6-BFC Groups.

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To Professor Guido Zappa on his 90th birthday

1. Introduction and preliminaries.

A group is said to be BFC if its conjugacy classes (of elements) are boundedly finite, and $n$-BFC if the largest conjugacy classes have order $n$. B.H. Neumann proved in [4] that a group is BFC if and only if its derived group is finite; in [8], the second author showed that the derived group of an $n$-BFC group is of order bounded in terms of $n$. He formulated there the following conjecture, in which $\lambda(n)$ stands for the number of prime factors of $n$, multiplicities included.

**Conjecture.** For every $n$-BFC group $G$, the order of $G'$ is at most $n^{\frac{1}{2}(1+\lambda(n))}$.

There are nilpotent groups of class 2 and arbitrarily large $n$ where this bound is achieved. Further, it is proved in [8] that the conjecture is true when $n$ is prime and when $n = 4$, in which case $G'$ is of order 4 or 8. There is a wide literature attacking this problem; the best bound achieved so far is that of Segal and Shalev [5], namely $n^{\frac{1}{2}(13+\log_2(n))}$. Vaughan-Lee [6] established the conjecture for nilpotent groups. The smallest value of $n$ for which the conjecture is not known to be true is 6, and the aim of this note is to rectify this by proving the following result.

**Theorem.** Let $G$ be a 6-BFC group. Then $G'$ is either $C_6$ or $Q_8$.

Throughout, notation is standard unless otherwise stated. For example, we write $n = \beta(G)$ for an $n$-BFC-group $G$. It is easy to see [8] that the proof

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of the theorem reduces to one for finite groups, and henceforward we shall consider finite groups only.

We state some obvious facts here.

1.1. For any groups $A$ and $B$, $\beta(A \times B) = \beta(A)\beta(B)$.

1.2. If $N$ is normal subgroup of a group $G$ and $g$ an element of $G$, then the number of conjugates of $gN$ in $G/N$ divides the number of conjugates of $g$ in $G$.

A useful tool in the proof of the Theorem is this. Let $a$ be an element of a group $G$ with exactly $t$ conjugates $a = a_1, a_2, \ldots, a_t$. Then the map $\mu(a)$ of $G$ into the symmetric group $S_t$ defined by

$$g\mu(a) = \begin{pmatrix} a_1 & a_2 & \ldots & a_t \\ a_1^g & a_2^g & \ldots & a_t^g \end{pmatrix}$$

is a homomorphism of $G$ onto a transitive subgroup of $S_t$.

1.3. The kernel of $\mu(a)$ is $\text{Core}_G C(a)$.

Note that both alternatives for $G'$ mentioned in the Theorem occur. There are in fact four 6-BFC group of order 24, the smallest possible order. Three of them have derived group cyclic of order 6, collected together as Examples 1.4; see Coxeter and Moser [1].

**Example 1.4.** The dihedral group of order 24 is 6-BFC and its derived group is $C_6$. The same is true of the dicyclic group of order 24 and the group with the following presentation:

$$\langle a, b : a^4 = b^6 = (ab)^2 = (a^{-1}b)^2 = 1 \rangle.$$

**Example 1.5.** The binary tetrahedral group, with presentation

$$\langle a, b, c : a^2 = b^2 = [a, b], c^3 = 1, a^c = b, b^c = a^{-1}b^{-1} \rangle,$$

is 6-BFC and has derived group $Q_8$.

Finally in this section, we show that the problem reduces to one in $\{2, 3\}$-groups. Specifically, we have the following result.

**Lemma 1.6.** (1) Every 6-BFC group is soluble.

(2) If $G$ is a 6-BFC group, then $G = A \times B$, where $A$ is a $\{2, 3\}$-group and $B$ is an abelian $\{2, 3\}'$-group.
Proof. The proof of (1) is almost immediate. If there is an insoluble 6-BFC group, some composition factor $H$ of it is a nonabelian simple group with $\beta(H) \leq 6$. By the remarks preceding 1.3, there is an isomorphism of $H$ onto a simple subgroup of $S_6$; these are $A_5$ and $A_6$, and they have conjugacy classes of order more than 6.

Now let $G$ be any 6-BFC group. All Sylow $p$-subgroups of $G$ with $p > 5$ are central. Otherwise, there is an element $g$ of $p$-power order not commuting with some element $a$ of $G$, and the conjugates $a^g$ of $a$ with $0 \leq i < p$ are all distinct, too many for a 6-BFC group.

Less obvious but still easy is that the Sylow 5-subgroup is central. We shall show first that every element $x$ of 5-power order in $G$ commutes with all elements having 1, 2, 3, 4, or 6 conjugates. The first four are proved in a manner like that in the previous paragraph. Now suppose that $y$ has 6 conjugates. Then, by 1.3 above, $G/\text{Core } C(y)$ is a transitive group of degree 6. If $y$ fails to commute with $x$, then the element $x \text{Core } C(y)$ is of order 5, and thus $G/\text{Core } C(y)$ is doubly transitive and therefore primitive; but it is soluble and thus imprimitive because 6 is not a prime-power. Thus $x$ does indeed commute with all elements having 1, 2, 3, 4 or 6 conjugates. If $x$ is not central, then $G$ must be generated by elements having exactly 5 conjugates, namely the elements outside the subgroup generated by elements having 1, 2, 3, 4 or 6 conjugates. Finally, suppose that $u$ is an element of 3-power order not commuting with an element $v$ with 5 conjugates. Then $G/\text{Core } C(v)$ is a transitive group of degree 5 having the non-trivial element $u \text{Core } G(v)$ of order 3; thus it is either $S_5$ or $A_5$ and that is impossible because $\beta(G) = 6$. So every element $u$ of 3-power order does commute with every element with 5 conjugates; since these generate $G$, we have that the Sylow 3-subgroup $P$ is central. This is the final contradiction, as then $G = P \times R$ with $R$ a $3'$-group and so by 1.1, $G$ cannot have an element with 6 conjugates. Thus the elements of 5-power order are central, which is all that is needed to complete the proof of part (2) of the lemma. All Sylow $p$-subgroups with $p > 3$ are central and the direct decomposition stated in (2) is obvious.

Thus $G' = A'$ and $\beta(G) = \beta(A)$, so we have reduced the problem to one in $\{2, 3\}$-groups.

2. Proof of the Theorem.

The proof goes by induction on the group order. The theorem holds for the smallest 6-BFC groups, namely those in Examples 1.4 and 1.5. So we
assume that $G$ is of order more than 24 and that the theorem holds for smaller groups. We divide the proof into several sections. Recall that we may assume that $G$ is a $\{2,3\}$-group.

2.1. Suppose that $G$ is nilpotent. Then $G = A \times B$, where $A$ is a 2-group and $B$ a 3-group. Then $A$ must be 2-BFC and $B$ must be 3-BFC by 1.1, so that $A'$ is of order 2 and $B'$ of order 3, and $G' = A' \times B'$ is cyclic of order 6, as required.

2.2. Suppose that $G$ has $A_4$ as a homomorphic image, and let $N$ be such that $G/N \cong A_4$. We shall show that $N = Z$, the centre of $G$, and that $G' \cong Q_8$. The conjugacy classes of $A_4$ are: the identity class, the three elements of order 2 in $A'$, and two classes consisting of four elements of order 3 outside $A'$. The possible conjugacy class sizes of $G$ are 1, 2, 3, 4, 6; by 1.2, elements with 1, 2, 3 or 6 conjugates map to $A_4$ mod $N$. The set of elements of $G$ mapping to $A_4$ is $G/N$, and the elements outside $G/N$ have four conjugates each and map to elements with 4 conjugates each outside $A_4$. Take $a$ in $G - G'N$. Then $|G : C(a)| = 4$ and $|G/N : C(aN)| = 4$, which means that $C(a)$ contains $N$. But $G$ is generated by elements like $a$, so that $N$ is central. As $G/N$ is $A_4$, this means that $N = Z$, as claimed, and $G/Z \cong A_4$. Next, $G'/(G' \cap Z) \cong A_4' \cong C_2 \times C_2$. Further, $G' \cap Z$ is isomorphic to a subgroup of the Schur multiplier of $A_4$, that is, of $C_2$. Indeed it must be of order 2, else $G'$ is of order 4, too small for a 6-BFC group. Thus $G'$ is of order 8. Let $H$ be a stem-group [3] in the isoclinism class of $G$. Then $Z(H) \subseteq H'$, $H' \cong G'$, $G/Z(G) \cong H/Z(H)$ by definition, and it is very easy to see that $\beta(G) = \beta(H)$. As $H$ is not nilpotent, this means that $Z$ is of order 2 and $H$ of order 24. It is now an easy matter to check that the only group with the required properties is that in Example 1.5, and so $G'$ is $Q_8$, as claimed.

Thus we may now assume from now on:

2.3. $G$ is not nilpotent and does not have $A_4$ as a homomorphic image.

Lemma 2.3.1. $G$ is not generated by the set of all elements with 1, 2, or 4 conjugates.

Proof. Suppose the contrary, namely that $G$ is so generated. The elements with 1 conjugate are central. If $a$ has 2 conjugates, then $G' \subseteq C(a)$ as $|G : C(a)| = 2$. If $a$ has 4 conjugates, then $G/\text{Core } C(a)$ is a transitive group on 4 symbols that is at most 6-BFC. Since $A_4$ is not a candidate, the only possibilities are $D_8, C_4, C_2 \times C_2$. Thus $G/\text{Core } C(a)$ is nilpotent of class
at most 2 and so $\gamma_3(G) \subseteq C(a)$ in all cases considered. But then $\gamma_3(G)$ is central and $G$ is nilpotent, which is not allowed. This proves the lemma.

Thus $G$ is generated by the set of all elements with 3 or 6 conjugates, and we prove:

**Lemma 2.3.2.** The Sylow subgroups of $G/Z$ are elementary abelian, and the derived group is a 3-group.

**Proof.** As before, we may assume that $Z \subseteq G'$; that is, we replace $G$ if necessary by the stem-group of its isoclinism class. Let $S$ be the set of all elements with 3 or 6 conjugates. Then $Z = \bigcap_{g \in S} C(g) = \bigcap_{g \in S} \text{Core } C(g)$, so that $G/Z$ is a subgroup of the direct product $\text{Dr } G/\text{Core } C(g)$; so all we have to do is to establish that each $G/\text{Core } C(g)$ has elementary abelian Sylow subgroups and that the derived group is a 3-group.

When $g$ has 3 conjugates, the result is clear since the choices for $G/\text{Core } C(g)$ are $C_3$ and $S_3$. When $g$ has 6 conjugates, $G/\text{Core } C(g)$ is a soluble transitive group of degree 6 which is at most 6-BFC, and the only such groups are (abstractly) $A_4$, $S_3$, $S_3 \times C_2$, $C_2 \wr C_3$, $C_3 \wr C_2$, $C_6$. But $C_2 \wr C_3$ maps to $A_4$, so in our case $G/\text{Core } C(g)$ must be one of $C_6$, $S_3$, $S_3 \times C_2$, $C_3 \wr C_2$ and so it has Sylow subgroups of the required type.

To sum up: $G$ is a $(2, 3)$-group, not nilpotent, does not map to $A_4$, the Sylow subgroups are elementary abelian, and $(G/Z)'$ is a 3-group.

**Case 1.** $Z \neq 1$. Then $G/Z$ is smaller than $G$ and is not nilpotent. Further, it is not 2-BFC nor 4-BFC since the derived groups of such groups are 2-groups [8]. Further, it is not 6-BFC either, as if it were the induction hypothesis would give that $(G/Z)'$ is $C_6$ or $Q_8$, neither of which are 3-groups. Thus $G/Z$ is 3-BFC and by [8] again, $(G/Z)'$ is of order 3. In particular, $G'$ is abelian.

Suppose first that $Z$ contains an element $x$ of order 2. If $Z = \langle x \rangle$, we have $G' \cong C_6$ as $G'/Z \cong C_3$, and all is well. If $\langle x \rangle \neq Z$, then $G'/\langle x \rangle$ is of order more than 3 and so $G/\langle x \rangle$ is 6-BFC; it cannot be 3-BFC since its derived group is of order more than 3, and it cannot be 2-BFC nor 4-BFC as $G'/\langle x \rangle$ is not a 2-group. The induction hypothesis now applies to give that $G'/\langle x \rangle$ is $C_6$, since it cannot be $Q_8$ as $G$ is metabelian. Thus $G'$ is of order at most 12. If $G'$ as order less that 12, it must have order 6 and so it is $C_6$ since $S_3$ is not a derived group. Thus we may assume that $G'$ is of order 12, and we can write $G' = A \times Z$, where $A$ is of order 3 and the centre $Z$ of $G$ is of order 4. Then $(G/A)'$ is of order 4, so $G/A$ is 4-BFC; it is also nilpotent of
class 2 since its derived group is $ZA/A$. The nilpotent residual of $G$ is contained in $A$, and indeed it must be $A$ since $A$ is of order 3 and $G$ is not nilpotent. By [2], $G$ splits over $A$, say $G = AU$, where $A \cap U = 1$. Note that $U \cong G/A$, so $U$ is 4-BFC and $U$ is nilpotent of class 2. The Sylow 3-subgroup $X$ of $U$ is abelian and central in $U$, being a direct factor; since $A$ is normal and of order 3, $X$ centralizes $A$. Thus $X$ is in the centre of $G$ and is therefore trivial since the centre is of order 4. So $U$ is a 2-group. By Lemma 2.3 of [7], $U$ is generated by elements with four conjugates; since $A$ does not centralize $U$, it must fail to centralize an element $u$ in $U$ with four $U$-conjugates. Let $a$ be a generator of $A$. Since $[a, u]$ is not 1, it must be $a$ since $a^u = a^{-1}$. Thus $u$ does not commute with any element of $G$ of the form $av$, with $v$ in $U$; that is, the $G$-centralizer of $u$ is in $U$ and thus it is the $U$-centralizer of $u$. This is a contradiction: $|G : C_G(u)| = |G : C_U(u)| = 3|U : C_U| = 12$, which gives $u$ 12 conjugates, impossible as $G$ is 6-BFC.

So in Case 1, we may assume that $Z$ is a 3-group and thus that $G'$ is a 3-group since $G'/Z$ is of order 3. We shall show that $Z$ must be of order 3. If not, there is an element $y$ of order 3 such that $\langle y \rangle \neq Z$; as above, the factor-group $G/\langle y \rangle$ must be 6-BFC and by induction this is a contradiction since $G'$ is a 3-group. Thus $Z$ is of order 3 and $G'$ of order 9.

If $G'$ is cyclic, say $G' = \langle t \rangle$, then $Z = \langle t^3 \rangle$. For every $g$ in $G$, $t^G = t^m$ for some integer $m$ and so $t^3 = (t^3)^g = t^{3m}$, which means that $9|3(m - 1)$ and $3|(m - 1)$, say $m = 3r + 1$ and then $[t, g] = t^{-1}g = t^{3m} \in Z$. Thus $[G', G]$ is central, a contradiction as $G$ is not nilpotent. Thus $G'$ is not cyclic. There are two possibilities for the nilpotent residual of $G$. It is either $G'$ or a non-central subgroup of order 3 in $G'$. We deal with the two cases separately.

Suppose first that $G'$ is the nilpotent residual of $G$. By [2] again, $G$ splits over $G'$, say $G = G'U$ where $G' \cap U = 1$ and thus $U$ is abelian. We shall show that $G$ has order at most 54 in these circumstances. The 6 non-central elements in $G'$ can split into $G$-conjugacy classes only of the following sizes: 6; 3, 3; 4, 2; 2, 2, 2. Suppose that $G'$ contains an element $x$ with just two conjugates. Then $|G : C(x)| = 2$ and thus $|UC(x) : C(x)| = 6$ is at most 2, that is, $|U : U \cap C(x)|$ is at most 2. But $G' = \langle Z, x \rangle$ so $C(G') = C(x)$ and thus $U \cap C(G')$ has index at most 2 in $U$. But $U$ is abelian and $G$ is generated by $U$ and $G'$, so that $U \cap C(G')$ is central. But $U \cap Z = 1$, so $U$ has order at most 2 and $G$ has order at most 18. Such a group cannot be 6-BFC since it has centre of order 3: all centralizers are bigger than the centre. If there is a conjugacy class of size 3, a similar argument shows that $G$ has order 27, impossible as groups of that order cannot be 6-BFC. Thus we may assume that the 6 non-central elements in
$G'$ form a conjugacy class, and an argument like the one above shows that $U$ has order at most 6, and the only non-trivial case is where it has order 6 and so $G$ has order $9.6 = 54$. A rather fussy argument now completes the proof. If $u$ is an element of $U$ of order 2, then its centralizer has index a $3$-power and therefore index 3; so $u$ centralizes a subgroup $X$ of order 9. If the Sylow 3-subgroup $P$ is abelian, this means that $X$ is central, impossible as $Z$ has order 3. Otherwise $P$ is one of the two non-abelian groups of order 27, and it is readily proved that it does not have an automorphism of order 2 such that the splitting extension of $P$ by it produces a group with the required properties.

Thus we may assume that the nilpotent residual $V$ has order 3, and thus $G' = Z \times V$. Again by [2], $G = VU$ for some subgroup $U$ with $V \cap U = 1$. Then $G' = V'U'[V, U] = VU'$ and $U'$ is of order 3, so $U$ is 3-BFC. Some element $u$ of $U$ with 3 conjugates in $U$ must fail to commute with a generator $v$ of $V$. It follows as above that $C_G(u) = C_U(u)$ and so $|G : C_G(u)| = 9$, false as $G$ is 6-BFC. This completes Case 1.

**Case 2.** $Z = 1$.

By Lemma 2.3.2, $G$ has elementary abelian Sylow subgroups and $G'$ is a 3-group. Let $P$ be the Sylow 3-subgroup. Then $P = G' \times L$ for some subgroup $L$. By Maschke’s theorem, $L$ can be chosen to be normal in $G$. But then $L = 1$ since $L$ is a normal subgroup missing $G'$ and therefore central, and we have $G' = P$. Next, $C(G')$ is nilpotent and therefore of the form $G' \times X$, where $X$ is a 2-group characteristic in $G'$ and therefore normal in $G$. Since it misses $G'$, it too is trivial and so $C(G') = G'$.

We claim that $G'$ contains a normal subgroup of order 3. Let $M$ be a maximal subgroup of $G$ containing $G'$. Then $M$ is normal and thus of index 2. It is smaller than $G$ and therefore, by the induction hypothesis, it is not 6-BFC since its derived group is a 3-group. It is not 4-BFC nor 2-BFC, because such groups have 2-group derived groups. If it is 1-BFC, that is, abelian, then $M$ is $G'$ because its 2-part is normal and therefore trivial. So when $M$ is abelian, $G$ is $G'(a)$, where $a$ is of order 2. The centralizer of $a$ has 3-power index and so is of index 3; as $G'$ is evidently of order more than 3 (at least 6 since $G$ is 6-BFC), this means that $a$ centralizes a non-trivial subgroup $Y$ of $G'$. But $Y$ is central since it centralizes $a$ and $G'$, and this is a contradiction. Thus $M$ must be 3-BFC, its derived group $M'$ is the normal subgroup of $G$ of order 3 that we claimed exists. Note that $G/M'$ is smaller than $G$, and the by now familiar argument shows that it is 3-BFC, which means that $G'/M'$ is of order 3 and $G'$ of order 9.
Further, $M'$ is a direct factor of $G'$, and so by Maschke again, there is a $G$-normal subgroup $A$ such that $G' = M' \times A$. Further, $G/C(G')$ is an elementary 2-group; as a subgroup of $\text{Aut}(C_3 \times C_3)$, it has order at most 4 and $G$ has order at 18 or 36 since $C(G') = G'$. When $G$ has order 18, we have $G = G'\langle a \rangle$ for some element $a$ of order 2; as in the previous case, $G$ has non-trivial centre and this is a contradiction. Suppose finally that $G'$ has order 36. We have that $G'$ is the direct product of two $G$-normal subgroups $\langle a \rangle, \langle b \rangle$, of order 3. A Sylow 2-subgroup of $G$ is a four-group $\langle c, d \rangle$. As above, $c$ has centralizer of index 3 and must centralize a subgroup of order 3, generating the centre of $\langle a, b, c \rangle$ and therefore normal in $G$. Without loss, we may assume that $c$ centralizes $a$; since $a$ is not central, (conjugation by) $d$ must invert $a$. Since $c$ is not central, it must invert $b$. If $d$ inverts $b$, then $d$ inverts everything in $G'$ and so has centralizer of order 4, meaning that it has 9 conjugates, which is impossible. Thus $d$ centralizes $b$ and inverts $a$. But then $cd$ inverts $a$ and $b$ and so has too many conjugates. This completes the Case 2 and the theorem is proved.

The next lowest value of $n$ for which the conjecture is not known is $n = 8$. To confirm it in this case would be a much longer undertaking than that in this short note.

REFERENCES


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