Rend. Sem. Mat. Univ. Padova, Vol. 116 (2006)

A Septic with 99 Real Nodes.

Oliver Labs (*)

Abstract - We find a surface of degree 7 in $\mathbb{P}^3(\mathbb{R})$ with 99 real nodes within a family of surfaces with dihedral symmetry: First, we consider this family over some small prime fields, which allows us to test all possible parameter sets using computer algebra. In this way we find some examples of 99-nodal surfaces over some of these finite fields. Then, the examination of the geometry of these surfaces allows us to determine the parameters of a 99-nodal septic in characteristic zero. This narrows the possibilities for $\mu(7)$, the maximum number of nodes on a septic, to: $99 \leq \mu(7) \leq 104$. When reducing our surface modulo 5, we even obtain a 100-nodal septic in $\mathbb{P}^3(\mathbb{F}_5)$.

Introduction.

The study of surfaces in $\mathbb{P}^3(\mathbb{C})$ of some degree d w.r.t. the possible combinations of singularities on them is one of the most classical subjects in algebraic geometry. Schläfli [17] already established a complete classification for the case of cubic surfaces (i.e., $d = 3$) in 1863. It took more than 130 years until the corresponding problem was solved for quartics ($d = 4$) using K3 lattice theory. Most of this work was done by Nikulin, Urabe, and finally Yang [19].

For surfaces of higher degree, we cannot hope to establish an analogous classification at the moment, even for $d = 5$, because the knowledge on surfaces of general type is still far from sufficient for such a purpose. Kummer [10] had a similar problem for the case of $d = 4$ in 1864. So, he restricted himself to the question on the maximum number $\mu(d)$ of nodes (i.e. singularities of type A_1, also called ordinary double points, locally

(*) Indirizzo dell'A.: Mathematik und Informatik, Gebäude E2.4, Universität des Saarlandes, D-66123 Saarbrücken, Germany.
E-mail: Labs@math.uni-sb.de, mail@OliverLabs.net
Primary 14J17, 14Q10; Secondary 14G15

given by $x^2 + y^2 + z^2 = 0$) which can occur on a surface of degree d in $\mathbb{P}^3(\mathbb{C})$. Kummer proved $\mu(4) = 16$ by noticing that Fresnel's Wave Surface actually had 16 nodes and that one could use a formula for the degree of the dual of a surface to show that there could not be more such singularities.

Dispite many attempts to determine $\mu(d)$ in the 20th century, this number is only known for $d \leq 6$ until now (for a historical overview of about 50 pages, see [13]). All restrictions established so far are summarized in the table below:

degree	1	2	3	4	5	6	7	8	9	10	11	12	d
$\mu(d) \geq$	0	1	4	16	31	65	93	168	216	345	425	600	$\approx \frac{5}{12} d^3$
$\mu(d) \leq$	0	1	4	16	31	65	104	174	246	360	480	645	$\approx \frac{4}{9} d^3$

In the present article we show:

$$\mu(7) \geq 99. \tag{1}$$

The best known upper bound in the case of septics is given by Varchenko's spectral bound [18]: $\mu(7) \leq 104$. Notice that Miyaoka's bound [14] yields 112, but Givental's bound [6] also computes to 104. The previously known septic with the greatest number of nodes was the example of Chmutov [2] with 93 nodes: Chmutov's construction works for any degree d. For $d \leq 5$ and the even degrees $d = 6, 8, 10, 12$ there are examples exceeding Chmutov's lower bound: [1], [4], [16]. These had been obtained by using some beautiful geometric arguments based on the idea to globalize the local equation of a node. This method goes back at least to Rohn who constructed quartics with $8, 9, \dots, 16$ nodes in the 19th century [15].

In the present note, we consider a family of surfaces depending on some parameters which is based on Rohn's construction and whose generic member has 63 nodes. Given an explicit equation of a family of nodal hypersurfaces, there is in fact an algorithm in characteristic zero to find those examples with the greatest number of nodes: We applied this succesfully in [12], but we cannot use this technique in the present case because of computer performance restrictions.

Here, we choose a more geometric approach to study the family: The main obstacle towards the construction of the recent examples which achieve the currently greatest number of nodes ($d = 6, 8, 10, 12$) was to have a good intuitive idea about the geometry of a potentially existing surface. Our starting point is to replace this intuition by a computer search over all possible parameters over some finite fields of prime order. Once we have found some examples of septics with many nodes over these fields explicitly, we study their geometry and use this to construct the corresponding septic in characteristic zero. The 99-nodal septic which we find in this way is the first surface of odd degree greater than 5 that exceeds Chmutov's general lower bound. Moreover, the idea to use the geometry of prime field experiments can certainly be applied to many other problems in constructive algebraic geometry.

I thank D. van Straten for his permanent motivation and many valuable discussions. Furthermore, I thank W. Barth for his invitation to Erlangen which was a good motivation to complete this work. I thank S. Cynk for helpful discussions. Finally, I thank St. Endra for discussions, motivation and his Ph.D. thesis which is a great source for dihedral-symmetric surfaces with many singularities.

1. The Family

Inspired by many authors (see in particular: [15], [1], [3], [4]), we look for septics with many nodes in $\mathbb{P}^3(\mathbb{C})$ within a 7-parameter family of surfaces $S_{a_1,a_2,\dots,a_7} := P - U_{a_1,a_2,\dots,a_7}$ of degree 7 admitting the dihedral symmetry D_7 of a 7-gon:

$$\begin{aligned} P :=& 2^6 \cdot \Pi_{j=0}^{6}\left[\cos\left(\frac{2\pi j}{7}\right)x + \sin\left(\frac{2\pi j}{7}\right)y - z\right] \\ =& x \cdot \left[x^6 - 3\cdot 7\cdot x^4y^2 + 5\cdot 7\cdot x^2y^4 - 7\cdot y^6\right] \\ &+ 7\cdot z\cdot\left[\left(x^2+y^2\right)^3 - 2^3\cdot z^2\cdot\left(x^2+y^2\right)^2 + 2^4\cdot z^4\cdot\left(x^2+y^2\right)\right] - 2^6\cdot z^7, \end{aligned}$$

$$U_{a_1,a_2,\dots,a_7} := (z + a_5w)\left(a_1z^3 + a_2z^2w + a_3zw^2 + a_4w^3 + (a_6z + a_7w)(x^2+y^2)\right)^2.$$

P is the product of 7 planes in $\mathbb{P}^3(\mathbb{C})$ meeting in the point $(0:0:0:1)$ and admitting D_7-symmetry with rotation axes $\{x = y = 0\}$: In fact, P is invariant under the map $y \mapsto -y$ and $P \cap \{z = z_0\}$ is a regular 7-gon for $z_0 \neq 0$. U is also D_7-symmetric, because x and y only appear as $x^2 + y^2$.

A generic surface S has nodes at the $3 \cdot 21 = 63$ intersections of the $\binom{7}{2} = 21$ doubled lines of P with the doubled cubic of U. We are looking for parameters $a_1, a_2, \dots, a_7$, s.t. the corresponding surface has 99 nodes.

As $S_{a_1,a_2,\dots,a_7}(x,y,z,\lambda w) = S_{a_1,\lambda a_2,\lambda^2 a_3,\lambda^3 a_4,\lambda a_5,a_6,\lambda a_7}(x,y,z,w)\ \forall \lambda \in \mathbb{C}^*$, we choose $a_7 := 1$. Moreover, experiments over prime fields suggest that the maximum number of nodes on such surfaces is 99 and that such examples exist for $a_6 = 1$. As we are mainly interested in finding an example with 99 nodes, we restrict ourselves to the sub-family:

$$S := S_{a_1,a_2,a_3,a_4,a_5,1,1} = P - U_{a_1,a_2,a_3,a_4,a_5,1,1}.$$

Some other cases, e.g. $a_6 = 0$, also lead to 99-nodal septics (see [13]).

2. Reduction to the Case of Plane Curves

To simplify the problem of locating examples with 99 nodes within our family S, we restrict our attention to the $\{y = 0\}$-plane and search for plane curves $S|_{y=0}$ (we write S_y for short) with many nodes. This is motivated by the symmetry of the construction:

LEMMA 1 (see [3]). *A member $S = S_{a_1,a_2,a_3,a_4,a_5,1,1}$ of our family of surfaces has only ordinary double points as singularities, if $(1 : i : 0 : 0) \notin S$ and the surface does only contain ordinary double points as singularities in the plane $\{y = 0\}$. If the plane septic S_y has exactly n nodes and if exactly n_{xy} of these nodes are on the axes $\{x = y = 0\}$ then the surface S has exactly $n_{xy} + 7 \cdot (n - n_{xy})$ nodes and no other singularities. Each singularity of S_y which is not on $\{x = y = 0\}$ gives an orbit of 7 singularities of S under the action of the dihedral group D_7.*

PROOF. Because of the D_7-symmetry of the construction, we only have to show that there are no other singularities than the claimed ones. It is easy to prove (see [3, p. 18, cor. 2.3.10] for details) that any isolated singularity of S which is not contained in one of the orbits of the nodes of S_y would yield a non-isolated singularity which intersects the plane $\{y = 0\}$. But this contradicts the assumption that the surface S does only contain ordinary double points on $\{y = 0\}$. □

So, we first look for septic plane curves of the form S_y with many nodes, then we verify that these singularities are indeed also nodes of the surface. Via the lemma, we are then able to conclude that the surface has only ordinary double points. In order to understand the geometry of the plane septic S_y better we look at the singularities that occur for generic values of the parameters. First, we compute:

$$P|_{y=0} = x^7 + 7\cdot x^6 z - 7\cdot 2^3\cdot x^4 z^3 + 7\cdot 2^4\cdot x^2 z^5 - 2^6\cdot z^7$$
$$= \frac{(x-z)}{2^4}\cdot\big(\underbrace{x+(-\rho)z}_{=:L_1}\big)^2\cdot\big(\underbrace{2x+(\rho^2+4\rho)z}_{=:L_2}\big)^2\cdot\big(\underbrace{2x+(-\rho^2-2\rho+8)z}_{=:L_3}\big)^2,$$
$$U|_{y=0} = (z+a_5 w)\big(\underbrace{(z+w)x^2+a_1 z^3+a_2 z^2 w+a_3 z w^2+a_4 w^3}_{=:C}\big)^2,$$

where ρ satisfies:

$$\rho^3 + 2^2\rho^2 - 2^2\rho - 2^3 = 0. \tag{2}$$

The three points G_{ij} of intersection of C with the line L_i are ordinary double points of the plane septic $S_y = P|_{y=0} - U|_{y=0}$ for generic values of the parameters, s.t. we have $3\cdot 3 = 9$ *generic* singularities (see fig. 1).

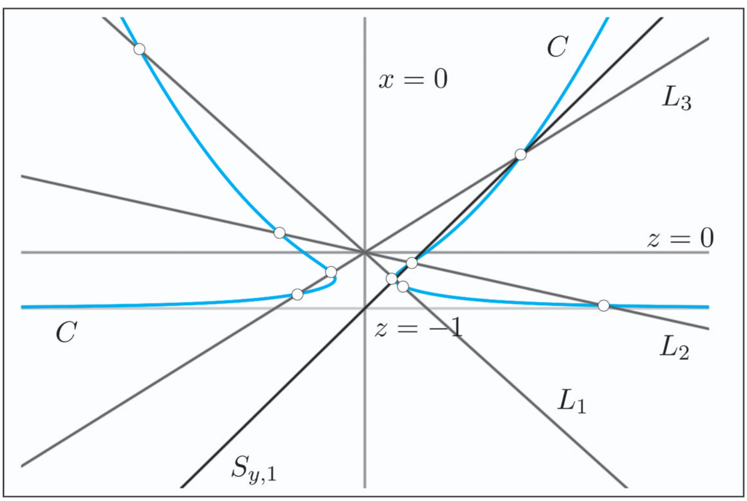

Fig. 1. – The three doubled lines L_i and the doubled cubic C intersect in $3\cdot 3 = 9$ points G_{ij}. These are the *generic* singularities of the plane septic S_y.

3. Finding Solutions over some Prime Fields

By running over all possible parameter combinations over some small prime fields $\mathbb{F}_p$ using the computer algebra system Singular [8], we find some 99-nodal surfaces over these fields: For a given set of parameters $a_1, a_2, \dots, a_5$, we can easily check the actual number of nodes on the corresponding surface using computer algebra (see [7, appendix A, p. 487]).

As indicated in the previous section, we work in the plane $\{y = 0\}$ for faster computations. It turns out that the greatest number of nodes on S_y is 15 over the small prime fields $\mathbb{F}_p$, $11 \le p \le 53$: See table 1 on the next page. The prime fields $\mathbb{F}_p$, $2 \le p \le 7$, are not listed because they are special cases: These primes appear as coefficients or exponents in the equation of our family. In each of the cases we checked, one of the 15 singular points lies on the axes $\{x = 0\}$, such that the corresponding surface has exactly $14 \cdot 7 + 1 = 99$ nodes and no other singularities.

4. The Geometry of the 15-nodal septic Plane Curve

To find parameters $a_1, a_2, \dots, a_5$ in characteristic 0 we want to use geometric properties of the 15-nodal septic plane curve S_y. But as we do not know any such property yet, we use our prime field examples to get some good ideas:

Observation 1. *In all our prime field examples of* 15*-nodal plane septics* S_y*, we have:*

1) S_y *splits into a line* $S_{y,1}$ *and a sextic* $S_{y,6}$*:* $S_y = S_{y,1} \cdot S_{y,6}$. *The plane curve* $S_{y,6}$ *of degree* 6 *has* $15 - 6 = 9$ *singularities. Note that this property is similar to the one of the* 31*-nodal* D_5*-symmetric quintic in* $\mathbb{P}^3(\mathbb{C})$ *constructed by W. Barth: See* [3, p. 27-32] *for a description.*

The line and the sextic have some interesting geometric properties (see fig. 3*):*

2) $S_{y,1} \cap S_{y,6} = \{R, G_{1j_1}, G_{2j_2}, G_{3j_3}, O_1, O_2\}$*, where* R *is a point on the axes* $\{x = 0\}$ *and the* G_{ij_k} *are three of the* 9 *generic singularities* G_{ij} *of* S_y*, one on each line* L_i*, and* O_1, O_2 *are some other points that neither lie on* $\{x = 0\}$*, nor on one of the* L_i*.*

3) *The sextic* $S_{y,6}$ *has the six generic singularities* G_{ij}*,* $(i,j) \in \{1,2,3\}^2 \setminus \{(1,j_1),(2,j_2),(3,j_3)\}$*, and three exceptional singularities:* E_1, E_2, E_3*.*

In many prime field experiments, we have furthermore:

4) *In the projective* x, z, w*-plane, the point* R *has the coordinates*

$(0:-1:1)$, *s.t. the line* $S_{y,1}$ *has the form* $S_{y,1}:\ z+t\cdot x+w=0$ *for some parameter* t *(see also table* 1*).*

The other cases $(R=(0:c:1), c\neq -1)$ *lead to more complicated equations and will not be discussed here.*

TABLE 1. A few examples of parameters giving 15-nodal septic plane curves (and 99-nodal surfaces) over prime fields.

Field	a_1	a_2	a_3	a_4	a_5	$S_{y,1}$	α
$\mathbb{F}_{11}$	2	3	5	2	-5	$z=x-w$	$\alpha=-3$
$\mathbb{F}_{19}$	-7	-2	7	1	8	$z=8x-w$	$\alpha=7$
$\mathbb{F}_{19}$	2	0	1	9	7	$z=9x-w$	$\alpha=-4$
$\mathbb{F}_{19}$	5	-9	7	-3	-1	$z=2x-w$	$\alpha=-3$
$\mathbb{F}_{23}$	-5	11	10	1	7	$z=-9x-w$	$\alpha=-2$
$\mathbb{F}_{31}$	-15	-13	-5	13	-10	$z=-2x-w$	$\alpha=-13$
$\mathbb{F}_{31}$	1	-2	14	-9	11	$z=15x-w$	$\alpha=-11$
$\mathbb{F}_{31}$	14	-10	-13	-14	-11	$z=-13x-w$	$\alpha=-7$
$\mathbb{F}_{43}$	-11	15	0	-13	-6	$z=-6x-w$	$\alpha=7$
$\mathbb{F}_{43}$	20	16	-1	-14	10	$z=-12x-w$	$\alpha=14$
$\mathbb{F}_{43}$	-9	3	-3	-11	5	$z=18x-w$	$\alpha=-21$
$\mathbb{F}_{53}$	-8	20	14	18	11	$z=25x-w$	$\alpha=4$
$\mathbb{F}_{53}$	-2	-10	-14	-26	16	$z=-9x-w$	$\alpha=24$
$\mathbb{F}_{53}$	10	25	-4	22	25	$z=-16x-w$	$\alpha=25$

Using this observation as a guess for our septic in characteristic 0, we obtain several polynomial conditions on the parameters. Using SINGULAR to eliminate variables, we find the following relation between the parameters a_4 and t:

$$t\cdot\big(\underbrace{a_4t^3+t}_{=:\alpha}\big)^2+t-1=0, \tag{3}$$

which can be parametrized by α: $t=-\frac{1}{1+\alpha^2}$, $a_4=(\alpha(1+\alpha^2)-1)(1+\alpha^2)^2$. Further eliminations allow us to express all the other parameters in terms of α:

- $a_1=\alpha^7+7\alpha^5-\alpha^4+7\alpha^3-2\alpha^2-7\alpha-1$,
- $a_2=(\alpha^2+1)(3\alpha^5+14\alpha^3-3\alpha^2+7\alpha-3)$,
- $a_3=(\alpha^2+1)^2(3\alpha^3+7\alpha-3)$,
- $a_5=\frac{1+\alpha^2}{\alpha^2}$.

5. The 1-parameter Family of Plane Sextics

Once more we use our explicit examples of 15-nodal septic plane curves over prime fields to finally be able to write down a condition for α in characteristic 0.

First, note that we can now easily obtain the equation of $S_{y,6}$ by dividing the equation of our septic curve S_y by the equation of the line $S_{y,1} = z + tx + w = z - \frac{1}{1+\alpha^2}x + w$. $S_{y,6}$ is a sextic which has 6 nodes for generic α, but should have 9 double points for some special values of α. One idea to determine these particular values is to find a geometric relation between the 6 generic singular points and the 3 exceptional ones.

5.1 – *Three Conics*

Looking at the equations describing the singular points of our examples of 9-nodal sextics $S_{y,6}$ over the prime fields, we see the following:

Observation 2. *For all our 9-nodal examples of plane sextics over prime fields, there are three conics through six of these points each (see fig. 2):*

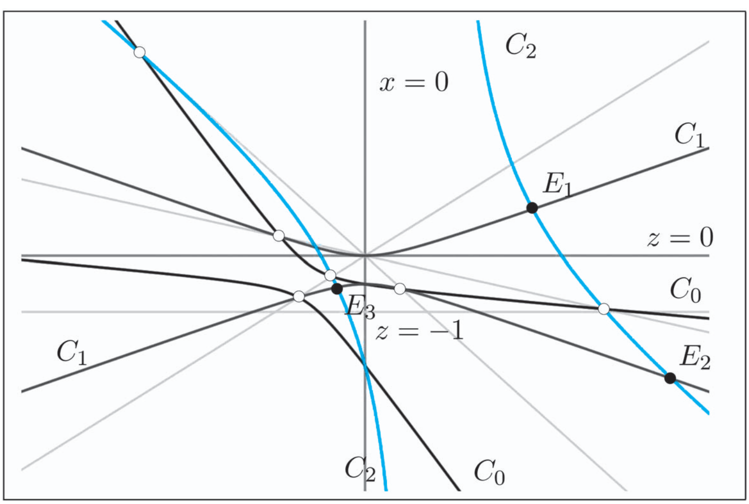

Fig. 2. – Three conics relating the 9 double points of the sextic $S_{y,6}$. E_1, E_2, and E_3 (black) are the exceptional singularities (i.e. they do not lie on one of the lines L_i, see fig. 1). The white points are the generic singularities, coming from the intersection of the doubled cubic C with the three doubled lines L_i.

1) *one conic* C_0 *through the* 6 *generic singularities,*

2) *one conic* C_1 *through the* 3 *exceptional singularities and* 3 *of the generic ones,*

3) *one conic* C_2 *through the* 3 *exceptional singularities and the other* 3 *generic ones.*

Moreover, over the prime fields the three conics have the following properties:

4) C_1 *has the form:*

$$C_1 : \ x^2 + kz^2 + (k+4)zw = 0, \tag{4}$$

where k is a still unknown parameter. In particular, C_1 is symmetric with respect to $x \mapsto -x$ and contains the point $(0:0:1)$.

5) C_0 *intersects the other two conics on the* $\{x = 0\}$ *-axes (see fig.* 2 *on the preceding page):*

$$X_1 := C_0 \cap C_1 \cap \{x = 0\}, \quad X_2 := C_0 \cap C_2 \cap \{x = 0\}. \tag{5}$$

To determine the new parameter k in equation (4), we will use (5). We compute the two points of C_0 on the $\{x = 0\}$-axes explicitly using SINGULAR: First, the ideal $I^{gen}_{S_{y,6}}$ describing the six generic singularities of $S_{y,6}$ can be computed from the ideal $I^{gen}_{S_y} := (C, L_1L_2L_3)$ describing the 9 generic singularities of S_y by calculating the following ideal quotient: $I^{gen}_{S_{y,6}} = I^{gen}_{S_y} : S_{y,1}$. Now, the equation of C_0 can be obtained by taking the degree-2-part of the ideal $I^{gen}_{S_{y,6}}$:

$$\begin{aligned} C_0 : \quad & \alpha x^2 + (\alpha^3 + 5\alpha - 1)xz + (\alpha^3 + \alpha - 1)xw \\ & (\alpha^5 + 6\alpha^3 - \alpha^2 + \alpha - 1)z^2 + (2\alpha^5 + 8\alpha^3 - 2\alpha^2 + 6\alpha - 2)zw \\ & + (\alpha^5 + 2\alpha^3 - \alpha^2 + \alpha - 1)w^2 = 0. \end{aligned} \tag{6}$$

Thus, $\{P^+, P^-\} := C_0 \cap \{x = 0\} = \left\{ \left(0 : \frac{-2(\alpha^3+3\alpha-1)(1+\alpha^2)\beta(\alpha)}{2(\alpha^5+6\alpha^3-\alpha^2+\alpha-1)} : 1 \right) \right\}$, where

$$\beta(\alpha)^2 \quad := \quad 16\alpha(2\alpha^5 + 4\alpha^3 - \alpha^2 + 2\alpha - 1). \tag{7}$$

C_1 intersects the $\{x = 0\}$-axes in exactly two points: $(0:0:1)$ and X_1. Hence, we can determine the two possibilities for the parameter $k \in \mathbb{Q}(\alpha, \beta(\alpha))$ in equation (4) for C_1: Together with the z and w-coordinates of the points $P^\pm$, $C_1 \cap \{x = 0\} = \{kz(z+w) + 4zw = 0\}$ leads to the following two possibilities:

$$C_1 : \quad x^2 + \frac{-4P_z^\pm}{P_z^\pm(P_z^\pm + 1)} z(z+w) + 4zw = 0. \tag{8}$$

5.2 – *The Condition on α*

The equations of the conics C_0 and C_1 will allow us to compute the condition on α, s.t. the sextic $S_{y,6}$ has 9 singularities, using the following (see observation 2 and fig. 2):

- C_0 intersects the three doubled lines L_i exactly in the six generic singularities.
- C_1 intersects the three doubled lines L_i exactly in three of these six generic singularities and the origin (which counts three times).

Thus, the set of z-coordinates of the three points $(C_1 \cap L_1L_2L_3) \setminus \{(0:0:1)\}$ has to be contained in the set of z-coordinates of the six points $C_0 \cap L_1L_2L_3$. This means that the remainder q of the following division (res_x denotes the resultant with respect to x)

$$(9) \qquad res_x(C_0,\, L_1L_2L_3) = p(z) \cdot \left(\frac{1}{z^3} \cdot res_x(C_1,\, L_1L_2L_3)\right) + q(z)$$

should vanish: $q = 0$.

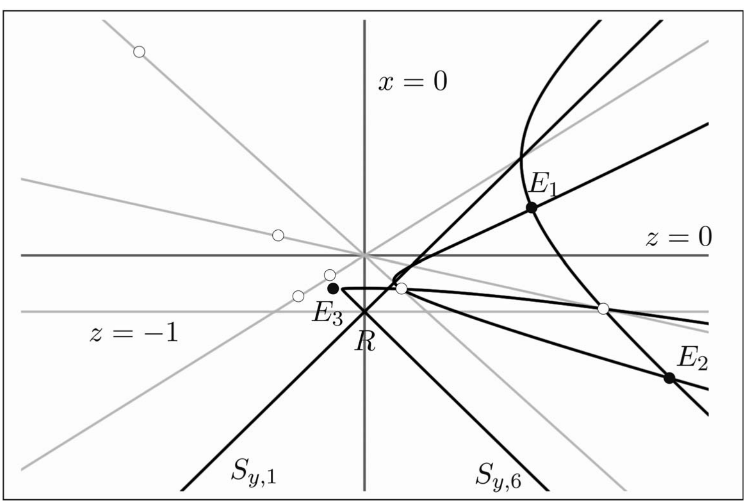

Fig. 3. – The 15-nodal plane septic $S_{y_{\alpha_R}} = S_{y,1_{\alpha_R}} \cdot S_{y,6_{\alpha_R}}$ (see (11)); the singularities of the sextic $S_{y,6_{\alpha_R}}$ are marked by large circles: The three exceptional singularities E_1, E_2, E_3 are marked in black, the generic singularities in white. The five left-most nodes are real isolated ones. Only five of the six intersections of the line $S_{y,1_{\alpha_R}}$ and the sextic $S_{y,6_{\alpha_R}}$ are visible because we just show a small part of the whole (x, z)-plane.

As the degree of the remainder is $\deg(q) = 2$, this gives 3 conditions on α and $\beta(\alpha)$, coming from the fact that all the 3 coefficients of $q(z)$ have to vanish. It turns out that it suffices to take one of these, the coefficient of z^2, which can be written in the form $c(\alpha) + \beta(\alpha)d(\alpha)$, where $c(\alpha)$ and $d(\alpha)$ are polynomials in $\mathbb{Q}[\alpha]$. As a condition on α only we can take:

$$\mathit{cond}(\alpha) := \big(c(\alpha) + \beta(\alpha)d(\alpha)\big) \cdot \big(c(\alpha) - \beta(\alpha)d(\alpha)\big) \in \mathbb{Q}[\alpha],$$

which is of degree 150.

This condition $\mathit{cond}(\alpha)$ vanishes for those α for which the corresponding surface has 99 nodes and for several other α. To obtain a condition which exactly describes those α we are looking for, we factorize $\mathit{cond}(\alpha) = f_1 \cdot f_2 \cdots f_k$ (e.g., using SINGULAR again). Substituting in each of these factors our solutions over the prime fields, we see that the only factor that vanishes is: $7\alpha^3 + 7\alpha + 1 = 0$.

6. The Equation of the 99-nodal Septic

Up to this point, it is still only a guess — verified over some prime fields — that the values α satisfying the condition above give 99-nodal septics in characteristic 0. But we have indeed:

THEOREM 1 [99-nodal Septic]. *Let $\alpha \in \mathbb{C}$ satisfy:*

$$7\alpha^3 + 7\alpha + 1 = 0. \tag{10}$$

Then the surface S_α in $\mathbb{P}^3(\mathbb{C})$ of degree 7 with equation $S_\alpha := P - U_\alpha$ has exactly 99 ordinary double points and no other singularities, where

$$\begin{aligned} P :=\, & x \cdot \big[x^6 - 3\cdot 7\cdot x^4y^2 + 5\cdot 7\cdot x^2y^4 - 7\cdot y^6\big] \\ & + 7\cdot z\cdot \Big[\big(x^2+y^2\big)^3 - 2^3\cdot z^2\cdot\big(x^2+y^2\big)^2 + 2^4\cdot z^4\cdot\big(x^2+y^2\big)\Big] - 2^6\cdot z^7, \\ U_\alpha :=\, & (z + a_5w)\big((z+w)(x^2+y^2) + a_1z^3 + a_2z^2w + a_3zw^2 + a_4w^3\big)^2, \end{aligned}$$

$$a_1 := -\frac{12}{7}\alpha^2 - \frac{384}{49}\alpha - \frac{8}{7}, \quad a_2 := -\frac{32}{7}\alpha^2 + \frac{24}{49}\alpha - 4,$$

$$a_3 := -4\alpha^2 + \frac{24}{49}\alpha - 4 \qquad a_4 := -\frac{8}{7}\alpha^2 + \frac{8}{49}\alpha - \frac{8}{7},$$

$$a_5 := 49\alpha^2 - 7\alpha + 50.$$

There is exactly one real solution $\alpha_{\mathbb{R}} \in \mathbb{R}$ *to the condition* (10),

$$\alpha_{\mathbb{R}} \approx -0.14010685, \tag{11}$$

and all the singularities of $S_{\alpha_{\mathbb{R}}}$ *are also real.*

PROOF. By computer algebra. The total tjurina number (i.e., 99) of S_α can be computed as follows:

```
ring r = (0, alpha), (x, y, w, z), dp; minpoly = 7*alpha^3 + 7*alpha + 1;
poly S_alpha = ...;
ideal sl = jacob(S_alpha); option(redSB); sl = std(sl);
degree(sl);        // gives: proj. dim: 0, mult: 99
```

Using the hessian criterion, we can check in a similar way that the singularities are all nodes:

```
matrix mHess = jacob(jacob(S)); ideal nonnodes = minor(mHess,2), sl;
nonnodes = std(nonnodes); degree(nonnodes);        // gives: proj. dim: -1
```

See [11] for the complete SINGULAR code and for more information which may help you to verify the result by hand. Using the geometric description of the singularities of the plane septic given in the previous sections, it is straightforward to verify the reality assertion (see fig. 4 for a visualization). □

7. Concluding Remarks

The existence of the real $\alpha_{\mathbb{R}}$ allows us to use the program SURFEX [9] (which uses SURF [5]) to compute an image of the 99-nodal septic $S_{\alpha_{\mathbb{R}}}$ (fig. 4 on the following page). When denoting the maximum number of real singularities a septic in $\mathbb{P}^3(\mathbb{R})$ can have by $\mu^{\mathbb{R}}(7)$, we get, with the remarks mentioned in the introduction:

COROLLARY 2.

$$99 \leq \mu^{\mathbb{R}}(7) \leq \mu(7) \leq 104.$$

Note that the previously known lower bounds were reached by S. V. Chmutov (93 complex nodes: [2]) and D. van Straten (84 real nodes: a variant of Chmutov's construction using regular polygons instead of folding polynomials).

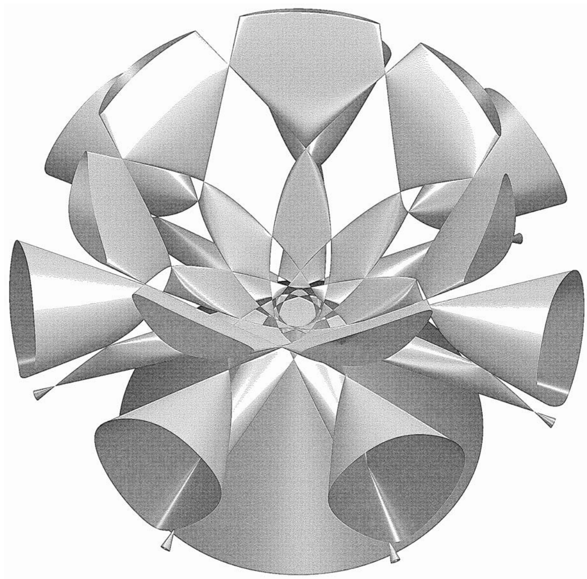

Fig. 4 – A part of the affine chart $w = 1$ of the real septic with 99 nodes, see [11] for more images and movies.

As it can be computed using deformation theory and SINGULAR that the space of obstructions for globalizing all local deformations is zero — this is based on ideas of D. van Straten, details will be published elsewhere — we obtain:

COROLLARY 3. *There exist surfaces of degree* 7 *in* $\mathbb{P}^3(\mathbb{R})$ *with exactly* k *real nodes and no other singularities for* $k = 0, 1, 2, \ldots, 99$.

Recently there has been some interest in surfaces that do exist over some finite fields, but which are not liftable to characteristic 0. The reduction of our 99-nodal septic S_α modulo 5 (note: $1 \in \mathbb{F}_5$ satisfies (10): $7 \cdot 1^3 + 7 \cdot 1 + 1 \equiv 0$ modulo 5) neither gives a 99-nodal surface nor a highly degenerated one as one might expect because the exponent 5 appears several times in the defining equation. Instead, we can easily verify the following using computer algebra:

COROLLARY 4. *For* $\alpha_5 := 1 \in \mathbb{F}_5$ *the surface* $S_{\alpha_5} \subset \mathbb{P}^3(\mathbb{F}_5)$ *defined as in the above theorem has* 100 *nodes and no other singularities.*

Of course, not all the coordinates of its singularities are in $\mathbb{F}_5$, but in some algebraic extension. The septic has similar geometric properties as our 99-nodal surface; in addition it has one node at the intersection of the $\{x = y = 0\}$ axes and $\{w = 0\}$. Until now, we were not able to determine if this 100-nodal septic defined over $\mathbb{F}_5$ can be lifted to characteristic zero.

We hope to be able to apply our technique for finding surfaces with many nodes within families of surfaces to similar problems. E.g., it should be possible to study surfaces with dihedral symmetry of degree 9 and 11 with many ordinary double points using the same ideas. Another application could be the search for surfaces with many cusps. We already studied families of such surfaces succesfully using computer algebra in simpler cases [12].

REFERENCES

[1] W. Barth, *Two Projective Surfaces with Many Nodes, Admitting the Symmetry of the Icosahedron*, J. Algebraic Geom., **5**, 1 (1996), pp. 173–186.
[2] S. W. Chmutov, *Examples of Projective Surface with Many Singularities*, J. Algebraic Geom., **1**, 2 (1992), pp. 191–196.
[3] S. Endraß, *Simmetrische Flächen mit vielen gewöhnlichen Doppelpunkten*, Ph.D. thesis, Fr.-A.-Universität Erlangen-Nürnberg, 1996.
[4] S. Endraß, *A Projective Surface of Degree Eight with 168 Nodes*, J. Algebraic Geom., **6**, 2 (1997), pp. 325–334.
[5] S. Endraß, Surf *1.0.3*, http://surf.sourceforge.net, 2001, A Computer Software for Visualising Real Algebraic Geometry.
[6] A. B. Giventai, *Maximum number of singular points on a projective hypersurface*, Funct. Anal. Appl., **17** (1984), pp. 223–225.
[7] G.-M. Greuel - G. Pfister, *A Singular Introduction to Commutative Algebra*, Springer-Verlag Berlin Heidelberg, 2002.
[8] G.-M. Greuel - G. Pfister - H. Schönemann, Singular *2.0*, A Computer Algebra System for Polynomial Computations, Centre for Computer Algebra, Univ. Kaiserslautern, 2001, http://www.singular.uni.kl.de
[9] S. Holzer - O. Labs - R. Morris, Surfex - *Visualization of Real Algebraic Surfaces*, www.surfex.AlgebraicSurface.net, 2005.
[10] E. Kummer, *Über die Flächen vierten Grades mit sechzehn singulären Punkten*, Collected Papers, vol. **2**, Springer-Verlag (1975), pp. 418–432.
[11] O. Labs, *Algebraic Surface Homepage. Information, Images and Tools on Algebraic Surfaces*, www.AlgebraicSurface.net, 2003.
[12] O. Labs, *A Sextic with* 35 *Cusps*, Preprint, math.AG/0502520, 2005.
[13] O. Labs, *Hypersurfaces with Many Singularities*, Ph.D. thesis, Johannes Gutenberg Universität Mainz, 2005, available from www.OliverLabs.net
[14] Y. Miyaoka, *The Maximal Number of Quotient Singularities on Surfaces with Given Numerical Invariants*, Math. Ann., **268** (1984), pp. 159–171.

[15] K. Rohn, *Die Flächen vierter Ordnung hinsichtlich ihrer Knotenpunkte und ihrer Gestaltung*, Preisschriften der Fürstlich Jablonowski'schen Gesellschaft, no. IX, Leipzig, 1886.
[16] A. Sarti, *Pencils of symmetric surfaces in* $\mathbb{P}^3$, J. Algebra, **246**, 1 (2001), pp. 429–452.
[17] L. Schläfli, *On the Distribution of Surfaces of the Third Order into Species, in Reference to the Presence or Absence of Singular Points and the Reality of their Lines*, Philos. Trans. Royal Soc., **CLIII** (1863), 193–241.
[18] A. N. Varchenko, *On the Semicontinuity of the Spectrum and an Upper Bound for the Number of Singular Points of a Projective Hypersurface*, J. Soviet Math. **270,** (1983), 735–739.
[19] J.-G. Yang, *Enumeration of Combinations of Rational Double Points on Quartic Surfaces*, AMS/IP Studies in Advanced Mathematics, **5** (1997), pp. 275–312.

Manoscritto pervenuto in redazione il 21 novembre 2005.