Factorization of Mappings and General Existence Theorems in Locally Convex Spaces.

Tullio Valent (*)

0. Introduction.

Here, we need a notion of transpose for an arbitrary map \( u : X \to Y \), with \( X, Y \) two sets. The (real) transpose of \( u \) will be the (linear) map

\[
^t u : \mathbb{R}^Y \to \mathbb{R}^X
\]
defined by putting \(^t u(f) = f \circ u \) \( \forall f \in \mathbb{R}^Y \).

All linear spaces considered in this paper are over the real field \( \mathbb{R} \).

This work starts from a simple algebraic remark: given two maps \( u_1 : X \to X_1, \ u_2 : X \to X_2 \), with \( X_1, X_2 \) linear spaces and \( X \) a set, the inclusion

\[
^t u_1(X_1^*) \subseteq ^t u_2(X_2^*)
\]
where \( F \) is some linear subspace of \( \text{Lip}(u_2(X), \mathbb{R}) \) (= the space of Lipschitzian maps from \( u_2(X) \) into \( \mathbb{R} \)) containing the dual \( X_2^* \) of \( X_2 \).

Evidently, (in view of the Hahn-Banach theorem) a first consequence of the inclusion (0.1) is that there is a unique map \( \varphi : u_2(X) \to X_1 \) such that \( u_1 = \varphi \circ u_2 \). In the case \( F = X_2^* \) we prove that (0.1) implies that \( \varphi \) has a weakly continuous, linear extension to the subspace \( \langle u_2(X) \rangle \) of \( X_2 \) generated by \( u_2(X) \); this extension of \( \varphi \) is continuous if \( \langle u_2(X) \rangle \) is a Mackey space (Theorem 3.1). If \( F \) is the space of all bounded, linear forms on \( \langle u_2(X) \rangle \), then condition (0.1) assures that \( \varphi \) has a bounded, linear extension to \( \langle u_2(X) \rangle \) (Theorem 3.7), while if (0.1) holds with \( F = \text{Lip}(u_2(X), \mathbb{R}) \) then \( \varphi \) is Lipschitzian (Theorem 4.1).

(*) Indirizzo dell’A.: Dipartimento di Matematica Pura ed Applicata dell’Università di Padova, Via Trieste 63 - 35121, Padova.
Successively, we replace (0.1) with a (weaker) inclusion of the type

$$(0.2) \quad {^t}u_1(H^0) \subseteq {^t}u_2(F),$$

where $H$ is a fixed linear subspace of $X$, and $H^0$ is the polar of $H$, (i.e. $H^0 = \{x'_1 \in X'_1 : x'_1(h) = 0 \ \forall h \in H\}$). An important choice of $H$ is $H = \langle u_1(\text{Ker} u_2) \rangle$ (see Remark 3.6). Sometimes, in concrete situations, it occurs that $H$ is the kernel of a continuous, linear map defined in $X_1$ (see section 6).

Stronger properties of the function $\varphi$ can be deduced from the inclusions (0.1) and (0.2) when $X_1$ and $X_2$ are Fréchet spaces. (See Corollaries 5.2, 5.3, Theorem 6.1, and Corollaries 6.2, 6.3). The well-known “Theorem on the surjections of Fréchet spaces”, together with its generalizations, are contained, as particular cases, in Theorems 3.5 and 6.1. Corollary 5.5 provides a useful necessary and sufficient condition for a Lipschitzian map between Fréchet spaces to be bijective with inverse which is Lipschitzian.

We observe that, in the case when $X$ is a linear space and the maps $u_1$, $u_2$ are linear, Corollary 3.4 extends to the locally convex spaces an important theorem which was proved by G. Fichera for Banach spaces (see G. Fichera [1] and [2]). We also remark that Corollary 6.2 was obtained, in a different way, by G. Zampieri, who used such result in proving that semiglobal $\text{C}^\infty$-solvability in an open subset of $\mathbb{R}^n$ for overdetermined systems $Pu = f$, $Qu = 0$ with constant coefficients and $Q$ elliptic, implies the global $\text{C}^\infty$-solvability. (See G. Zampieri [5] and [6]).

1. An algebraic remark.

**Remark 1.1.** Let $X$ be a set, let $X_1$, $X_2$ be linear spaces, and let $u_1 : X \rightarrow X_1$, $u_2 : X \rightarrow X_2$ be two maps. The following statements are equivalent:

(a*) $\quad {^t}u_1(X_1^*) \subseteq {^t}u_2(X_2^*)$,

(b*) there is a unique linear map $\varphi : \langle u_2(X) \rangle \rightarrow X_1$ such that $u_1 = \varphi \circ u_2$,

where $X_1^*$ and $X_2^*$ are the duals of $X_1$ and $X_2$, and $\langle u_2(X) \rangle$ denotes the linear subspace of $X_2$ generated by $u_2(X)$.

**Proof.** Property (a*) says that for each $x_1^* \in X_1^*$ there is $x_2^* \in X_2^*$ such that $x_1^* \circ u_1 = x_2^* \circ u_2$. Then it is evident that (b*) $\Rightarrow$ (a*), because if $\varphi : \langle u_2(X) \rangle \rightarrow X_1$ is a linear map such that $u_1 = \varphi \circ u_2$, and $x_2^*$ is a linear
extension of \( x_1^* \circ \varphi \) to \( X_2 \), then \( x_1^* \circ u_1 = x_2^* \circ u_2 \). Now, let us prove that \((a^*) \Rightarrow (b^*)\). Obviously, if \((a^*)\) holds then

\[
(1.1) \quad u_2(x) = u_2(\xi), \; x, \xi \in X \Rightarrow u_1(x) = x_1(\xi),
\]

and thus we can consider the map \( x_2(u) \mapsto u_1(x) \), from \( u_2(X) \) into \( X_1 \). By \((a^*)\) this map has a linear extension to \( \langle u_2(X) \rangle \); this is the (linear) map \( \varphi : \langle u_2(X) \rangle \to X_1 \) defined by putting

\[
(1.2) \quad \varphi \left( \sum_j \lambda_j u_2(x_j) \right) = \sum_j \lambda_j u_1(x_j), \quad \text{(with } \lambda_j \in \mathbb{R} \text{ and } x_j \in X).\]

This definition of \( \varphi \) has a sense, from \((a^*)\) it follows that if

\[
(1.3) \quad \sum_j \lambda_j u_2(x_j) = \sum_k \mu_k u_2(\xi_k), \quad \text{(with } \mu_k \in \mathbb{R} \text{ and } \xi_k \in X),
\]

then

\[
(1.4) \quad \sum_j \lambda_j u_1(x_j) = \sum_k \mu_k u_1(\xi_k).\]

In order to prove this observe that, in view of \((a^*)\), for each \( x_1^* \in X_1^* \) there is \( x_2^* \in X_2^* \) such that \( x_1^*(u_1(x)) = x_2^*(u_2(x)) \) \( \forall x \in X \); in particular

\[
\begin{align*}
  x_1^*(u_1(x_j)) &= x_2^*(u_2(x_j)) \\
  x_1^*(u_1(\xi_k)) &= x_2^*(u_2(\xi_k))
\end{align*}
\]

for all \( j \) and \( k \), which implies

\[
\begin{align*}
  x_1^* \left( \sum_j \lambda_j u_1(x_j) \right) &= x_2^* \left( \sum_j \lambda_j u_2(x_j) \right) \\
  x_1^* \left( \sum_k \mu_k u_1(\xi_k) \right) &= x_2^* \left( \sum_k \mu_k u_2(\xi_k) \right).
\end{align*}
\]

Then, if \((1.3)\) holds, then

\[
x_1^* \left( \sum_j \lambda_j u_1(x_j) \right) - \left( \sum_k \mu_k u_1(\xi_k) \right) = 0
\]

for all \( x_1^* \in X_1^* \), and so \((1.4)\) holds. \( \square \)

Observe that, if \( X \) is a linear space and the mappings \( u_1 \) and \( u_2 \) are linear, then \((b^*)\) is equivalent to

\[ \text{Ker } u_2 \subseteq \text{Ker } u_1. \]
2. Testing with continuous linear forms.

For any locally convex (topological, linear) space X the usual notation will be used: in particular $\sigma(X, X')$ and $\tau(X, X')$ will denote the weak and the Mackey topologies on X with respect to the natural duality between X and its dual $X'$. X is called a Mackey space if its topology coincides with $\tau(X, X')$. $X^*$ will denote the set of all linear forms on X, while $X^*$ will denote the set of those linear forms on X that are bounded; so we have $X' \subseteq X^* \subseteq X^*$.

The following proposition is well-known (see, for instance, A. Grothendieck [3, 2.16], or H. Jarchow [7, 8.6]).

**Proposition 2.1.** Let $X, Y$ be Hausdorff locally convex spaces. For any linear mapping $\varphi : X \to Y$ the following three statements are equivalent:

(I) for each $y' \in Y'$ the map $y' \circ \varphi$ is continuous [i.e. $\varphi(y') \in X'$];

(II) $\varphi$ is weakly continuous [i.e. $\varphi$ is continuous for the topologies $\sigma(X, X')$ on $X$ and $\sigma(Y, Y')$ on $Y$];

(III) $\varphi$ is Mackey continuous [i.e. $\varphi$ is continuous for the topologies $\tau(X, X')$ on $X$ and $\tau(Y, Y')$ on $Y$].

It is also well-known that the bounded subsets of a Hausdorff locally convex space are the same for all locally convex Hausdorff topologies on $X$ which are compatible with the natural duality between $X$ and $X'$. So, the following proposition holds.

**Proposition 2.2.** A subset $B$ of a Hausdorff locally convex space $X$ is bounded if and only if every element $x'$ of $X'$ is bounded on $B$ (namely, if and only if $B$ is bounded for the topology $\sigma(X, X')$).

**Corollary 2.3.** Let $X, Y$ be locally convex spaces, with $Y$ a Hausdorff space. A mapping $f : X \to Y$ is bounded [i.e. $f$ maps each bounded subset of $X$ to a bounded subset of $Y$] if and only if for each $y' \in Y'$, $y' \circ f$ is bounded.

When $X$ is a Hausdorff locally convex space, $Y$ is a topological linear space and $U$ is a subset of $X$, we say that a map $f : U \to Y$ is Lipschitzian (respectively, locally Lipschitzian) if for each absolutely convex, bounded subset $B$ of $X$ the set $\{(f(x_1) - f(x_2))/\|x_1 - x_2\|_B : x_1, x_2 \in U \cap X_B, x_1 \neq x_2\}$ is bounded in $Y$ (respectively, locally bounded in $Y$). Here $X_B$ denotes the linear subspace of $X$ spanned by $B$, equipped with the norm $\| \cdot \|_B$ defined by $\|x\|_B = \inf \{\lambda > 0 : x \in \lambda B\}$. The set of all Lipschitzian maps from $U$ into $Y$ will be denoted by $\text{Lip}(U, Y)$. 
Let us emphasize the following consequence of Proposition 2.2.

**Proposition 2.4.** Let \( X, Y \) be Hausdorff locally convex spaces, and let \( U \) be a subset of \( X \). A map \( f: U \to Y \) is Lipschitzian if and only if for each \( y' \in Y' \) the (scalar) map \( y' \circ f: U \to \mathbb{R} \) is Lipschitzian.

**Proof.** If \( f: X \to Y \) is Lipschitzian and \( y' \in Y' \) then, obviously, the composite \( y' \circ f \) is Lipschitzian. Conversely, suppose that for each \( y' \circ Y \) the map \( y' \circ f \) is Lipschitzian; then for each \( y' \in Y' \) and each absolutely convex, bounded subset \( B \) of \( X \) there is a number \( c > 0 \) such that 
\[
|y'(f(x_1) - f(x_2))/\|x_1 - x_2\|_B| \leq c \quad \forall x_1, x_2 \in U \cap X_B \quad \text{with} \quad x_1 \neq x_2,
\]
namely \( y' \) is bounded on the subset \( \{f(x_1) - f(x_2)/\|x_1 - x_2\|_B: x_1, x_2 \in U \cap X_B, x_1 \neq x_2 \} \) of \( Y \). Then, in view of Proposition 2.2, such subset of \( Y \) is bounded, thus \( f \) is Lipschitzian. \( \square \)

3. Surjectivity criteria.

**Theorem 3.1.** Let \( X \) be a set, let \( X_1, X_2 \) be Hausdorff locally convex spaces, and let \( u_1: X \to X_1, u_2: X \to X_2 \) be two maps. The following statements are equivalent:

(a') \( \overset{!}{u_1}(X_1') \subseteq \overset{!}{u_2}(X_2') \), (i.e. \( \forall x'_1 \in X_1' \exists x'_2 \in X_2' \) such that \( x'_1 \circ u_1 = x'_2 \circ u_2 \));

(b') there is a unique weakly continuous, linear map \( \varphi: \langle u_2(X) \rangle \to X_1 \) such that \( u_1 = \varphi \circ u_2 \).

Moreover, if the subspace \( \langle u_2(X) \rangle \) of \( X_2 \), spanned by \( u_2(X) \), is a Mackey space then the properties (a') and (b') are equivalent to

(c') there is a unique continuous, linear map \( \varphi: \langle u_2(X) \rangle \to X_1 \) such that \( u_1 = \varphi \circ u_2 \).

**Proof.** (b') \( \Rightarrow \) (a') because if \( \varphi: \langle u_2(X) \rangle \to X_1 \) is a weakly continuous, linear map such that \( u_1 = \varphi \circ u_2 \), then for each \( x'_1 \in V_1' \) then linear form \( x'_1 \circ \varphi \) on \( \langle u_2(X) \rangle \) is continuous, and hence (by the Hahn-Banach theorem) it has a continuous, linear extension \( x'_2 \) on \( X_2 \), which evidently is related to \( x'_1 \) by the equality \( x'_1 \circ u_1 = x'_2 \circ u_2 \). Let us prove that (a') \( \Rightarrow \) (b'). Proceeding as in the proof of Remark 1.1 and using the Hahn-Banach theorem one shows that if (a') holds then there is a unique linear map \( \varphi: \langle u_2(X) \rangle \to X_1 \) such that \( u_1 = \varphi \circ u_2 \). Note that from (a') it follows also that for each \( x'_1 \in X_1' \) the linear form \( x'_1 \circ \varphi \) on \( \langle u_2(X) \rangle \) is the restriction to \( \langle u_2(X) \rangle \) of some element \( x'_1 \).
of $X'_1$, and so it is continuous; thus we have
\[ x'_1 \circ \varphi \in \langle u_2(X) \rangle' \quad \forall x'_1 \in X'_1, \]
which means, in view of Proposition 2.1, that $\varphi$ is weakly continuous. It
remains to prove that, if $\langle u_2(X) \rangle$ is a Mackey space, then $(b')$ is equivalent to
$(c')$. To do this it suffices to observe that if $\varphi$ is continuous it is also weakly
continuous, and that, in view of Proposition 2.1, $\varphi$ is weakly continuous if and
only if it is Mackey continuous.
\[ \square \]

**Remark 3.2.** The fact that $X_2$ is a Mackey space does not imply that the
subspace $\langle u_2(X) \rangle$ of $X_2$ is a Mackey space. However, the subspace $\langle u_2(X) \rangle$ of
$X_2$ is a Mackey space provided one of the following conditions is satisfied:

(i) $X_2$ is metrizable;

(ii) $X_2$ is barrelled, and $\langle u_2(X) \rangle$ is a countable codimension linear
    subspace of $X_2$;

(iii) $X_2$ is bornological, and $\langle u_2(X) \rangle$ is a finite codimension linear
    subspace of $X_2$.

**Proof.** If $X_2$ is metrizable then $\langle u_2(X) \rangle$ is a Mackey space, because any
metrizable locally convex space is a Mackey space. If (ii) holds then $\langle u_2(X) \rangle$
is a Mackey space, because a countable codimension subspace of a barreled
space is barrelled (see J. C. Ferrando - M. Lopez Pellicer - L. M. Sanchez
Rui [8, Prop. 1.1.15]), and so it is a Mackey space. Finally, (iii) implies that
$\langle u_2(X) \rangle$ is bornological (see H. Jarchow [7, Theorem 13.5.2]), and hence it is
a Mackey space.
\[ \square \]

Taking, in Theorem 3.1, $X \subseteq X_2$ and $u = \text{identity map}$, we obtain the
following

**Corollary 3.3 (Continuous, linear extension).** Let $X_1$, $X_2$ be Haus-
dorff locally convex spaces, and let $X$ be a subset of $X_2$ such that the
subspace $\langle X \rangle$ of $X_2$ generated by $X$ is a Mackey space. A map $w : X \to X_1$
has a continuous, linear extension to $\langle X \rangle$ if and only if for each $x'_1 \in X'_1$
the (scalar) map $x'_1 \circ u$ has a continuous, linear extension to $\langle X \rangle$.

In the case when $X$ is a linear space and $u_1, u_2$ are linear, Theorem 3.1 yields

**Corollary 3.4 (The linear case).** Let $X$ be a linear space, let $X_1$, $X_2$ be
Hausdorff locally convex spaces, let $u_1 : X \to X_1$, $u_2 : X \to X_2$ be linear maps,
and let $P_1$ be a family of seminorms on $X$ defining the topology of $X_1$ and $P_2$
a filtering family of seminorms on $X_2$ defining the topology of $X_2$. If the subspace $\langle u_2(X) \rangle$ of $X_2$ is a Mackey space, then (a′) is satisfied if and only if

$$\text{(3.1)} \quad \text{for each } p_1 \in \mathcal{P}_1 \text{ there is } p_2 \in \mathcal{P}_2 \text{ and a number } c > 0 \text{ such that } p_1(u_1(x)) \leq c p_2(u_2(x)) \forall x \in X.$$ 

**Proof.** It suffices to observe that, when $X$ is a linear space and $u_1$, $u_2$ are linear, the property (c′) can be expressed in the form

$$\text{Ker } u_2 \subseteq \text{Ker } u_1, \text{ and the map } u_2(x) \mapsto u_1(x), \ x \in X, \text{ from the subspace } u_2(X) \text{ of } X_2 \text{ into } X_1, \text{ is continuous,}$$

namely, in the from (3.1). \hfill \square

We recall that the statement of Corollary 3.4 was proved by Valent [4]. It generalizes to the case of locally convex spaces a theorem proved by G. Fichera for Banach spaces (see G. Fichera [1], [2]). Note also that no completeness hypothesis on $X_1$ or $X_2$ is required in Corollary 3.3. (The proof by Fichera needs the completeness of $X_1$ and $X_2$, because it makes use of the closed graph theorem).

In the following theorem condition (a′) is replaced by the (weaker) condition $(a_H)$.

**Theorem 3.5.** Let $X$, $X_1$, $X_2$, $u_1$, $u_2$ be as in the statement of Theorem 3.1, and let $H$ be a linear subspace of $X_1$. The following statements are equivalent

(a$_H$) \( t^{\prime}u_1(H^0) \subseteq t^{\prime}u_2(X'_2), \) (i.e., for each $x'_1 \in X'_1$, such that $x'_1(\hat{h}) = 0$ $\forall \hat{h} \in H$, there is $x'_2 \in X'_2$ such that $x'_1 \circ u_1 = x'_2 \circ u_2$);

(b$_H$) there is a unique weakly continuous linear map $\varphi_H: \langle u_2(X) \rangle \to X_1/\overline{H}$ such that $\pi_H \circ u_1 = \varphi_H \circ u_2$, where $\pi_H$ denotes the canonical projection of $X_1$ onto $X_1/\overline{H}$.

Moreover, if the subspace $\langle u_2(X) \rangle$ of $X_2$ is a Mackey space, then (a$_H$) and (b$_H$) are equivalent to

(c$_H$) there is a unique continuous linear map $\varphi_H: \langle u_2(X) \rangle \to X_1/\overline{H}$ such that $\pi_H \circ u_1 = \varphi_H \circ u_2$.

**Proof.** It suffices to apply Theorem 3.1 with $\pi_H \circ u_1$ instead of $u_1$ and observe that

$$t^{\prime}(\pi_H \circ u_1)((X_1/\overline{H})) = t^{\prime}u_1(H^0)$$
**Remark 3.6.** A remarkable choice of $H$ in the statement of Theorem 3.5 is

$$H = \langle u_1(Ker u_2) \rangle.$$  

For this choice of $H$ the property $(a_H)$ becomes: for each $x'_1 \in X'_1$ satisfying

$$u_1(Ker u_2) \subseteq Ker x'_1$$

there is $x'_2 \in X'_2$ such that $x'_1 \circ u_1 = x'_2 \circ u_2$. Note that (3.2) is a necessary condition on $x'_1 \in X'_1$ in order $(a')$ to be satisfied.

**Theorem 3.7.** Let $X, X_1, X_2, u_1, u_2$ be as in the statement of Theorem 3.1. The following statements are equivalent:

(a') $\langle ^tu_1(X'_1) \rangle \subseteq \langle ^tu_2(u_2(X)) \rangle$;

(b') there is a unique bounded, linear map $\varphi: \langle u_2(X) \rangle \to X_1$ such that $u_1 = \varphi \circ u_2$.

**Proof.** Obviously (b') implies (a'). In order to prove that (a') implies (b') one can essentially proceed as in the proof of Remark 1.1 in showing that if (a') holds then there is a unique linear map $\varphi: \langle u_2(X) \rangle \to X_1$ such that $u_1 = \varphi \circ u_2$. Then it suffices to observe that, by Corollary 2.3, (a') implies that the linear map $\varphi$ is bounded. \qed

**Remark 3.8.** From Theorem 3.7 a bounded, linear extension result like Corollary 3.3 can be deduced.

4. Other surjectivity criteria. The Lipschitzian case.

**Theorem 4.1.** Let $X, X_1, X_2, u_1, u_2$ be as in the statement of Theorem 3.1. The following statements are equivalent

(a$_L$) $\langle ^tu_1(X'_1) \rangle \subseteq \langle ^tu_2(Lip(u_2(X), R)) \rangle$;

(b$_L$) there is a unique map $f \in Lip(u_2(X), X_1)$ such that $u_1 = f \circ u_2$.

**Proof.** (b$_L$) $\Rightarrow$ (a$_L$) because $X'_1 \subseteq Lip(X, R)$ and the composed of two Lipschitzian map is a Lipschitzian map. In order to prove that (a$_L$) $\Rightarrow$ (b$_L$) we first observe that (by the Hahn-Banach theorem) (a$_L$) implies that (1.1) holds and hence one can define a map $f: u_2(X) \to X_1$ by putting

$$f(u_2(x)) = u_1(x) \quad \forall x \in X.$$
moreover, from \((a_\ell)\) it follows that \(x_1' \circ f \in Lip(u_2(X), \mathbb{R}) \forall x_1' \in X_1',\) and by Proposition 2.4 this implies that \(f\) is Lipschitzian. 

A consequence of Theorem 4.1 is the following

**Corollary 4.2.** Let \(X, X_1, X_2, u_1, u_2\) be as in the statement of Theorem 3.1, and let \(\mathcal{P}\) be a family of seminorms on \(X_1\) defining its topology. Then the condition \((a_\ell)\) in the statement of Theorem 4.1 is satisfied if and only if

\[
(4.1) \quad \text{for each absolutely convex, bounded subset } B \text{ of } X_2 \text{ and each } p \in \mathcal{P} \text{ there is a number } c_{B,p} > 0 \text{ such that}
\]

\[
p(u_1(x) - u_1(\xi)) \leq c_{B,p} \|u_2(x) - u_2(\xi)\|_B \quad \forall x, \xi \in u_2^{-1}(X_{2,B}),
\]

where \(X_{2,B}\) is the linear subspace of \(X_2\) spanned by \(B\), equipped with the norm \(\| \cdot \|_B\) defined by \(\|x\|_B = \inf \{ \lambda > 0 : x \in \lambda B \}\).

**Proof.** A subset of \(X_1\) is bounded if and only if every element of \(\mathcal{P}\) is bounded on it; then it is easy to see that the condition \((b_\ell)\) in the statement of Theorem 4.1 is equivalent to (4.1).

A result of type Theorem 3.5 for Lipschitzian maps is the following

**Theorem 4.3.** Let \(X, X_1, X_2, u_1, u_2, H, \pi_H\) be as in the statement of Theorem 3.5. The following statements are equivalent:

\((a_H)\) \(\ ^\prime u_1(H) \subseteq \ ^\prime u_2(Lip(u_2(X), \mathbb{R}))\);  
\((b_H)\) there is a unique Lipschitzian map \(f_H : u_2(X) \to X_1/H\) such that \(\pi_H \circ u_1 = f \circ u_2\).

Theorem 4.3 follows from Theorem 4.1 in the same way as Theorem 3.5 follows from Theorem 3.1.

**Remark 4.4.** The statements of Theorems 4.1 and 4.3 hold also when one replace Lipschitzian maps with locally Lipschitzian maps.

5. Some particularizations of Theorems 3.5 and 4.1.

Let \(X, Y\) be Hausdorff locally convex spaces, and let \(u : X \to Y\) be a weakly continuous linear map. So \(\text{Ker } u\) is weakly closed, namely closed in \(X\). (Recall that the closure of a convex subset of \(X\) is the same for all
Hausdorff locally convex topologies on $X$ which are compatible with the natural duality between $X$ and $X'$.

Let us denote by $\tilde{u}: X/	ext{Ker} u \to u(X) \subset Y$ the natural bijection (associated with $u$) such that $u = \pi \circ \tilde{u}$ where $\pi$ is the projection of $X$ onto $X/	ext{Ker} u$. Let $\tilde{\varphi}: u(X) \to X/	ext{Ker} u$ be the inverse of $\tilde{u}$.

Clearly, $u$ is open if and only if $\tilde{\varphi}$ is continuous; furthermore, $u$ is weakly open if and only if $\tilde{\varphi}$ is weakly continuous. From Theorem 3.5 (applied to the case when $H = (\text{Ker} u)$, $u$ is the identity map $id: X \to X$ and $u_2 = u$) it follows that $\tilde{\varphi}$ is weakly continuous if and only if $^t \text{id}((\text{id}(\text{Ker} u))^0) \subseteq ^t u(Y')$, i.e.,

\begin{equation}
(\text{Ker} u)^0 \subseteq ^t u(Y'),
\end{equation}

where $(\text{Ker} u)^0 := \{ x' \in X': \text{Ker} u \subseteq \text{Ker} x' \}$ is the polar of $\text{Ker} u$. As $(\text{Ker} u)^0$ coincides with the weak closure of $^t u(Y')$ in $X'$, we get that $u$ is weakly open if and only if $^t u(Y')$ is weakly closed in $X'$. From Theorem 3.5 it also follows that, if the subspace $u(X)$ of $Y$ is a Mackey space, then $\tilde{\varphi}$ is continuous if and only if (5.1) is satisfied. Thus the following result holds.

**Corollary 5.1.** A weakly continuous linear map $u: X \to Y$, with $X$, $Y$ Hausdorff locally convex spaces, is weakly open if and only if $^t u(Y')$ is weakly closed in $X'$. If the subspace $u(X)$ of $Y$ is a Mackey space, then $u$ is open if and only if $^t u(Y')$ is weakly closed in $X'$.

If $X, Y$ are Fréchet spaces and $u$ is continuous, then (in view of the open mapping theorem) $u$ is open if and only if $u(X)$ is complete (namely closed) in $Y$. Therefore a consequence of Corollary 5.1 is

**Corollary 5.2.** If $X, Y$ are Fréchet spaces and $u: X \to Y$ is a continuous linear map, then $u(X)$ is closed in $Y$ if and only if $^t u(Y')$ is weakly closed in $X'$.

It is well-known that $u(X)$ is dense in $Y$ if and only if the mapping $^t u: Y' \to X'$ is one-to-one. Then Corollary 5.2 yields immediately the following result which is known as the “theorem on the surjections of Fréchet spaces”.

**Corollary 5.3.** If $X, Y$ are Fréchet spaces, and $u: X \to Y$ is a continuous linear map, then $u$ is onto if and only if $^t u$ is one-to-one and $^t u(Y')$ is weakly closed in $X'$.

Let us now consider a particularization of Theorem 4.1 (concerning the Lipschitzian case).
Corollary 5.4. Let $X$, $Y$ be Hausdorff locally convex spaces, and $U$ a subset of $X$. For any map $u: X \to Y$ the following statements are equivalent:

(j) for each $x' \in X'$ there is $l \in \text{Lip}_0(u(U), \mathbb{R})$ such that $x' = l \circ u$;

(jj) $u$ is one-to-one and its left inverse $u^{-1}: u(U) \to U$ is Lipschitzian.

This corollary can be obtained by applying Theorem 4.1 to the case when $X_1 = X$, $X_2 = Y$, $u_1 = \text{id}$: $U \to X$, and $u_2 = u$.

Let us now denote that $\text{Lip}_0(X, Y)$ [respectively $\text{Lip}_0(Y, X)$, and $\text{Lip}_0(Y, \mathbb{R})$] the set of elements $f$ of $\text{Lip}(X, Y)$ [respectively $\text{Lip}(Y, X)$, and $\text{Lip}(Y, \mathbb{R})$] such that $f(0) = 0$.

Corollary 5.5. Let $u \in \text{Lip}_0(X, Y)$, with $X$, $Y$ Hausdorff locally convex spaces, and suppose that $X$ is complete. The following statements are equivalent:

(j)$_0$ for each $x' \in X'$ there is a unique $l \in \text{Lip}_0(Y, \mathbb{R})$ such that $x' = l \circ u$;

(jj)$_0$ $u$ is bijective, and $u^{-1} \in \text{Lip}_0(Y, X)$.

Proof. Evidently (jj)$_0 \Rightarrow$ (j)$_0$. Suppose that (j)$_0$ holds. Then, by Corollary 5.4, $u$ is one-to-one and its left inverse $u^{-1}: u(X) \to X$ is Lipschitzian. It follows that $u(X)$ is complete and so it is closed in $Y$. It remains to prove that $u(X)$ is dense in $Y$. This is true, because if $y' \in Y'$ and $y' \circ u = 0$ then $y' = 0$ in view of (j)$_0$, and so $u(X)$ is dense in $Y$ by the Hahn-Banach theorem.

6. Other consequences of Theorem 3.5.

In this section we emphasize some consequences of Theorem 3.5 in the case when the subspace $H$ of $X_1$ is the kernel of a continuous, linear map $\nu_1: X_1 \to Y$, with $Y$ a Hausdorff locally convex space. Let us denote by $\overline{\nu_1(Y')}^c$ the closure of $\nu_1(Y')$ in $(X'_1, \sigma(X'_1, X_1))$.

Since

\begin{equation}
\overline{\nu_1(Y')} = (\text{Ker} \nu_1)^0,
\end{equation}

from Theorem 3.5 it follows that, if the subspace $\langle u_2(X) \rangle$ of $X_2$ is a Mackey space, then the inclusion

$$(a_{\nu_1}) \quad \nu_1(\overline{\nu_1(Y')}^c) \subseteq \nu_2(X'_2)$$

is equivalent to the property
(c_{v_1}) there is a unique continuous, linear map \( \varphi: \langle u_2(X) \rangle \to X_1/\text{Ker } v_1 \) such that \( \pi_1 \circ u_1 = \varphi \circ u_2 \), where \( \pi_1 \) denotes the canonical projection of \( X_1 \) onto \( X_1/\text{Ker } v_1 \).

Observe that, if \( v_1 \) is one-to-one, then \( \overline{v_1(Y')} = X'_1 \); thus in this case the conditions \((a_{v_1})\) and \((c_{v_1})\) coincide with \((a')\) and \((c')\) respectively, and so they do not involve the map \( v_1 \).

**Theorem 6.1.** Let \( X_1, X_2, Y \) be Hausdorff locally convex spaces, let \( X \) be a set, let \( u_1: X \to X_1 \) and \( u_2: X \to X_2 \) be two maps, and let \( v_1: X_1 \to Y \) be a continuous, linear map. If \( X_1 \) and \( X_2 \) are metrizable and complete, then \((a_{v_1})\) is satisfied if and only if

\[
(d_{v_1}) \text{ there is a continuous linear map } v_2: \overline{\langle u_2(X) \rangle} \to v_1(X_1) \text{ such that } v_1 \circ u_1 = v_2 \circ u_2.
\]

**Proof.** Let \( X_1 \) and \( X_2 \) be metrizable and complete. Since the subspace \( \overline{\langle u_2(X) \rangle} \) of \( X_2 \) is metrizable and hence a Mackey space, from Theorem 6.2 it follows (as we have remarked above) that \((a_{v_1})\) is equivalent to \((c_{v_1})\). Then, in order to prove the equivalence of \((a_{v_1})\) and \((d_{v_1})\), we shall show that \((c_{v_1})\) is equivalent to \((d_{v_1})\). We have \((c_{v_1}) \Rightarrow (d_{v_1})\) because (under our hypotheses) \( X_1/\text{Ker } v_1 \) is complete, and hence the continuous linear map \( \varphi: \overline{\langle u_2(X) \rangle} \to X_1/\text{Ker } v_1 \) can be extended to a continuous linear map from \( \overline{\langle u_2(X) \rangle} \) into \( X_1/\text{Ker } v_1 \). We now prove that \((d_{v_1}) \Rightarrow (c_{v_1})\). Accordingly, we denote by \( \tilde{v}_1: X/\text{Ker } v_1 \to v_1(X) \subseteq Y \) the bijection (associated with \( v_1 \)) such that \( v_1 = \pi_1 \circ \tilde{v}_1 \), and consider the map

\[
\tilde{v}_1^{-1} \circ v_1: \overline{\langle u_2(X) \rangle} \to X_1/\text{Ker } v_1.
\]

Note that the subspace \( \overline{\langle u_2(X) \rangle} \) of \( X_2 \) and the quotient space \( X_1/\text{Ker } v_1 \) are both metrizable and complete. Moreover (using the fact that the map \( v_1 \) is continuous) it is easy to prove that the graph of the map \( \tilde{v}_1^{-1} \circ v_1 \) is closed. Thus, in view of the closed graph theorem, the map \( \tilde{v}_1^{-1} \circ v_1 \) is continuous. Then \((c_{v_1})\) is satisfied with \( \varphi \) the restriction on \( \overline{\langle u_2(X) \rangle} \) of the continuous linear map \( \tilde{v}_1^{-1} \circ v_1 \). \( \square \)

Often, in concrete cases, two continuous linear maps \( v_1: X_1 \to Y \) and \( v_2: X_2 \to Y \) are assigned such that \( v_1 \circ u_1 = v_2 \circ u_2 \); thus condition \((d_{v_1})\) in the statement of Theorem 6.1 becomes

\[
v_2(\overline{\langle u_2(X) \rangle}) \subseteq v_1(X_1).
\]

Therefore the following corollary of Theorem 6.1 can be useful. It foun-
ishes another generalization of the “Theorem on the surjections of Fréchet spaces” (see Corollary 5.3).

**Corollary 6.2.** Let $X$ be a set, let $X_1, X_2, Y$ be Hausdorff locally convex spaces, let $u_1: X \to X_1$, $u_2: X \to X_2$ be two maps, and let $v_1: X_1 \to Y$, $v_2: X_2 \to Y$ be continuous linear maps such that $v_1 \circ u_1 = v_2 \circ u_2$. If $X_1$, $X_2$ are metrizable and complete, then the following statements are equivalent:

1. $\overline{t u_1(v_1(Y))} \subseteq \overline{t u_2(X_2')}$;
2. $v_2(\overline{u_2(X)}) \subseteq v_1(X_1)$.

We observe that, if $v_1$ is one-to-one (i.e., if $\overline{t u_1(Y)}$ is weakly dense in $X_1'$) and $\overline{u_2(X)}$ is dense in $X_2$, then (1) and (2) reduce respectively to

3. $\overline{t u_1(X_1')} \subseteq \overline{t u_2(X_2')}$

and

4. $v_2(X_2) \subseteq v_1(X_1)$.

We also remark that, taking in Corollary 6.2: $X = X_1$, $Y = X_2$, $u_1 = id_X$, $u_2 = v_1(= u)$, and $v_2 = id_Y$, one immediately obtains the statement of Corollary 5.2. Hence the so called “Theorem on the surjections of Fréchet spaces” is generalized by Corollary 6.2.

**Corollary 6.3.** Let $X, X_1, X_2$, be Hausdorff locally convex spaces, let $v_1: X_1 \to Y$ be a continuous linear map, and let $u_2: X_1 \to X_2$ be a map. If $X_1$ and $X_2$ are metrizable and complete, then the following statements are equivalent:

5. $(\text{Ker } v_1) = \overline{t u_2(X_2')}$, (i.e., for each $x_1' \in X_1'$ such that $x_1'(h) = 0$ for all $h \in \text{Ker } v_1$ there is $x_2' \in X_2'$ such that $x_1' = u \circ x_2'$);
6. there is a continuous linear map $v_2: \overline{u_2(X)} \to v_1(X_1)$ such that $v_1 = v_2 \circ u_2$.

**Proof.** It suffices to take $X = X_1$ and $u_1 = \text{identity map}$ in the statement of Theorem 6.1, and recall (6.1). \qed

**References**


Manoscritto pervenuto in redazione il 29 novembre 2005