(2 \cdot W(A_2^2)\),

(3) \((G, F)\) of type \(^2A_3\), \(H\) reductive of type \(2 \cdot W(A_2)\), \(K^o\) reductive of type \(2 \cdot W(A_2^2)\).

These cases will now be analyzed by a case-by-case analysis. If the characteristic of the field is different from 2, neither \(H_F\) nor \(K_F\) will contain regular unipotent elements (cf. [20]). Therefore, we may restrict our consideration to the case when the field of definition is of characteristic 2.

**Case (1):** Let \((G, F)\) be the simply-connected finite group of Lie type of type \(C_4\) defined over the algebraically closed field \(\bar{\mathbb{F}}_2\) of characteristic 2, and let \(H\) and \(K^o\) be \(F\)-stable reductive subgroups of maximal rank of type \(W(A_4 \times C_3)^W\) and \(2 \cdot W(C_2^2)^W\), respectively. We put \(K := N_G(K^o)\). In particular,

\[(4.1) \quad G_F \simeq Sp_8(q), \quad H_F \simeq Sp_2(q) \times Sp_6(q), \quad K_F \simeq 2 \cdot Sp_4(q^2)\]

for some 2-power \(q = 2^l\). One has:

**Proposition 4.2.** The pair of subgroups \(\{H_F, K_F\}\) is not a 2-covering of \(G_F\).

**Proof.** Assume that \(\{H_F, K_F\}\) is a 2-covering of \((G, F)\). Let \(x := x_s x_u \in G_F\) be an element where \(x_s\) is a semisimple element generating a Coxeter torus in \(Sp_4(q)\) and where \(x_u\) is a regular unipotent element in \(Sp_4(q)\). Hence \(x \in Sp_4(q) \times Sp_4(q)\), and \(Sp_4(q) \times Sp_4(q) \leq Sp_8(q)\) is stabilizing the 4-dimensional summands \(V_4\) and \(V_4'\) of an orthogonal decomposition of the natural module \(V = V_8\) of \(Sp_8(q)\). From the action of \(x\) on \(V\), it is obvious that \(x\) cannot be contained in any \(G_F\)-conjugate of \(H_F\). Hence it must be contained in a \(G_F\)-conjugate of \(K_F\), and thus we may assume that \(x \in K_F\). Since \(x_u\) is acting as a Jordan block of size 4 on \(V_4'\), \(x_u\) cannot be contained in \(K_F^o\). Moreover, one concludes that \(x_s \in C_{K_F}(x_u) \simeq Sp_4(q)\) is acting diagonally on \(V\). But this yields a contradiction, since we assumed that \(x_s\) is leaving a 4-dimensional subspace of \(V\) fixed.

**Case (2):** Let \(G\) be the reductive group \(GL_4\) defined over the algebraically closed field \(\bar{\mathbb{F}}_2\) of characteristic 2, and let \(H\) and \(K^o\) be \(F\)-stable reductive subgroups of maximal rank of type \(W(A_2)^W\) and \(2 \cdot W(A_2^2)^W\), respectively. We put also \(K := N_G(K^o)\). In particular,

\[(4.2) \quad G_F \simeq GL_4(q), \quad H_F \simeq GL_3(q) \times GL_1(q), \quad K_F \simeq 2 \cdot GL_2(q^2)\]
for some 2-power $q = 2^f$. Let $V_4$ denote the natural $\mathbb{F}_q[GL_4(q)]$-module, and let $V := V_2 \oplus V'_2$ be a direct decomposition in subspaces of $\mathbb{F}_q$-dimension 2. Then using the element $x := x_s x_u$, where $x_s$ is a semisimple element acting as a Singer cycle on $V_2$ and fixing $V'_2$, and where $x_u$ is a non-trivial unipotent element fixing $V_2$ and acting non-trivial on $V'_2$, the same argument which was used in the proof of Proposition 4.2 shows that $\{H_F, K_F\}$ is not a 2-covering of $G_F$. Moreover, since in this case $GL_4(q) = SL_4(q) \times (q - 1)$ this yields that $\{SL_4(q) \cap H_F, SL_4(q) \cap K_F\}$ is not a 2-covering of $SL_4(q)$. Hence one has:

**Proposition 4.3.** Let $(G, F)$ be a simply-connected group of Lie type of type $A_3$ defined over the algebraically closed field $\overline{\mathbb{F}}_2$ of characteristic 2. Let $H$ be an $F$-stable reductive subgroup of maximal rank of type $W(A_2)^W$, and let $K^o$ be an $F$-stable reductive subgroup of maximal rank of type $2.W(A_1^2)^W$. Then for $K := N_G(K^o)$, $\{H, K\}$ is not a generic 2-covering of $(G, F)$.

**Case (3):** Let $G$ be the reductive group $GL_4$ defined over the algebraically closed field $\overline{\mathbb{F}}_2$ of characteristic 2, and let $F$ be a Frobenius morphism which is the composition of a standard Frobenius morphism with a graph automorphism. Let $H$ and $K^o$ be $F$-stable reductive subgroups of type $W(A_2)^W$ and $2.W(A_1^2)^W$, respectively. There are obviously two $W$-conjugacy classes of type $2.W(A_1^2)^W$. The one we are considering is the one corresponding to the $G_F$-conjugacy class of Levi complements of the maximal parabolic subgroup of type $A_1 \times A_1$. Then for $K := N_G(K^o)$ one has:

$$G_F \simeq GU_4(q), \quad H_F \simeq GU_3(q) \times GU_1(q), \quad K_F \simeq 2.GL_2(q^2)^{\bar{2}}$$

for some 2-power $q = 2^f$, where $GL_2(q^2)^{\bar{2}}$ denotes the group of 2-by-2 matrices $X$ over $\mathbb{F}_{q^2}$ whose norm of the determinant is equal to 1, i.e., $N_{\mathbb{F}_{q^2}/\mathbb{F}_q}(\det(X)) = 1$.

Let $V_1$ be the natural $\mathbb{F}_q[GU_4(q)]$-module equipped with the non-degenerate hermitian form $(.,.)$. Let $V_4 = V_2 \oplus V'_2$ be an orthogonal decomposition into non-degenerate 2-dimensional $\mathbb{F}_q$-subspaces $V_2$ and $V'_2$. By choosing an element $x = x_s x_u$, where $x_u$ is acting as a non-trivial unipotent element on $V_2$ and fixing $V'_2$ pointwise, and where $x_s$ is acting as an element of order $q - 1$ on $V_2$ and fixing $V'_2$ pointwise, the same argument as presented for Case (1) shows that $\{H_F, K_F\}$ is not a 2-covering of $G_F = GU_4(q)$. Thus using Proposition 4.1 and applying the same technique
which was used to reduce to the simply-connected case (cf. Case (2)) one obtains the following:

**Proposition 4.4.** Let \( (G, F) \) be a simply-connected group of Lie type of type \( ^2A_3 \) defined over the algebraically closed field \( \overline{F}_2 \) of characteristic 2. Let \( H \) be an \( F \)-stable reductive subgroup of maximal rank of type \( 2W(A_2)^W \), and let \( K^\circ \) be an \( F \)-stable reductive subgroup of maximal rank of type \( 2W(A_2^2)^W \) corresponding to the Levi complement of the \( F \)-stable parabolic subgroup of type \( A_1^2 \). Then for \( K := N_G(K^\circ) \), \( \{H, K\} \) is not a generic 2-covering of \( (G, F) \).

4.2 – Generic 2-coverings

From Theorem 2.2 and 3.4 and the main result of [20] one concludes the following:

**Theorem 4.5.** Let \( \{H, K\} \) be a generic 2-covering of the finite group of Lie type \( (G, F) \). Then \( H \) and \( K \) are of maximal rank.

(a) Assume that for one subgroup say \( H \), \( R_u(H) \neq 1 \). Then \( (G, F) \) is of type \( A_1, A_2, A_3 \) or \( ^2A_3 \), \( H \) is maximal parabolic and \( K^\circ \) is reductive. Moreover, the types of the Levi complement \( H/R_u(H) \) of \( H \) and of \( K^\circ \) is as listed in Table 1.1.

(b) Assume that \( H^\circ \) and \( K^\circ \) are reductive. Then the types of \( (G, F) \), \( H^\circ \) and \( K^\circ \) and the characteristic of \( \overline{F} \) is as listed in Table 1.2.

**Proof.** By definition, one of the subgroups - say \( H \) - must contain a maximal torus, and thus it is of maximal rank. Then either \( K \) is also of maximal rank, or \( H \) must contain an \( F \)-stable maximal torus of each \( G_F \)-conjugacy class. Assume the latter case holds. If \( H^\circ \) is not reductive, by enlarging the group \( H^\circ \) we may assume that \( H^\circ \) is an \( F \)-stable parabolic subgroup of \( G \) (cf. [13, Thm. 30.4]). In particular, an \( F \)-stable Levi complement \( L \) must contain an \( F \)-stable maximal torus of each \( G_F \)-conjugacy class. Thus in either case \( G \) must contain an \( F \)-stable reductive subgroup \( R \) containing an \( F \)-stable maximal torus of each \( G_F \)-conjugacy class. From Proposition 3.1 and [25, Prop. 4] one concludes that this implies that \( (G, F) \) must be of twisted type. Moreover, the arguments used in §3.5 show that \( (G, F) \) cannot be of type \( ^2A_\ell, ^2D_\ell, \ell \) odd, \( ^2E_6 \) either. For none of the remaining types there exist an admissible 2-coset covering for the Weyl group coset (cf. Thm. 3.4). Hence
such an \( F \)-stable reductive subgroup cannot exist, and thus \( K \) is of maximal rank, too.

If \( H^o \) is not reductive, we enlarge it to an \( F \)-stable parabolic subgroup \( H^c \). Let \( L \leq H^c \) be an \( F \)-stable Levi complement of \( H^c \), and let \( n_{\varphi}.W(\Psi)^W \) denote its type. If \( H^o \) is reductive, we denote by \( n_{\varphi}.W(\Psi)^W \) the type of \( H^o \).

Second we do the same for \( K^c \) and denote by \( n_{\varphi}.W(\Xi)^W \) the type of either \( K^c \) - in case \( K^c \) is reductive - or of an \( F \)-stable Levi complement of an \( F \)-stable parabolic containing \( K^c \).

Thus, the hypothesis and Theorem 2.2 imply, that \((n_{\varphi}.W(\Psi), n_{\varphi}.W(\Xi))\) is an admissible 2-coset covering of the Weyl group coset \( \gamma.W \) (cf. § 2.1). Hence by Theorem 3.4, \( \gamma.W \) and the admissible 2-coset covering \((n_{\varphi}.W(\Psi), n_{\varphi}.W(\Xi))\) must occur in the list of Table 3.1.

The only cosets corresponding to Levi complements occur for the groups of Lie type \( A_1, A_2, A_3 \) or \( 2A_3 \). From this one concludes that if \( H^o \) is not reductive, \((G,F)\) must be of type \( A_1, A_2, A_3 \) or \( 2A_3 \) and \( K^c \) must be reductive. In particular, a direct verification shows that the groups \( \{H, K\} \) listed in Table 1.1 are the only possibilities in this case. This yields (a).

For (b) we may assume that \( H^o \) and \( K^c \) are reductive of maximal rank of one of the types listed in Table 3.1. From the covering property (ii) (cf. § 1) one concludes that either \( H \) or \( K \) must contain a regular unipotent element of \( G \). Hence from [20] one concludes that \( \{H, K\} \) is one of the groups listed in Table 1.2, or \((G,F)\) is of type \( A_3, 2A_3 \) or \( C_4 \), \( p = 2 \) and the type of \( H^c \) and \( K^c \) must be as indicated in Table 3.1. In the previous section (cf. §4.1) we showed that in neither of these cases \( \{H_F, K_F\} \) is a 2-covering for \( G_F \). This completes the proof. \( \square \)

5. Some 2-coverings for finite groups of Lie type

In this section we shall give an easy geometric argument showing the generic 2-covering property for the cases listed in Table 1.1 and for some of the cases listed in Table 1.2. The two remaining case will be handled in the two subsequent sections.

5.1 – Parabolic-type generic 2-coverings

For groups \( G \) of type \( A_\ell, \ell = 1, 2, 3 \), the covering property of the pair of subgroups \( \{H, K\} \) as listed in Table 1.1 was shown in [4, Prop. 4.2]. Therefore, it remains to show the generic 2-covering property for \( G \) of type
2Aₙ, and H and K given as described in Table 1.1. This property can be deduced from the following elementary geometric fact:

**Proposition 5.1.** Let $\mathbb{F}_{q^2}$ be a finite field with $q^2$ elements, where $q$ is a prime power. Let $(V, \langle ., . \rangle)$ be a 4-dimensional non-degenerate hermitian $\mathbb{F}_{q^2}$-space, and let $GU_4(q) := \text{Iso}(V, \langle ., . \rangle)$ denotes its isometry group. Then for an element $g \in GU_4(q)$ one (or both) of the following holds:

(i) There exists a 1-dimensional non-singular $g$-invariant $\mathbb{F}_{q^2}$-space $V_1$.

(ii) There exists a 2-dimensional singular $g$-invariant $\mathbb{F}_{q^2}$-space $V_2$.

**Proof.** Let $\mathcal{R}(g)$ be the set of all non-trivial $g$-invariant non-singular subspaces of $g$. Note that if $W \in \mathcal{R}(g)$ is directly indecomposable, $\dim_{\mathbb{F}_{q^2}}(W)$ must be odd. Furthermore, for $W \in \mathcal{R}(g)$, $W^\perp \in \mathcal{R}(g)$. This shows that either $\mathcal{R}(g) = \emptyset$, or there exists a 1-dimensional $g$-invariant subspace $W \in \mathcal{R}(g)$.

Assume that $g \in GU_4(q)$ is an element for which neither (i) nor (ii) holds, and let $W \leq V$ be a minimal non-trivial $g$-invariant subspace. Hence from the previous remark one concludes that $W$ must be singular. By hypothesis, $\dim_{\mathbb{F}_{q^2}}(W) = 1$. In particular, $W \leq W^\perp = \text{rad}(W)$ has codimension 2 in its radical.

Assume that there exists a 3-dimensional $g$-invariant subspace $W'$ different from $W^\perp$. By hypothesis, $\text{rad}(W') \neq W$ and $W'$ is not non-singular. Hence either $\text{rad}(W') \leq W^\perp$ and $\text{rad}(W') \oplus W$ is a 2-dimensional $g$-invariant singular subspace, or $W^\perp \oplus \text{rad}(W') = V$ and thus $U := W' \cap W^\perp$ is $g$-invariant and has dimension 2. Since $U$ cannot be non-singular, it contains either a $g$-invariant 1-dimensional subspace $U_0$ or it is a 2-dimensional singular subspace. Hence by assumption, $U_0$ is a 1-dimensional singular subspace, and $U_0 \oplus \text{rad}(W')$ is a singular 2-dimensional subspace, a contradiction. This shows that $W^\perp$ is a unique 3-dimensional $g$-invariant subspace, and also that $W$ is the unique 1-dimensional $g$-invariant subspace.

The element $g$ acts as unitary transformation on the induced hermitian space $W^\perp/W$, and thus has a 1-dimensional invariant subspace $R \leq W^\perp/W$. Let $R \leq W^\perp$ be its canonical preimage, i.e., $R$ is a $g$-invariant 2-dimensional subspace containing $W$ satisfying $R = R/R/W$. By hypothesis, $R$ cannot be singular and thus $R$ is a 1-dimensional non-singular space. In particular, $R \nleq R^\perp$ and this implies $W = R \cap R^\perp$. 
Let \( x_1 \in W \) be a vector spanning \( W \), and let \( y_1 \in V \) be such that \( x_1 \) and \( y_1 \) span a hyperbolic plane. Let \( x_2 \in R \) such that \( R = \text{span}_{\mathbb{F}_q}(x_1, x_2) \), and \( y_2 \in R^\perp \) such that \( R^\perp = \text{span}_{\mathbb{F}_q}(x_1, y_2) \). Then with respect to the ordered basis \( (y_1, y_2, x_2, x_1) \) the element \( g \) is represented by an upper triangular matrix over \( \mathbb{F}_q \). The entries on the diagonal of this matrix are \( \mu^{-1}, \lambda', \lambda, \mu \) for some \( \mu \in \mathbb{F}_q^\times = \{ a \in \mathbb{F}_q \mid a^q = a \} \), and \( \lambda, \lambda' \in \mathbb{F}_q^\times, \lambda^{q+1} = 1 \). From the Jordan normal form one concludes that either there exists another 1-dimensional \( g \)-invariant subspace different from \( W \), or \( \lambda = \mu \). Similarly, the non-existence of a 3-dimensional \( g \)-invariant subspace implies that \( \mu^{-1} = \lambda' \). Thus the semi-simple part of \( g \) has to be a central element of order 1 or 2. In particular, \( g \) satisfies (ii), a contradiction, and this yields the claim. \( \square \)

5.2 – Reductive-type generic 2-coverings for groups of type \( A_\ell \) or \( ^2A_\ell \)

**Case (1): G of type \( A_1 \), \( p = 2 \).** In case \((G, F)\) is of type \( A_1 \), it is well-known that \( \{H_F, K_F\} \), where \( H \) is the normalizer of an \( F \)-stable maximally split torus, and \( K \) is the normalizer of an \( F \)-stable maximally non-split torus, is a 2-covering for \( G = GL_2(q), q = 2^f \) (cf. [14]).

**Case (2): G of type \( A_3 \), \( p = 3 \).** Let \( G_F := GL_3(q), q = 3^f \), and let \( V = V_3 \) be its natural 3-dimensional \( \mathbb{F}_q[G] \)-module. We assume that \( H_F := GL_2(q) \times GL_1(q), K_F := 3.T_F, T_F \) a Singer torus. From Jordan’s normal form theorem one concludes that a non-trivial element \( g \) has either no proper \( g \)-invariant subspace - in which case it is \( G_F \)-conjugate to \( K_F \) - or it has a 1-dimensional and 2-dimensional \( g \)-invariant subspace, say \( V_1 \) and \( V_2 \). If there exist \( V_1 \) and \( V_2 \) with \( V_1 \cap V_2 = 0 \), \( g \) is \( G_F \)-conjugate to \( H_F \). Thus we may assume \( V_1 \leq V_2 \). Hence the same argument which was used in the proof of Proposition 5.1 shows that in this case \( g \) is either \( G_F \)-conjugate to \( GL_2(q) \times GL_1(q) \), or \( g \) has Jordan decomposition \( g = x_c x_u \) with \( x_c \) central and \( x_u \) regular unipotent. In particular, \( g \) is \( G_F \)-conjugate into \( K_F \), and this yields the claim in this case.

**Case (3): G of type \( ^2A_3 \), \( p = 3 \).** For \( G_F := GU_3(q), q = 3^f \), the same argument as was used in Case (2) shows the 2-covering property for the subgroups \( \{H_F, K_F\} \).

5.3 – Reductive-type generic 2-coverings in characteristic 2

**Case (4): G of type \( C_\ell \), \( p = 2 \).** Let \((G, F)\) be a finite group of Lie type of type \( C_\ell \) defined over the algebraically closed field \( \overline{\mathbb{F}_2} \) of characteristic 2 and
let $F: G \to G$ be a standard Frobenius morphism. Let $H^o$ be an $F$-stable reductive subgroup of maximal rank of type $W(D_\ell)^W$, and let $K^o$ be an $F$-stable reductive subgroup of maximal rank of type $2.W(D_\ell)^W$. Then by Theorem 2.2, 3.4 and Dye’s theorem, $\{H, K\}$, where $H := N_G(H^o)$, $K := N_G(K^o)$, is a generic 2-covering of $(G, F)$.

**Case (5): G of type $C_2$, $p = 2$.** Let $\ell = 2$ and $(G, F)$ as in Case (4). Let $H^o$ be an $F$-stable reductive subgroup of maximal rank of type $W(A_1^2)^W$, and let $K^o$ be an $F$-stable reductive subgroup of maximal rank of type $2.W(A_1^2)^W$. By Theorem 2.2 and 3.4, every maximal $F$-stable torus of $(G, F)$ is $G_F$-conjugate to a torus in $H^o$ or $K^o$. Let $H := N_G(H^o)$, $K := N_G(K^o)$. Then $a(H_F)$ and $a(K_F)$ are $G_F$-conjugate to the subgroups constructed in Case (4), where $a : G_F \to G_F$ denotes the non-trivial graph automorphism. Hence $\{H, K\}$ is a generic 2-covering of $(G, F)$.

**Case (6): G of type $G_2$, $p = 2$.** Let $\ell = 3$, and let $(G, F)$ be the finite group of Lie type of type $C_3$, and let $X \leq G$ be an $F$-stable subgroup of $G$ of type $G_2$. Let $Y^o \leq X$ be an $F$-stable reductive subgroup of maximal rank of type $W(A_2)^W$, and let $Z^o \leq X$ be an $F$-stable reductive subgroup of maximal rank of type $2.W(A_2)^W$. We put also $Y := N_X(Y^o)$, $Z := N_X(Z^o)$. Then $Y$ is contained in an $F$-stable subgroup $H$ as constructed in Case (4), and $Z$ is contained in an $F$-stable subgroup $K$ as constructed in Case (4). Moreover, since $X$ cannot be contained in $H$, and as $Y$ is maximal in $X$, one has $Y = X \cap H$. In a similar fashion one shows that $Z = X \cap K$.

By Theorem 2.2 and 3.4, every $F$-stable maximal torus of $X$ is $X_F$-conjugate to a maximal torus in $Y^o$ or $Z^o$. Moreover, a straightforward order argument shows that

\[
G_F = H_F.X_F, \quad G_F = K_F.X_F.
\]

Hence the following proposition shows that $\{Y, Z\}$ is a generic 2-covering for $(X, F)$:

**Proposition 5.2.** Let $A$ be a finite group with 2-covering $\{B, C\}$, and let $R \leq A$ be a subgroup satisfying $A = B.R = C.R$. Then $\{B \cap R, C \cap R\}$ is a 2-covering for $R$.

**Proof.** By hypothesis, $A = \bigcup_{r \in R} B^r \cup \bigcup_{r \in R} C^r$. Hence since $B^r \cap R = (B \cap R)^r$, this yields

\[
R \subseteq \bigcup_{r \in R} (B \cap R)^r \cup \bigcup_{r \in R} (C \cap R)^r.
\]
6. Groups of type $C_3$, $p = 3$

Throughout this section we assume that $(G, F)$ is a finite group of Lie type of type $C_3$ of adjoint type defined over the algebraically closed field $\overline{F}_3$, i.e., $G = \text{PS}L_3(C)$. Furthermore, we assume that $H = H^c$ is an $F$-stable reductive subgroup of maximal rank of type $W(C_2 \times A_1)^W$, and that $K^c$ is an $F$-stable reductive subgroup of maximal rank of type $3.W(A_2)^W$. Moreover, $K := N_G(K^c)$. We denote by $W(\Psi)$ and $n_{\Psi}.W(\Xi)$ the Weyl group coset corresponding to the $G_F$-orbit containing $H$ and $K^c$, respectively.

In order to show that $\{H, K\}$ is a generic 2-covering of $G$, it suffices to show by Theorem 3.4(b) that every element $x = x_s x_u \in G_F$ with non-trivial unipotent part $x_u$ is $G_F$-conjugate to an element in $H_F$ or $K_F$.

**Proposition 6.1.** Let $(G, F)$, $H$ and $K$ be given as described above. Let $x = x_s x_u \in G_F$ be an element satisfying:

(i) the semisimple part $x_s$ of $x$ is not contained in the center of $G_F$,

(ii) $C_G(x_s)^c$ is not of type $W(\tilde{A}_2)^W$.

Then $x$ is $G_F$-conjugate to an element of $H_F$ or $K_F^c$.

**Proof.** Since 3 is not a bad prime for $G$, every unipotent element of $C_G(x_s)$ is contained in $C_G(x_s)^c$ (cf. [24]). Thus $x \in C_G(x_s)^c_F$. If the root system of $C_G(x_s)^c$ is of type $\emptyset$, $x_s$ is a regular semisimple element and thus $x_u = 1$. Hence in this case Theorem 3.4 yields the claim. Thus it suffices to show that every $F$-stable reductive subgroup $C$ of maximal rank of type $n.W(\Pi)^W$ - with $\Pi$ not of type $\emptyset$, $\tilde{A}_2, C_3$ - is $G_F$-conjugate to a subgroup in $H$ or $K^c$. By [25, Prop. 4], it suffices to show that $n.W(\Pi)$ is $W$-conjugate to a subset in $W(\Psi)$ or $n_{\Psi}.W(\Xi)$.

In Table 6.1 we have listed all 3-saturated subrootsytem of the rootsysytem $\Phi$ of type $C_3$. In the second column we have indicated whether such a subrootsystem is $W$-conjugate to a subrootsystem in $W(\Psi)$. If this is the case, we assume $\Pi \subseteq \Psi$ and denote by $\tilde{N}_W(W(\Pi)) := N_W(W(\Pi))/W(\Pi)$ and $\tilde{N}_{W(\Psi)}(W(\Pi)) := N_{W(\Psi)}(W(\Pi))/W(\Pi)$ the respective outer normalizers. We used the following notation: $2^n$ denotes an elementary abelian 2-group of order $2^n$, $S_n$ denotes the symmetric group of degree $n$, $D(n)$ denotes the dihedral group of order $2n$.

Every 3-saturated subrootsystem $\Pi$ of $\Phi$ of type different from $\tilde{A}_2$ and $C_3$ is contained in a subrootsystem of type $C_2 \times A_1$. Thus we may assume that $\Pi$ is contained in $\Psi$. If $\tilde{N}_{W(\Psi)}(W(\Pi))$ contains an element of
Table 6.1. – $p$-Saturated subrootsystems of $\Phi = \Phi(C_3)$, $p$ odd.

<table>
<thead>
<tr>
<th>type $\Pi$</th>
<th>$\subseteq_W \Psi$</th>
<th>$\tilde{N}_W(W(\Pi))$</th>
<th>$\tilde{N}_{W(\Psi)}(W(\Pi))$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_3$</td>
<td>$-$</td>
<td>1</td>
<td>$-$</td>
</tr>
<tr>
<td>$C_2 \times A_1$</td>
<td>$+$</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$A_1^3$</td>
<td>$+$</td>
<td>$S_3$</td>
<td>2</td>
</tr>
<tr>
<td>$C_2$</td>
<td>$+$</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>$\tilde{A}_2$</td>
<td>$-$</td>
<td>2</td>
<td>$-$</td>
</tr>
<tr>
<td>$A_1^2$</td>
<td>$+$</td>
<td>$2^2$</td>
<td>$2^2$</td>
</tr>
<tr>
<td>$A_1 \times \tilde{A}_1$</td>
<td>$+$</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>$A_1$</td>
<td>$+$</td>
<td>$D(4)$</td>
<td>$D(4)$</td>
</tr>
<tr>
<td>$\tilde{A}_1$</td>
<td>$+$</td>
<td>$2^2$</td>
<td>$2^2$</td>
</tr>
<tr>
<td>$\emptyset$</td>
<td>$+$</td>
<td>$S_3 \times 2^2$</td>
<td>$D(4) \times 2$</td>
</tr>
</tbody>
</table>

every conjugacy class of $\tilde{N}_W(W(\Pi))$, then every $F$-stable reductive subgroup of maximal rank of type $\Pi$ is contained in a $G_F$-conjugate of $H$. Hence by Table 6.1, the claim follows for elements $x$ for which $C_G(x_0)^0$ is not of type $A_1^3$.

Let $x$ be an element such that $C_0 = C_G(x_0)^0$ has rootsystem $\Pi$ of type $\tilde{A}_1^3$. There are 3 $G_F$-conjugacy classes of $F$-stable subgroups $C$ with rootsystem of type $A_1^3$ depending whether for the corresponding $W$-orbit $n_{\Pi}.W(\Pi)^W$ the element $n_{\Pi}.W(\Pi) \in \tilde{N}_W(W(\Pi))$ has order 1, 2 or 3. In the first 2 cases $C$ is $G_F$-conjugate to a subgroup of $H$; while in the third case $C$ is $G_F$-conjugate to $K^\circ$ (cf. [25, Prop. 4]). This yields the claim. 

\[\underline{6.1 - G and G_F-conjugacy classes}\]

In order to relate $G_F$-conjugacy classes in $G_F$ to $G$-conjugacy classes we use the following lemma:

**Lemma 6.2.** Let $X$ be a connected affine algebraic group defined over the algebraic closure $\overline{\mathbb{F}}_p$ of a finite field $\mathbb{F}_p$, and let $F:X \rightarrow X$ be a Frobenius morphism.

(a) Let $x \in X$ be an element of $X$. Then the $X$-conjugacy class $x^X$ contains an $F$-fixed point, if and only if $F(x) \in x^X$. 

(b) Assume \( x \) is an \( F \)-fixed point, i.e., \( F(x) = x \). Then the mapping

\[
\rho: \{ x^g \mid g \in X, \ F(x^g) = x^g \} \rightarrow C_X(x)/C_X(x)^e =: \bar{C}_X(x),
\]

is surjective and yields a bijection \( \bar{\rho} \) between \( X_F \)-conjugacy classes of \( F \)-fixed points in \( x^X \) and \( F \)-conjugacy classes of \( \bar{C}_X(x) \). In particular, if \( C_X(x) \) is connected and if \( x \) is an \( F \)-fixed point, there exists a unique \( X_F \)-conjugacy class of \( F \)-fixed points in \( x^X \).

(c) Let \( Y \leq X \) be an \( F \)-stable closed connected subgroup and let \( x \in Y \) be an \( F \)-fixed point. Then if \( y^h = x^h \), \( h \in Y \), is an \( F \)-fixed point one has

\[
\tau(\rho_Y(y)) = \rho_X(y),
\]

where \( \tau: \bar{C}_Y(x) \rightarrow \bar{C}_X(x) \) is the canonical map.

**Proof.** (a) and (b) are direct consequence of the Lang-Steinberg theorem. (c) is a direct consequence of (b). \( \square \)

6.2 – **Semisimple elements with connected centralizer of type \( W(A_2)^W \) and \( 2.W(A_2)^W \)**

In view of Proposition 6.1 we need to take a closer look at semisimple elements \( x_3 \in G_F \) whose connected centralizer \( C_G(x_3)^e \) is of type \( W(A_2)^W \) or \( 2.W(A_2)^W \).

\[
\begin{array}{c}
1 \\
\circ \\
2 \\
\bullet
\end{array}
\]

Let \( \{ a_1, a_2, a_3 \} \) denote a basis of the rootsystem \( \Phi \) with numbering given as in (6.3), i.e., \( a_1, a_2 \) are short roots and \( a_3 \) is a long root. The subrootsystem \( \Delta \) spanned by \( a_1 \) and \( a_2 \) is of type \( \bar{A}_2 \). Let \( T \leq G \) be an \( F \)-stable maximal torus of \( G \) being contained in an \( F \)-stable Borel subgroup of \( G \), and let

\[
X := \text{Hom} (T, \bar{\Gamma}_3^e), \quad Y := \text{Hom} (\bar{\Gamma}_3^e, T)
\]

denote the character group and co-character group, respectively. We put

\[
C := \langle T, U_a \mid a \in \Delta \rangle.
\]

Note that \( C \) is a Levi complement of the maximal parabolic subgroup \( P_3 \), in particular \( C \simeq GL_3(\bar{\Gamma}_3)/\{ \pm 1 \} \) and thus \( C \) has precisely 3 unipotent con-
jugacy classes: regular unipotent elements, root elements, and the trivial element. Obviously, the centralizer in \( C \) for either of these unipotent elements is connected.

Let \( \tilde{e} \in \Phi^\vee \) be the sum of all short positive coroots. Then with respect to the canonical pairing, \( \Delta \subset \Phi \) is orthogonal to \( \tilde{e} \in \Phi^\vee \). In terms of affine group schemes the kernel of \( \tilde{e} \) is non-trivial, but coincides with the constant group scheme of order 3. However, on \( \tilde{\Phi}_3^e \)-rational points \( \tilde{e}: \tilde{\Phi}_3^e \to T \) is injective. Moreover, \( (\tilde{\Phi}_3^e)_F \) is a cyclic group of order \( q - 1 = 3^f - 1 \) and the induced map on \( F \)-fixed points \( \tilde{e}_F: (\tilde{\Phi}_3^e)_F \to T_F \) is also injective. Thus

\[
(6.6) \quad \text{im}(\tilde{e}) = Z(C), \quad \text{im}(\tilde{e}_F) = Z(C_F).
\]

Moreover, every semisimple element \( x_s \in G_F \) for which \( C_G(x_s)^o \) is of type \( W(\Delta)^W \) is \( G_F \)-conjugate to a non-trivial element in \( \text{im}(\tilde{e}_F) \).

Let \( (-1) \in W \) be the element acting as \( -id_X \) on \( X \), and let \( e \in G \) be an element such that \( F(e)e^{-1} \in N_G(T) \), \( \pi(F(e)e^{-1}) = (-1) \), where \( \pi: N_G(T) \to W \) denotes the canonical map. Then \( C^e \) is an \( F \)-stable reductive subgroup of maximal rank of \( G \). Let

\[
(6.7) \quad \tilde{e}^e: \tilde{\Phi}_3^e \longrightarrow T^e
\]

denote the composition of \( \tilde{e} \) with \( -e \). Then, if we let \( F_0 \) act on \( \tilde{\Phi}_3^e \) by \( F_0(x) = F(x^{-1}), x \in \tilde{\Phi}_3^e \), one has

\[
(6.8) \quad F \circ \tilde{e}^e = \tilde{e}^e \circ F_0.
\]

Moreover, \( \tilde{\Phi}_{F_0}^e \) is a cyclic group of order \( q + 1 = 3^f + 1 \) and one has

\[
(6.9) \quad \text{im}(\tilde{e}_{F_0}^e) = Z(C_F^e).
\]

Thus every semisimple element \( x_s \) for which \( C_G(x_s)^o \) is of type \((-1)W(\Delta)^W \) is \( G_F \)-conjugate to a non-trivial element in \( \text{im}(\tilde{e}_{F_0}^e) \).

**Proposition 6.3.** Let \((G,F)\), \( H \) and \( K \) be as defined above. Let \( x = x_r \cdot x_u \in G_F \) be an element such that \( C_G(x_r)^o \) is of type \( W(A_2)^W \) or \((-1)W(A_2)^W \).

(a) For \( x_u \) one of the following holds: (i) \( x_u \) is trivial, or (ii) \( x_u \) is a non-trivial short root element, or (iii) \( x_u \) is a regular unipotent element in \( A: = [C_G(x_r)^o, C_G(x_r)^o] \).

(b) If \( x_u \) is a short root element, then \( x \) is \( G_F \)-conjugate to an element in \( H_F \).

(c) If \( x_u \) is a regular unipotent element in \( A \), then \( x \) is \( G_F \)-conjugate to an element in \( K_F \).
PROOF. The type of $C_G(x_s)^o$ is either $W(Δ)^W$ or $(-1).W(Δ)^W$. Part (a) has been proved already in the discussion above. Thus it remains to establish (b) and (c).

(b) Assume that $C_G(x_s)^o$ is of type $W(Δ)^W$. Let $H' = \langle T, U_a \mid a \in Ψ' \rangle$, where $Ψ' \subset Φ$ is the subrootsystem spanned by $a_2, a_3$ and the highest long root $a_s$. Then $Ψ'$ is a subrootsystem of type $C_2 \times A_1$, $H'$ is an $F$-stable subgroup of $G$ of maximal rank of type $C_2 \times A_1$ and thus $G_F$-conjugate to $H$. Hence we may assume that $x_s \in im(\tilde{ε}) \leq H'_F$ and $C_G(x_s)^o = C$. As $U_{a_2} \leq C \cap H'$, there exists an element $h \in C_F$ such that $x^h_u = u \in (U_{a_2})_F$. Hence $x^h \in H'_F$ and the claim follows.

Assume that $C_G(x_s)^o$ is of type $(-1).W(Δ)^W$. As $(-1) \in W(Ψ')$, there exists an element $e \in H'$ such that $F(e)e^{-1} \in N_{H'}(T)$, $π(F(e)e^{-1}) = (-1)$. In this case we may assume that $x_s \in im(\tilde{ε}_F) \leq H'_F$. Moreover, $C_{H'}(x_s)^o$ is an $F$-stable reductive subgroup of $H'$ of maximal rank of type $(-1).W(\{± .a_2\})^W$. Hence $C_{H'}(x_s)_F^o$ contains short root elements, and there exists $h \in C_G(x_s)_F$ such that $x^h_u \in C_{H'}(x_s)_F$. Thus $x^h \in H'_F$ and the claim follows.

(c) Assume that $C_G(x_s)^o$ is of type $W(Δ)^W$. Then we may assume that $x_s \in im(\tilde{ε}_F)$, $C = C_G(x_s)^o$ and $x_u \in C_F$. In particular, $W(Δ)$ coincides with the stabilizer of $\tilde{ε}$ in $W$. Let $τ \in W(Δ)$ be an element of order 3. As $|T_F|$ is coprime to 3, there exists an element $u \in N_C(T)_F$ of order 3 with $π(u) = τ$. Moreover, $u$ is regular unipotent in $G$ (cf. [20]). Moreover, since $A = SL_3(F_3)$ has a unique conjugacy class of regular unipotent elements, and as $C_A(u)$ is connected, $x_u$ is $A_F$-conjugate to $u$ (cf. Lemma 6.2). We assume therefore that $x_u = u$. Since $C$ is connected, the Lang-Steinberg theorem implies the existence of an element $g \in A$ such that $F(g)g^{-1} = u$. Note that by construction $x^g_s = x_s$. In particular, $x$ is $A_F$-conjugate to $x^g$ (cf. Lemma 6.2).

Let $K_1^o := \langle T, U_a \mid a \in Ξ \rangle^g$. As $τ$ is stabilizing $Ξ$, $K_1^o$ is an $F$-stable reductive subgroup of $G$ of type $n_Ξ.W(Ξ)^W$ which is $G_F$-conjugate to $K^o$. Hence we may assume that $K_1^o$ coincides with $K^o$. By construction, $x_s = x^g_s \in K_1^o$, and also, $u^g \in N_G(K_1^o)$. This yields the claim in this case.

Assume that $C_G(x_s)^o$ is of type $(-1).W(Δ)^W$. Let $e \in G_F$ and $\tilde{ε}$ be given as in (6.7). Then we may assume that $x_s \in im(\tilde{ε}_F)$ and $C_G(x_s)^o = C^o$. As in the previous case one concludes that there exists a regular unipotent element $u \in N_C(T^e)_F$ of order 3 which centralizes $\tilde{ε} \in Y(T^o)$, and that we may assume that $x = x_s . u$. Let $g \in A^e$ such that $F(g)g^{-1} = u$. Then $u^g \in A^e_F$ and $x$ and $x^g$ are $A_F$-conjugate. Note that

$$F(eg)g^{-1}e^{-1} = F(e)e^{-1}eue^{-1} \in N_G(T).$$
Furthermore, \( \pi(F(eg)(eg)^{-1}) \) is an element of order 6 and \( K'' := \langle T, U_a \mid a \in \Xi \rangle^{\tau} \) is an \( F \)-stable reductive subgroup of \( G \) which is \( G_F \)-conjugate to \( K \). We therefore assume that \( K \) and \( K'' \) coincide. By construction, \( x_s^{\tau} = x_s \in K'' \), and \( u^\tau \in N_{G}(K'') \). This completes the proof. \( \square \)

6.3 – \( G_F \)-Conjugacy classes of unipotent elements

The unipotent \( G \)-conjugacy classes as well as their centralizers have been determined already in [24, § IV]. It turns out that in this case the Bala-Carter theorem (cf. [8, Thm. 5.9.6]) remains valid, and thus the \( G \)-conjugacy classes can be parameterized by certain weighted Dynkin diagrams corresponding to Levi subgroups of \( G \). We list the weighted Dynkin diagrams occurring in the 1st column of Table 6.2 (cf. [8, p. 174, p. 400]). Moreover, every such \( G \)-conjugacy class can be represented as a regular unipotent element of reductive subgroup of maximal rank, and the saturated subrootsystems corresponding to the respective conjugacy class are listed in the 2nd column. As a consequence, every \( G \)-conjugacy class \( u^G \) of unipotent elements contains an \( F \)-fixed point (cf. [24, III.1.19(a)]). In the 3rd column we describe the isomorphism type of

\[
(6.11) \quad \tilde{C}_G(u) = C_G(u)/C_G(u)^\circ
\]

Table 6.2. – Unipotent conjugacy classes

| type | reg | \(|\tilde{C}_G(u)|\) | \(J(\tilde{u})\) | \(u^\circ\) | \(H_F\) | \(K_F\) |
|------|-----|---------------------|-----------------|--------|--------|--------|
| \(\begin{array}{ccc} 2 & 2 & 2 \\ \circ & \circ & \circ \end{array}\) | \(C_3\) | 1 | 6 | \(\begin{array}{ccc} 2 & 2 & 2 \\ \circ & \circ & \circ \end{array}\) | – | + |
| \(\begin{array}{ccc} 2 & 0 & 2 \\ \circ & \circ & \circ \end{array}\) | \(C_2 \times A_1\) | 2 | 4, 2 | \(\begin{array}{ccc} 2 & 0 \end{array}\) | +/- | –/- |
| \(\begin{array}{ccc} 2 & 2 & \circ \\ \circ & \circ & \circ \end{array}\) | \(\tilde{A}_2\) | 1 | 4,1,1 | \(\begin{array}{ccc} 2 & 0 \end{array}\) | + | – |
| \(\begin{array}{ccc} 2 & 0 & \circ \\ \circ & \circ & \circ \end{array}\) | \(A_1 \times \tilde{A}_1\) | 1 | 3,3 | | – | + |
| \(\begin{array}{ccc} 2 & \circ & \circ \\ \circ & \circ & \circ \end{array}\) | \(\tilde{A}_1^2\) | 1 | 2, 2, 2 | | + | + |
| \(\begin{array}{ccc} 2 & \circ & \circ \\ \circ & \circ & \circ \end{array}\) | \(A_1^2\) | 2 | 2, 2, 1, 1 | | +/- | –/- |
| \(\begin{array}{ccc} 2 & \circ & \circ \\ \circ & \circ & \circ \end{array}\) | \(A_1\) | 1 | 2, 1, 1, 1, 1 | | + | – |
| \(\emptyset\) | \(\emptyset\) | 1 | 1, 1, 1, 1, 1 | | + | + |
for every \( G \)-conjugacy class of unipotent elements (cf. [8, p. 400], [24, IV.2]). From Lemma 6.2 one concludes that the \( F \)-fixed points of a unipotent \( G \)-conjugacy class form a single \( G_F \)-conjugacy class apart from regular unipotent elements of type \( C_2 \times \tilde{A}_1 \) and short root elements, where one has 2 \( G_F \)-conjugacy classes.

In the 4th column we list the size of the Jordan blocks of the unipotent element \( \tilde{u} \in S\rho_0(\tilde{F}_3) \) with \( r(\tilde{u}) = u \) in the natural 6-dimensional representation. Here \( r: S\rho_0(\tilde{F}_3) \to PCS\rho_0(\tilde{F}_3) \) denotes the canonical map. These values can be easily obtained from the 2nd column. In the 5th column the \( G \)-conjugacy class of \( u^3 \) is given terms of weighted Dynkin diagrams. In the 6th and 7th column we list whether an \( F \)-fixed point of the unipotent conjugacy class is contained in \( G_F \)-conjugate of \( H_F \) or \( K_F \). In case that \( C_G(u) \) is not connected, there will be 2 \( G_F \)-conjugacy classes, and this is the reason why for some \( G \)-conjugacy classes there occur two signs in this column.

**Proposition 6.4.** Let \( (G, F) \), \( H \) and \( K \) be given as before.

(a) If \( u \) is a unipotent element of type

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(cf. [8, p. 400]), where \( A_1 \) is a simple algebraic group of type \( A_1 \) whose rootgroups are long roots, and “\( \circ \)” stands for direct product modulo a finite central subgroup. As this group is isomorphic to \( C_G(u) / R_u(C_G(u)) \), one has \( C_G(u) = im(c_6) R_u(C_G(u)) \) and this yields the claim.

(b) Note that \( K_F \simeq PGL_2(3^k) \). Hence there exists a unique \( K_F \)-conjugacy class of elements of order 3 being contained in \( K_F \). This conjugacy class is of type \( \frac{2}{3} \). By [21, III.1, Lemma 1], \( K_F \) contains two conjugacy classes of elements of order 3 which are not contained in \( K_F \). Furthermore, there exist elements of order 9 (cf. [9, p. 18]), and for each element \( v \) of order 9, \( \gamma \beta \) is a unipotent element of type \( \frac{2}{3} \). Hence every element of order 9 is of type \( \frac{2}{3} \). This yields the claim.

\[ \square \]

7. Groups of type \( F_4 \), \( p = 3 \)

Let \( (G, F) \) be a finite group of Lie type of type \( F_4 \) defined over the algebraically closed field \( \overline{F}_3 \). Let \( H \) be an \( F \)-stable reductive group of maximal rank of type \( W(B_4)^W \), and let \( K^o \) be an \( F \)-stable reductive group of maximal rank of type \( 3.W(D_4)^W \). Note that \( G \) is simply-connected and of adjoint type. By \( W \) we denote its Weyl group. We denote by \( W(\Psi)^W \) the type of \( H \), and by \( n_{CE}.W(\Xi) \) the type of \( K^o \). We also assume that \( n_{CE} \) is an element of order 3.

Let \( \gamma \in K_F \) be an element of order 3 which is not contained in \( K_F \). Then \( \gamma \) is a unipotent element. Its \( G \)-conjugacy class is the one of type \( A_2 \) (cf. [8, p. 400]), and its \( G_F \)-conjugacy class corresponds to the class \( G_F^{G_F} \) in the list of T. Shoji (cf. [22, Table 6]). In particular, \( C_G(\gamma) \) is connected.

From Theorem 2.2(b) and 3.4 one knows already that every semisimple element \( s \in G_F \) is \( G_F \)-conjugate to an element in \( H_F \) or \( K_F \). Therefore, it suffices to consider elements with non-trivial unipotent part.

7.1 – Elements with non-trivial semisimple and unipotent part

In order to study elements with non-trivial semisimple and non-trivial unipotent part we determine first all \( W \)-orbits of saturated subrootsystems.

Proposition 7.1. Let \( \Pi \subseteq \Phi(F_4) \) be a saturated subrootsystem of the rootsystem of type \( F_4 \). Then the type of \( \Pi \) is as listed in Table 7.1. Furthermore, any two saturated subrootsystems of the same type are \( W \)-conjugate.
Table 7.1. – Saturated subroot systems of $\Phi = \Phi(F_4)$

<table>
<thead>
<tr>
<th>$\Pi$</th>
<th>$N_W(W(\Pi))$</th>
<th>$\Pi \subseteq \Phi$</th>
<th>$N_W(W(\Pi)) = N_{W(\Phi)}(W(\Pi))$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\emptyset$</td>
<td>$W(F_4)$</td>
<td>$\times$</td>
<td>no</td>
</tr>
<tr>
<td>$A_1$</td>
<td>$W(A_1 \times C_3)$</td>
<td>$\times$</td>
<td>no</td>
</tr>
<tr>
<td>$\tilde{A}_1$</td>
<td>$W(\tilde{A}_1 \times B_3)$</td>
<td>$\times$</td>
<td>yes</td>
</tr>
<tr>
<td>$A_1^2$</td>
<td>$W(B_2^2)$</td>
<td>$\times$</td>
<td>yes</td>
</tr>
<tr>
<td>$A_1 \times \tilde{A}_1$</td>
<td>$W(A_1^2 \times \tilde{A}_1^2)$</td>
<td>$\times$</td>
<td>yes</td>
</tr>
<tr>
<td>$A_2$</td>
<td>$(\pm id_E) \times W(A_2 \times \tilde{A}_2)$</td>
<td>$\times$</td>
<td>yes</td>
</tr>
<tr>
<td>$\tilde{A}_2$</td>
<td>$(\pm id_E) \times W(A_2 \times \tilde{A}_2)$</td>
<td>$\emptyset$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>$B_2$</td>
<td>$W(B_2^3)$</td>
<td>$\times$</td>
<td>yes</td>
</tr>
<tr>
<td>$A_1^3$</td>
<td>$W(A_1^3) : S_3$</td>
<td>$\times$</td>
<td>no</td>
</tr>
<tr>
<td>$A_2 \times \tilde{A}_1$</td>
<td>$W(B_2 \times \tilde{A}_1^2)$</td>
<td>$\times$</td>
<td>yes</td>
</tr>
<tr>
<td>$A_1 \times \tilde{A}_2$</td>
<td>$(\pm id_E) \times W(A_1 \times \tilde{A}_2)$</td>
<td>$\emptyset$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>$A_1 \times B_2$</td>
<td>$W(A_1^2 \times B_2)$</td>
<td>$\times$</td>
<td>yes</td>
</tr>
<tr>
<td>$\tilde{A}_1 \times A_2$</td>
<td>$(\pm id_E) \times W(\tilde{A}_1 \times A_2)$</td>
<td>$\times$</td>
<td>yes</td>
</tr>
<tr>
<td>$A_3$</td>
<td>$W(\tilde{A}_1 \times B_3)$</td>
<td>$\times$</td>
<td>yes</td>
</tr>
<tr>
<td>$B_3$</td>
<td>$W(\tilde{A}_1 \times B_3)$</td>
<td>$\times$</td>
<td>yes</td>
</tr>
<tr>
<td>$C_3$</td>
<td>$W(A_1 \times C_3)$</td>
<td>$\emptyset$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>$A_1^4$</td>
<td>$W(C_4)$</td>
<td>$\times$</td>
<td>no</td>
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<tr>
<td>$A_2 \times B_2$</td>
<td>$W(B_2^2)$</td>
<td>$\times$</td>
<td>yes</td>
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<tr>
<td>$A_1 \times C_3$</td>
<td>$W(A_1 \times C_3)$</td>
<td>$\emptyset$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>$\tilde{A}_1 \times A_3$</td>
<td>$W(\tilde{A}_1 \times B_3)$</td>
<td>$\times$</td>
<td>yes</td>
</tr>
<tr>
<td>$A_2 \times \tilde{A}_2$</td>
<td>$(\pm id_E) \times W(A_2 \times \tilde{A}_2)$</td>
<td>$\emptyset$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>$B_4$</td>
<td>$W(B_4)$</td>
<td>$\times$</td>
<td>yes</td>
</tr>
<tr>
<td>$D_4$</td>
<td>$W(F_4)$</td>
<td>$\times$</td>
<td>no</td>
</tr>
<tr>
<td>$F_4$</td>
<td>$W(F_4)$</td>
<td>$\emptyset$</td>
<td>$\emptyset$</td>
</tr>
</tbody>
</table>
PROOF. From the standard algorithm established by A. Borel and J. de Siebenthal and E. B. Dynkin (cf. [1], [11]) one deduces easily that the type of a subrootsystem \( \Psi \subseteq \Phi(F') \) must be one of the types listed in Table 7.1. Every \( W \)-orbits of subrootsystem consisting entirely of long roots or short roots is uniquely determined by its type (cf. [6, Table 4]). Since subrootsystems of type \( B_2, B_3, C_3 \) are the saturation of subrootsystems of type \( \tilde{A}_1^2, \tilde{A}_1^3, A_3 \), this yields the claim.

In the 3rd column of Table 7.1 we have indicated which subrootsystems are \( W \)-conjugate to a subrootsystem of \( \Psi \). In the 4th column we specified the subrootsystems for which there exists a containment \( \Pi \subseteq \Psi \), satisfying

\[
N_W(W(\Pi)) = N_{W(\Psi)}(W(\Pi)).
\]

However, the reader should be warned: \( \Psi \) contains two \( W(\Psi) \)-orbits of subrootsystems of type \( A_1^2 \) and \( A_3 \). Only for one such \( W(\Psi) \)-orbit (7.1) is satisfied.

From Proposition 7.1 one deduces the following property:

**Proposition 7.2.** Let \( (G,F), H \) and \( K \) be as defined above. Let \( x = x_s x_u \in G_F \) be an element with non-trivial semisimple part. Then \( x \) is contained in \( H_F \) or \( K_F \), or \( C_G(x_s) \) is of type \( W(\tilde{A}_2)^W \), \( W(A_1 \times \tilde{A}_2)^W \), \( W(C_3)^W \) or \( W(A_1 \times C_3)^W \).

**Proof.** Since \( G \) is simply-connected, Steinberg’s theorem implies that \( C := C_G(x_s) \) is an \( F \)-stable reductive group of maximal rank containing \( x \). Let \( n_{\Pi}, W(\Pi)^W \) denote the type of \( C \). In particular, \( \Pi \subseteq \Phi \) is a saturated subrootsystem and thus one of the subrootsystems listed in Table 7.1. We may assume that \( \Pi \neq \emptyset, \Phi \).

Suppose \( \Pi \) is \( W \)-conjugate to a subrootsystem of \( \Psi \). We may assume that \( \Pi \subseteq \Psi \). In case that \( N_W(W(\Pi)) \simeq N_{W(\Psi)}(W(\Pi)), C_F \) is \( G_F \)-conjugate to a subgroup of \( H_F \) (cf. [25, §1]) and the conclusion follows.

In case \( \Pi \) is of type \( A_1, A_2, A_3, A_4 \) or \( D_4, \Pi \) is also a subrootsystem of \( \Phi_{long} = \Xi \). Since \( W = W(\Xi) : S_3 \), this shows that for every element \( n \in N_W(W(\Pi)), n.W(\Pi) \) is either contained in one of the \( W \)-conjugates of \( W(\Psi) \), or in one of the \( W \)-conjugates of the coset \( 3.W(\Xi) \).

As the defining characteristic equals 3, the reductive group with root-system of type \( A_2 \times \tilde{A}_2 \) is semisimple with trivial center. Hence \( C \) cannot be of type \( W(A_2 \times \tilde{A}_2)^W \) leaving the four possibilities.

From the 2-covering property for groups of type \( C_3 \) (cf. §6) one deduces the following:
PROPOSITION 7.3. Let \((G, F)\), \(H\) and \(K\) be defined as above. Let \(C \leq G\) be a reductive subgroup of maximal rank of type \(W(A_1 \times C_3)^W\) or \(W(C_3)^W\). Then every element \(x \in C_F\) is \(G_F\)-conjugate to an element in \(H_F\) or \(K_F\).

PROOF. If \(C\) is of type \(C_3\), \(C' := C \circ C_G([C, C])\) is an \(F\)-stable reductive subgroup of maximal rank of type \(A_1 \times C_3\). Hence it suffices to prove the claim for \(C\) of type \(W(A_1 \times C_3)^W\).

The character group \(X_C\) has index 2 in the character group \(X\) of \(G\). Hence \(C \cong SL_2(\mathbb{F}_3) \circ Sp_6(\mathbb{F}_3)\) is isomorphic to the central product of \(SL_2(\mathbb{F}_3)\) with \(Sp_6(\mathbb{F}_3)\), and thus is neither adjoint nor simply-connected. From the 2-covering property of groups of Lie type \(C_3\) one deduces that every element in \(C_F\) is \(G_F\)-conjugate to an element in \(X_F\), \(X\) an \(F\)-stable reductive subgroup of \(G\) of maximal rank of type \(W(A_1^2 \times B_2)^W\), or \(Y_F\), \(Y = N_G(Y^o)\), \(Y^o\) an \(F\)-stable reductive subgroup of maximal rank of type \(3.W(A_1^2)^W\). In the proof of Proposition 7.2 it was shown that \(X_F\) is \(G_F\)-conjugate to a subgroup of \(H_F\), and \(Y_F\) is a \(G_F\)-conjugate to a subgroup of \(K_F\). This yields the claim. \(\square\)

From the 2-covering property for groups of type \(A_2\) and \(^2A_2\) (cf. § 5.2) one concludes:

PROPOSITION 7.4. Let \((G, F)\), \(H\) and \(K\) be defined as above. Let \(C \leq G\) be a reductive subgroup of maximal rank of type \((\pm 1).W(A_2 \times 2A_2)^W\) or \((\pm 1).W(A_1 \times 2A_2)^W\). Then every element \(x \in C_F\) is \(G_F\)-conjugate to an element in \(H_F\) or \(K_F\).

PROOF. Assume that \(C\) is of type \(W(A_2 \times 2A_2)^W\). Hence \(C = S_1 \times S_2\), \(S_1 \cong SL_3\) is generated by long root groups, and \(S_2 \cong SL_3\) is generated by short root groups. In particular, \(C_F \cong SL_3(q) \times SL_3(q)\). Let \(x = x_1x_2\), \(x_1 \in (S_1)_F\), \(x_2 \in (S_2)_F\). From the 2-covering property for the groups of type \(A_2\) one concludes that \(x \in C_F\) is \(G_F\)-conjugate to

(i) \(X_F\), \(X\) an \(F\)-stable reductive subgroup of maximal rank of type \(W(A_2 \times 2A_1)^W\), or

(ii) \(Y_F\), \(Y\) an \(F\)-stable reductive subgroup of maximal rank of type \(3.W(A_2)^W\), or

(iii) \(x_2\) is a regular unipotent element in \(SL_3\) generated by short roots.

So by Proposition 7.2 only the last case has to be considered. Since \(x_2\) is of type \(A_2\), we may assume that \(x_2 = \bar{v}\).
Let \( S_1^g \) be an \( F \)-stable \( G \)-conjugate of \( S_1 \) centralizing \( \tilde{\gamma} \). As \( C_G(S_1) = S_2 \) and \( C_G(S_2) = S_1, \tilde{\gamma}^{-1} \in S_2 \). Hence there exists \( s \in S_2 \) such that \( \tilde{\gamma}^{-1} s = \tilde{\gamma} \), and thus \( S_1^g = S_1^{-1} g \). In particular, \( S_1^g \) and \( S_1 \) are conjugate by an element in \( C_G(\tilde{\gamma}) \). Since \( C_G(\tilde{\gamma}) \) is connected, the Lang-Steinberg theorem implies that \( S_1^g \) and \( S_1 \) are conjugate by an element in \( C_G(\tilde{\gamma})_F \).

As \( C_K(\tilde{\gamma}) \cong G_2 \), it contains an \( F \)-stable subgroup which is \( G_F \)-conjugate to \( S_1 \). Hence the previous remark yields the claim in this case. The other three cases follow by a similar argument and are therefore left to the reader. \( \square \)

7.2 – Regular unipotent elements

For our purpose it is important to know whether \( K_F \) contains regular unipotent elements of \( G \) or not. In [20, Lemma 3.2] it was shown that the algebraic group \( K \) contains regular unipotent elements of \( G \). However, the argument used there cannot be applied in our case.

From [16] (cf. [20, Lemma 3.2]) one knows that regular unipotent elements are the only elements of order 27 in \( G \). Thus it suffices to show that \( K_F \) contains elements of order 27.

Let \( A = \{ e_1 - e_2, e_2 - e_3, e_3 - e_4, e_3 + e_4 \} \) be the standard basis of the root system of \( \Phi_{\text{long}} = \Xi \), and let \( \gamma \) be the graph automorphism with cycle decomposition \((e_2 - e_3)(e_1 - e_2, e_3 + e_4)\). We assume that \( \tilde{\gamma} \in \text{Aut}(K_F) \) is acting on \( \Phi \) through \( \gamma \).

For simplicity we use the same notation as was introduced in [22], i.e., \( 1 - 2 \) is a short form of the root \( e_1 - e_2 \), etc. For \( s \in F_q^* \) and \( t \in F_{q^3}^* \) we define the root elements

\[
\begin{align*}
(7.2) \quad u_1(s) &= x_{2-3}(s), & v_1(t) &= x_{1-2}(t) \cdot x_{3+4}(t^q) \cdot x_{3-4}(t^{q^2}), \\
(7.3) \quad u_2(s) &= x_{1+3}(s), & v_2(t) &= x_{1-3}(t) \cdot x_{2+4}(t^q) \cdot x_{2-4}(t^{q^2}), \\
(7.4) \quad u_3(s) &= x_{1+2}(s), & v_3(t) &= x_{1+4}(t) \cdot x_{2+3}(t^q) \cdot x_{1-4}(t^{q^2}),
\end{align*}
\]

which are obviously contained in \( K_F^\circ \) (cf. [7, § 13.6]), and which generate the 3-Sylow subgroup of \( K_F^\circ \). In particular, for \( i = 1, 2, 3 \) one has

\[
\begin{align*}
(7.5) \quad u_i(s)^{\tilde{\gamma}} &= u_i(s), \\
(7.6) \quad v_i(t)^{\tilde{\gamma}} &= v_i(t^{q^2}).
\end{align*}
\]

If one chooses the sign of the structure constants as in [22, Tab. 1] one deduces the following commutator formula from Chevalley’s commutator
formula (cf. [7, Thm. 5.2.2]). Here \([x, y] = x^{-1}y^{-1}xy\) denotes the standard commutator.

**Proposition 7.5.** For the elements as defined in (7.2) one has the following formulae:

\[
[u_1(s), u_2(s')] = u_2(-ss'),
\]

\[
[u_1(s), v_1(t)] = v_2(-st) \cdot v_3(-st^{t^q}) \cdot u_2(-sN(t)) \cdot u_3(s^2N(t)),
\]

\[
(v_1(t), v_2(\omega)) = \frac{1}{\omega} \cdot u_2(-sT(t + \omega^t)) \cdot u_2(-sT(t + \omega^t)),
\]

\[
(v_1(t), v_3(\omega)) = u_2(-Tr(t + \omega^t)),
\]

\[
[v_2(t), v_3(\omega)] = u_2(-Tr(t + \omega^t)),
\]

where \(s, s' \in \mathbb{F}_q^\ast, t, \omega \in \mathbb{F}_{q^2}, \) \(\mathbb{N} : \mathbb{F}_{q^2} \to \mathbb{F}_q\) denotes the norm map and \(Tr: \mathbb{F}_{q^2} \to \mathbb{F}_q\) denotes the trace map. Furthermore, any pair of elements not occurring in (7.7) commutes.

**Proof.** This follows by a lengthy but straightforward calculation. \(\square\)

**Lemma 7.6.** Let \(a, b, c\) be elements of a group \(A\) with the following properties:

(i) \(a^3 = b^3 = c^3 = [b, a]^3 = [c, b]^3 = [c, a]^3 = [[[b, a], b]^3 = 1,\)

(ii) \([c, b], [c, a], [[b, a], b] \in Z(A),\)

(iii) \([[b, a], a], b] = 1.\)

Then \((abc)^3 = [[[b, a], a][[b, a], b]^3.\)

**Proof.** This follows by a straightforward calculation using the basic commutator identities (cf. [18, § 5.1.5]). \(\square\)

We finally obtain the following property:

**Proposition 7.7.** Let \((G, F), H\) and \(K\) be as above. Then \(K_F\) contains regular unipotent elements of \(G.\)

**Proof.** Let \(g = \tilde{v} \cdot u_1(s) \cdot v_1(t).\) Then

\[
g^3 = v_1(Tr(t)) \cdot v_2(s(t^q - t)^q) \mod U_0,
\]

where \(U_0 = \langle u_2(s), u_3(s), v_3(t) \mid s \in \mathbb{F}_q, t \in \mathbb{F}_{q^2} \rangle.\) Hence

\[
g^3 = v_1(Tr(t)) \cdot v_2(s(t^q - t)^q) \cdot v_3(\omega) \cdot u_2(s) \cdot u_3(s').
\]
for suitable numbers \( \omega \in \mathbb{F}_{q^2}, s, s' \in \mathbb{F}_q \). Since \( u_2(s) \) and \( u_3(s') \) centralize the subgroup generated by \( v_1(t), v_2(t'), v_3(t''), t, t', t'' \in \mathbb{F}_{q^2} \) (cf. (7.7)), one has

\[
(7.10) \quad g^9 = \left( v_1(Tr(t)) \cdot v_2(s(t' - t)^q) v_3(\omega) \right)^3.
\]

It is easy to check that Lemma 7.6 applies for \( a := v_1(Tr(t)), b := v_2(s(t'- t)^q) \) and \( c := v_3(\omega) \). Moreover, for \( a, b, c \) as defined above, one has

\[
(7.11) \quad [[b, a], b] = u_3(2s^2 Tr(t)(Tr(t'^{-1} - Tr(t^2))),
\]

\[
[[b, a], a] = u_2(2s Tr(t^2 Tr(t'' - t)) = 1.
\]

For \( q \neq 3 \), \( Tr(t)(Tr(t'^{q+1} - Tr(t^2)) \) is a polynomial in \( t \) of degree \( 3q^2 \) and thus there exists an element \( t \in \mathbb{F}_{q^2} \) with \( Tr(t)(Tr(t'^{q+1} - Tr(t^2)) \neq 0 \). Hence the claim follows. For \( q = 3 \), the polynomial \( Tr(t)(Tr(t'^{q+1} - Tr(t^2)) \) as a function on \( \mathbb{F}_{27} \) is equivalent to a polynomial of degree 21, and hence the claim follows also in this case.

\[\square\]

7.3 - \( G_F \)-conjugacy classes of unipotent elements

From the information obtained in [22] one concludes the following:

**Proposition 7.8.** Let \( (G, F) \), \( H \) and \( K \) be as above. Then every unipotent element in \( G_F \) is \( G_F \)-conjugate to an element in \( H_F \) or \( K_F \).

**Proof.** The unipotent \( G_F \)-conjugacy classes in the finite group of Lie type \( G_F \) of type \( F_4 \) defined over a finite field of characteristic 3 were determined by T. Shoji (cf. [22]). There are 28 unipotent \( G_F \)-conjugacy classes \( x_i^{G_F} \), \( i = 0, \ldots, 27 \), where \( x_i \) is the representative given in [22, Tab. 6].

From the definition of the elements \( x_i \) it is obvious that unipotent elements in the classes \( x_j^{G_F} \),

\[
(7.12) \quad j \in \{ 0, 1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 13, 14, 15, 16, 17, 19, 23, 24 \}
\]

are \( G_F \)-conjugate to an element in \( H_F \).

Furthermore, for \( j \in \{20, 21, 22\} \), \( x_j \) is \( G_F \)-conjugate to the group \( C_F \) of \( F \)-fixed points in \( C \), where \( C \) is an \( F \)-stable reductive subgroup of maximal rank of type \( W(A_1 \times C_3)^W \). Hence by Proposition 7.3, \( x_j \) is \( G_F \)-conjugate to an element in \( H_F \) or \( K_F \).

The unipotent element \( \bar{y} \in K_F \) is \( G_F \)-conjugate to \( x_7 \). A similar argument shows that \( \bar{y}, x_a(1), a \) the simple root fixed by \( \bar{y} \) is \( G_F \)-conjugate to \( x_{11} \).
By Proposition 7.7, \( K_F \) contains a regular unipotent element \( u \) of \( G \). Note that there exists a unique \( G_F \)-conjugacy class of \( F \)-stable reductive groups of maximal rank of type \( n \mathbb{Z}, W(\mathbb{Z})^W \). Since 3 is a bad prime for \( F_4, C_G(u)/C_G(u)^o \) is a cyclic group of order 3 being generated by \( u \) (cf. [24, 1.14(d)]). For \( k = 1, 2 \), let \( g_k \in G \) be such that \( F(g_k)g_k^{-1} = u^k \). Then \( K^{g_k} \) is the normalizer of an \( F \)-stable reductive group of maximal rank of type \( n \mathbb{Z}, W(\mathbb{Z})^W \) and thus \( G_F \)-conjugate to \( K \). By construction, \( (K^{g_k})_F \) contains the regular unipotent element \( u^{g_k} \) fixed by \( F \) corresponding to the \( G_F \)-conjugacy class \( u^k.C_G(u)^o \). This shows that \( K_F \) contains an element of either of the \( G_F \)-conjugacy classes of regular unipotent elements \( x_{25}^{G_F}, x_{26}^{G_F}, x_{27}^{G_F} \) (cf. Lemma 6.2(c)).

A similar argument shows that \( K_F \) contains also an element of the conjugacy class \( x_{18}^{G_F} \). This yields the claim. \( \square \)

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