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Compact Subgroups of $\mathrm{GL}_n(\mathbb{C})$.

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1. Introduction.

Let $G \subset \mathrm{GL}_n(\mathbb{C})$ be a compact subgroup. Using the Haar measure on G one obtains a positive definite Hermitian form on $\mathbb{C}^n$ which is invariant under G. In other words, G is conjugated, with respect to $\mathrm{GL}_n(\mathbb{C})$, to a subgroup of the standard unitary group $U_n(\mathbb{C})$. In particular, every $g \in G$ is semisimple and all its eigenvalues have absolute value 1.

The inverse problem was posed by K. Millet and I. Kaplansky (see [Ba]):

Suppose that the subgroup $G \subset \mathrm{GL}_n(\mathbb{C})$ has the property that every $g \in G$ is semisimple and all its eigenvalues have absolute value 1. *Is G conjugated to a subgroup of $U_n(\mathbb{C})$?*

For $n = 1, 2$ the answer is positive. A counterexample for $n \geq 3$ is given in ([Ba], Counterexample 1.10, p. 19). However, using the techniques of Burnside, it is shown in ([Ba], Corollary 1.8, p. 18), that G is isomorphic to a subgroup of $U_n(\mathbb{C})$. The aim of this paper is to present a proof of the following positive result.

THEOREM 1.1. *Suppose that the subgroup $G \subset \mathrm{GL}_n(\mathbb{C})$ satisfies:*

(i) *Every element of G is semisimple and all its eigenvalues have absolute value* 1.

(ii) *G is* closed *with respect to the ordinary topology of* $\mathrm{GL}_n(\mathbb{C})$.

Then G is conjugated in $\mathrm{GL}_n(\mathbb{C})$ *to a subgroup of* $U_n(\mathbb{C})$ *and therefore compact.*

The theorem has an almost immediate consequence.

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Corollary 1.2. *Let E be an n-dimensional affine euclidean space and G a closed subgroup of the group of all isometries of E. Suppose that each element of G has a fixed point. Then the group G is compact and has a fixed point.*

Proof. The action of $g \in G$ on $\mathbb{R}^n$ is given by $X \in \mathbb{R}^n \mapsto gX = UX + A$ with $U \in O_n(\mathbb{R})$, $A \in \mathbb{R}^n$. One associates to $g \in G$ the matrix $M(g) = \begin{pmatrix} U & A \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_{n+1}(\mathbb{R})$. All eigenvalues of $M(g)$ have absolute value 1. Since U is semisimple, $M(g)$ is semisimple if and only if there exists a vector $X \in \mathbb{R}^n$ such that $M(g)\begin{pmatrix} X \\ 1 \end{pmatrix} = \begin{pmatrix} X \\ 1 \end{pmatrix}$. This property of X is equivalent to X is a fixed point for g. It follows that $M(g)$ is semisimple if and only if g has a fixed point. The theorem implies that $\{M(g)|g \in G\}$ is compact. Then G is compact and has a fixed point. □

2. A result on real Lie algebras.

Proposition 2.1. *V is a complex vector space of dimension $n \geq 1$. Let $\mathfrak{g}$ be a* real *Lie subalgebra of* $\mathrm{End}_{\mathbb{C}}(V)$ *satisfying:*

(a) $\mathbf{i} \cdot 1_V \notin \mathfrak{g}$

(b) *If $V = V_1 \oplus V_2$ with V_1, V_2 complex vector spaces invariant under $\mathfrak{g}$, then $V_1 = 0$ or $V_2 = 0$.*

(c) *Every element of $\mathfrak{g}$ is semisimple and all its eigenvalues are in $\mathbf{i} \cdot \mathbb{R}$.*

Then the following holds:

(1) *$\mathfrak{g}$ is a real semisimple Lie algebra, $\mathfrak{G} := \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{g}$ is a complex semisimple Lie algebra and the canonical map $\mathfrak{G} \to \mathrm{End}_{\mathbb{C}}(V)$ is injective.*

(2) *Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Then $\mathfrak{H} := \mathbb{C} \otimes \mathfrak{h}$ is a Cartan subalgebra for the complex Lie algebra of $\mathfrak{G}$. Let R be the set of roots for the pair $(\mathfrak{G}, \mathfrak{H})$. Then*

(2a) *$\alpha(h) \in \mathbf{i} \cdot \mathbb{R}$ for every $h \in \mathfrak{h}$ and $\alpha \in R$,*

(2b) *for every $\alpha \in R$, the real Lie subalgebra of $\mathfrak{g}$, generated by $\mathfrak{g} \cap (\mathfrak{G}_\alpha \oplus \mathfrak{G}_{-\alpha})$ is isomorphic to $\mathfrak{su}_2$.*

(3) *There exists a positive definite Hermitian form F such that for all $x, y \in V$ and $g \in \mathfrak{g}$ one has $F(gx, y) + F(x, gy) = 0$.*

PROOF. (1). Suppose that $\mathfrak{g}$ is not semisimple. Then $\mathfrak{g}$ has a non zero solvable ideal. Let $\mathfrak{a} \neq 0$ be a minimal solvable ideal, then $[\mathfrak{a}, \mathfrak{a}] = 0$. Since the elements of $\mathfrak{a}$ are semisimple and commute there is a decomposition $V := \mathbb{C}^n = V_1 \oplus \cdots \oplus V_r$ and there are distinct $\mathbb{R}$-linear maps $\boldsymbol{\lambda}_j : \mathfrak{a} \to \mathbf{i} \cdot \mathbb{R}$ such that the action of $\mathfrak{a}$ on V is given by

$$a\left(\sum_{j=1}^{r} v_j\right) = \sum \boldsymbol{\lambda}_j(a) v_j \text{ for } a \in \mathfrak{a} \text{ and } v_j \in V_j \text{ for all } j\ .$$

Choose an element $a \in \mathfrak{a}$ such that, say, $\boldsymbol{\lambda}_1(a) = \mathbf{i}$ and the $\boldsymbol{\lambda}_j(a)$ are distinct. For $g \in \mathfrak{g}$ one writes $b := [g, a] = ga - ag \in \mathfrak{a}$. Consider for a given $u \in V_j$ the expression $g(u) = \sum_k v_k$ with all $v_k \in V_k$. Now $\boldsymbol{\lambda}_j(b)u = b(u) = (ga - ag)(u) = \boldsymbol{\lambda}_j(a) \sum_k v_k - \sum_k \boldsymbol{\lambda}_k(a) v_k$. This implies $v_k = 0$ for $k \neq j$ and $\boldsymbol{\lambda}_j(b)u = 0$. Thus the spaces V_j are invariant under $\mathfrak{g}$. Condition (b) implies $r = 1$. Then $a = \mathbf{i} \cdot 1_V$, which contradicts condition (a). One concludes that $\mathfrak{g}$ is semisimple.

According to [F-H], $\mathfrak{G} := \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{g}$ is semisimple, too. An element of $\mathfrak{G}$ can uniquely be written as $1 \otimes a + \mathbf{i} \otimes b$ with $a, b \in \mathfrak{g}$. If the image of this element is 0 in $\mathrm{End}_{\mathbb{C}}(V)$, then $a = -\mathbf{i}b$. This implies $a = b = 0$ since a and b have their eigenvalues in $\mathbf{i} \cdot \mathbb{R}$ and are semisimple.

(2). The first statement of (2) is immediate. We recall (see [F-H]) that the Cartan decomposition (or root decomposition) $\mathfrak{G} = \mathfrak{H} \oplus (\oplus_{\boldsymbol{\alpha}} \mathfrak{G}_{\boldsymbol{\alpha}})$ has the following properties: For any non zero linear map $\boldsymbol{\alpha} : \mathfrak{H} \to \mathbb{C}$ one has

$$\mathfrak{G}_{\boldsymbol{\alpha}} := \{g \in \mathfrak{G} \mid [h, g] = \boldsymbol{\alpha}(h) g \text{ for all } h \in \mathfrak{H}\}\ .$$

If $\mathfrak{G}_{\boldsymbol{\alpha}} \neq 0$ then $\boldsymbol{\alpha}$ is called a root and in that case $\dim_{\mathbb{C}} \mathfrak{G}_{\boldsymbol{\alpha}} = 1$. If $\boldsymbol{\alpha}$ is a root, then $c\boldsymbol{\alpha}$ with $c \in \mathbb{C}^*$ is a root if and only if $c = \pm 1$.

Fix an element $h \in \mathrm{End}(V)$. The eigenvalues of the linear map $\mathrm{End}(V) \to \mathrm{End}(V)$, defined by $g \mapsto ad(h)(g) := [h, g]$, are the differences of the eigenvalues of h. In particular for $h \in \mathfrak{h}$ and $\boldsymbol{\alpha} \in R$ one has $\boldsymbol{\alpha}(h) \in \mathbf{i} \cdot \mathbb{R}$. This proves (2a).

One writes $\boldsymbol{\alpha}_1, -\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_r, -\boldsymbol{\alpha}_r$ for the roots. Any $g \in \mathfrak{g}$ has a unique decomposition $g = g_0 + \sum_{j=1}^{r} (g_{\boldsymbol{\alpha}_j} + g_{-\boldsymbol{\alpha}_j})$ with $g_0 \in \mathfrak{H},\ g_{\pm \boldsymbol{\alpha}_j} \in \mathfrak{G}_{\pm \boldsymbol{\alpha}_j}$.

Choose a generic element $h_0 \in \mathfrak{h}$, i.e., the $2r$ elements $\pm \boldsymbol{\alpha}_j(h_0) \in \mathbf{i} \cdot \mathbb{R}^*$ are distinct. For $m \geq 1$ one has

$$ad(h_0)^m(g) = \sum_j \boldsymbol{\alpha}_j(h_0)^m g_{\boldsymbol{\alpha}_j} + (-\boldsymbol{\alpha}_j(h_0))^m g_{-\boldsymbol{\alpha}_j} \in \mathfrak{g}.$$

Using this relation for $m = 2n$, $n = 1, \ldots, r$ and observing that the $\boldsymbol{a}_j(h_0)^2 \in \mathbb{R}^*$, $j = 1, \ldots, r$ are distinct, one finds that all $g_{\boldsymbol{a}_j} + g_{-\boldsymbol{a}_j}$ are in $\mathfrak{g}$. Then also $g_0 \in \mathfrak{g}$. Similarly, one finds that each $\mathbf{i}g_{\boldsymbol{a}_j} - \mathbf{i}g_{-\boldsymbol{a}_j} \in \mathfrak{g}$.

Now we study the real vector space $T_j := \mathfrak{g} \cap (\mathfrak{G}_{\boldsymbol{a}_j} + \mathfrak{G}_{-\boldsymbol{a}_j})$. As shown above, any element of $\mathfrak{G}_{\pm\boldsymbol{a}_j}$ is nilpotent. Since the elements of $\mathfrak{g}$ are semisimple one has $\mathfrak{g} \cap \mathfrak{G}_{\pm\boldsymbol{a}_j} = 0$. In particular the two projections $T_j \to \mathfrak{G}_{\pm\boldsymbol{a}_j}$ are injective. We conclude from this that T_j has a real basis of the form $X_{\boldsymbol{a}_j} + X_{-\boldsymbol{a}_j}, \mathbf{i}X_{\boldsymbol{a}_j} - \mathbf{i}X_{-\boldsymbol{a}_j}$, where $X_{\pm\boldsymbol{a}_j}$ are non zero elements of $\mathfrak{G}_{\pm\boldsymbol{a}_j}$.

The complex Lie algebra generated by $X_{\pm\boldsymbol{a}_j}$ is easily seen to be the complex Lie algebra $\mathfrak{sl}_{2,\mathbb{C}}$. One easily verifies that the real Lie algebra generated by $X_{\boldsymbol{a}_j} + X_{-\boldsymbol{a}_j}, \mathbf{i}X_{\boldsymbol{a}_j} - \mathbf{i}X_{-\boldsymbol{a}_j}$ is isomorphic to $\mathfrak{su}_2$. This proves (2b).

(3). One applies [F-H], Proposition 26.4. The condition (i) of that proposition is (2a) and (2b). The equivalent condition (iii) states that the real Lie algebra associated to $\mathfrak{g}$ is compact. This implies the existence of a positive definite Hermitian form F on V such that $F(gx, y) + F(x, gy) = 0$ holds for all $x, y \in V$ and $g \in \mathfrak{g}$. □

3. Proof of the theorem.

The case G connected.

Put $\mathfrak{g} := \{A \in \mathrm{Matr}_n(\mathbb{C}) |\, \exp(tA) \in G \text{ for all } t \in \mathbb{R}\}$. According to ([M-T], Proposition 3.4.2 and 3.4.2.1.), $\mathfrak{g}$ is a real Lie subalgebra of $\mathrm{Matr}_n(\mathbb{C})$ and moreover G is generated by $\{\exp(g) |\, g \in \mathfrak{g}\}$. The elements $g \in \mathfrak{g}$ are clearly semisimple and all their eigenvalues are in $\mathbf{i} \cdot \mathbb{R}$.

Let $V := \mathbb{C}^n = V_1 \oplus \cdots \oplus V_r$ denote a maximal decomposition into (non trivial) complex subspaces invariant under $\mathfrak{g}$. This decomposition is also invariant under the action of G. It suffices to prove the theorem for the restriction of G to each V_j. In other words we may suppose that $r = 1$. Thus $\mathfrak{g}$ satisfies the conditions (b) and (c) of Proposition 2.1.

If $\mathbf{i} \cdot 1_V \in \mathfrak{g}$, then one replaces $\mathfrak{g}$ by $\mathfrak{g}^* := \{g \in \mathfrak{g} \mid Tr(g) = 0\}$. The latter is again a real Lie algebra, satisfies (a)–(c) and moreover $\mathfrak{g} = \mathfrak{g}^* \oplus \mathbb{R}\mathbf{i} \cdot 1_V$. The positive definite Hermitian form of part (3) of Propostion 2.1 has clearly the property $F(gx, gy) = F(x, y)$ for all $g \in G$ and $x, y \in V$.

The general case.

Now G is a closed subgroup of $\mathrm{GL}_n(\mathbb{C})$ (for the ordinary topology) such that every element of G is semisimple and such that all its eigenvalues have absolute value 1. Let G^o denote the component of the identity of G. According to the previous case, the group G^o is compact.

LEMMA 3.1. *G/G^o is a torsion group, i.e., all its elements have finite order.*

PROOF. Let g be an element of G. Choose a basis $e_1, \ldots, e_n$ of eigenvectors of g. The group T, consisting of all elements $t \in \mathrm{GL}_n(\mathbb{C})$ such that $te_j = c_j e_j,\ |c_j| = 1$ for all j, is compact. The topological closure $H \subset \mathrm{GL}_n(\mathbb{C})$ of the group generated by g is a closed subgroup of T and therefore compact. The component of the identity H^o of H has finite index in H, since H is compact. Moreover, $H^o \subset G^o$. It follows that the image of g in G/G^o has finite order. □

The group G^o is conjugated to a subgroup of $U_n(\mathbb{C})$ and hence compact. One considers the real vector space *Herm* consisting of the Hermitian forms F on V. The group G acts linearly on *Herm* by $(gF)(x, y) := F(gx, gy)$. The real linear subspace $Herm_{G^o}$ consisting of the G^o-invariant Hermitian forms is not 0 and contains in fact a positive definite Hermitian form. The space $Herm_{G^o}$ is invariant under G, since G^o is a normal subgroup of G. The action of G on $Herm_{G^o}$ induces a homomorphism $G \to \mathrm{GL}(Herm_{G^o})$ with kernel G^+ and image I. Since $G^+ \supset G^o$ the group I is a torsion group. G^+ leaves a positive definite Hermitian form invariant and is closed. Therefore G^+ is compact.

We will need the following classical result and refer to ([Fr], p. 209, or [C-R] p. 252, or [S]) for a proof.

LEMMA 3.2 (Schur's theorem). *Let H be a torsion subgroup of* $\mathrm{GL}_n(F)$, *for some field F. Then:*

Any finitely generated subgroup J of H is finite. As a consequence, H is the filtered union of its finite subgroups.

We apply the lemma to I. Let $J \subset I$ be a finite subgroup. Its preimage $J^* \subset G$ is compact and the subspace $Herm_{J^*}$ of the the J^*-invariant elements of *Herm* is not 0 and contains a positive definite Hermitian form. For finite subgroups $J_1 \subset J_2$ of I one has $Herm_{J_1^*} \supset Herm_{J_2^*}$. Since the spaces $Herm_{J^*}$ have finite dimension and I is the filtered union of its finite subgroups, there exists a finite subgroup J_0 of I such that $Herm_{J_0^*} = Herm_{K^*}$ for every finite subgroup $K \subset I$, containing J_0. This implies the existence of a positive definite Hermitian form invariant under G.

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