REND. SEM. MAT. UNIV. PADOVA, Vol. 117 (2007)

Factorization of Mappings and General Existence Theorems in Locally Convex Spaces.

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0. Introduction.

Here, we need a notion of transpose for an arbitrary map $u : X \to Y$, with X, Y two sets. The (real) *transpose* of u will be the (linear) map

$$^t u : \mathbb{R}^Y \to \mathbb{R}^X$$

defined by putting $^t u(f) = f \circ u \; \forall f \in \mathbb{R}^Y$.

All linear spaces considered in this paper are over the real field $\mathbb{R}$.

This work starts from a simple algebraic remark: given two maps $u_1 : X \to X_1$, $u_2 : X \to X_2$, with X_1, X_2 linear spaces and X a set, the inclusion $^t u_1(X_1^*) \subseteq {}^t u_2(X_2^*)$, where X_1^* and X_2^* denote the duals of X_1 and X_2, implies that there is a *linear* map $\varphi : X_2 \to X_1$ such that $u_1 = \varphi \circ u_2$.

Then we suppose that X_1, X_2 are Hausdorff locally convex (topological, linear) spaces and consider inclusions of the type

$$^t u_1(X_1') \subseteq {}^t u_2(F) \tag{0.1}$$

where F is some linear subspace of $Lip\,(u_2(X), \mathbb{R})$ (= the space of Lipshitzian maps from $u_2(X)$ into $\mathbb{R}$) containing the dual X_2' of X_2.

Evidently, (in view of the Hahan-Banach theorem) a first consequence of the inclusion (0.1) is that there is a unique map $\varphi : u_2(X) \to X_1$ such that $u_1 = \varphi \circ u_2$. In the case $F = X_2'$ we prove that (0.1) implies that φ has a weakly continuous, linear extension to the subspace $\langle u_2(X) \rangle$ of X_2 generated by $u_2(X)$; this extension of φ is continuous if $\langle u_2(X) \rangle$ is a Mackey space (Theorem 3.1). If F is the space of all bounded, linear forms on $\langle u_2(X) \rangle$, then condition (0.1) assures that φ has a bounded, linear extension to $\langle u_2(X) \rangle$ (Theorem 3.7), while if (0.1) holds with $F = Lip\,(u_2(X), \mathbb{R})$ then φ is Lipschitzian (Theorem 4.1).

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Successively, we replace (0.1) with a (weaker) inclusion of the type

$$ {}^t u_1(H^0) \subseteq {}^t u_2(F), \tag{0.2} $$

where H is a fixed linear subspace of X, and H^0 is the polar of H, (i.e. $H^0 = \{x'_1 \in X'_1 : x'_1(h) = 0 \ \forall h \in H\}$). An important choice of H is $H = \langle u_1(Ker\ u_2)\rangle$ (see Remark 3.6). Sometimes, in concrete situations, it occurs that H is the kernel of a continuous, linear map defined in X_1 (see section 6).

Stronger properties of the function φ can be deduced from the inclusions (0.1) and (0.2) when X_1 and X_2 are Fréchet spaces. (See Corollaries 5.2, 5.3, Theorem 6.1, and Corollaries 6.2, 6.3). The well-known "Theorem on the surjections of Fréchet spaces", together with its generalizations, are contained, as particular cases, in Theorems 3.5 and 6.1. Corollary 5.5 provides a useful necessary and sufficient condition for a Lipschitzian map between Fréchet spaces to be bijective with inverse which is Lipschitzian.

We observe that, in the case when X is a linear space and the maps u_1, u_2 are linear, Corollary 3.4 extends to the locally convex spaces an important theorem which was proved by G. Fichera for Banach spaces (see G. Fichera [1] and [2]). We also remark that Corollary 6.2 was obtained, in a different way, by G. Zampieri, who used such result in proving that semiglobal C$^\infty$-solvability in an open subset of $\mathbb{R}^n$ for overdeterminated systems $Pu = f$, $Qu = 0$ with constant coefficients and Q elliptic, implies the global C$^\infty$-solvability. (See G. Zampieri [5] and [6]).

1. An algebraic remark.

REMARK 1.1. *Let X be a set, let X_1, X_2 be linear spaces, and let $u_1 : X \to X_1$, $u_2 : X \to X_2$ be two maps. The following statements are equivalent:*

- (a^*) ${}^t u_1(X_1^*) \subseteq {}^t u_2(X_2^*)$,
- (b^*) *there is a unique linear map* $\varphi\colon \langle u_2(X)\rangle \to X_1$ *such that* $u_1 = \varphi \circ u_2$,

where X_1^* and X_2^* are the duals of X_1 and X_2, and $\langle u_2(X)\rangle$ denotes the linear subspace of X_2 generated by $u_2(X)$.

PROOF. Property (a^*) says that for each $x_1^* \in X_1^*$ there is $x_2^* \in X_2^*$ such that $x_1^* \circ u_1 = x_2^* \circ u_2$. Then it is evident that (b^*) $\Rightarrow$ (a^*), because if $\varphi : \langle u_2(X)\rangle \to X_1$ is a linear map such that $u_1 = \varphi \circ u_2$, and x_2^* is a linear extension of $x_1^* \circ \varphi$ to X_2, then $x_1^* \circ u_1 = x_2^* \circ u_2$. Now, let us prove that $(a^*) \Rightarrow (b^*)$. Obviously, if (a^*) holds then

$$u_2(x) = u_2(\xi),\ x, \xi \in X \Rightarrow u_1(x) = x_1(\xi), \tag{1.1}$$

and thus we can consider the map $x_2(u) \mapsto u_1(x)$, from $u_2(X)$ into X_1. By (a^*) this map has a linear extension to $\langle u_2(X) \rangle$; this is the (linear) map $\varphi : \langle u_2(X) \rangle \to X_1$ defined by putting

$$\varphi\left(\sum_j \lambda_j u_2(x_j)\right) = \sum_j \lambda_j u_1(x_j), \qquad (\text{with } \lambda_j \in \mathbb{R} \text{ and } x_j \in X). \tag{1.2}$$

This definition of φ has a sense, from (a^*) it follows that if

$$\sum_j \lambda_j u_2(x_j) = \sum_k \mu_k u_2(\xi_k), \qquad (\text{with } \mu_k \in \mathbb{R} \text{ and } \xi_k \in X), \tag{1.3}$$

then

$$\sum_j \lambda_j u_1(x_j) = \sum_k \mu_k u_1(\xi_k). \tag{1.4}$$

In order to prove this observe that, in view of (a^*), for each $x_1^* \in X_1^*$ there is $x_2^* \in X_2^*$ such that $x_1^*(u_1(x)) = x_2^*(u_2(x))\ \forall x \in X$: in particular

$$\begin{cases} x_1^*(u_1(x_j)) = x_2^*(u_2(x_j)) \\ x_1^*(u_1(\xi_k)) = x_2^*(u_2(\xi_k)) \end{cases}$$

for all j and k, which implies

$$\begin{cases} x_1^*\Big(\sum_j \lambda_j u_1(x_j)\Big) = x_2^*\Big(\sum_j \lambda_j u_2(x_j)\Big) \\ x_1^*\Big(\sum_k \mu_k u_1(\xi_k)\Big) = x_2^*\Big(\sum_k \mu_k u_2(\xi_k)\Big). \end{cases}$$

Then, if (1.3) holds, then

$$x_1^*\Big(\sum_j \lambda_j u_1(x_j)\Big) - \Big(\sum_k \mu_k u_1(\xi_k)\Big) = 0$$

for all $x_1^* \in X_1^*$, and so (1.4) holds. □

Observe that, *if X is a linear space and the mappings u_1 and u_2 are linear, then (b^*) is equivalent to*

$$Ker\ u_2 \subseteq Ker\ u_1.$$

2. Testing with continuous linear forms.

For any locally convex (topological, linear) space X the usual notation will be used: in particular $\sigma(X, X')$ and $\tau(X, X')$ will denote the weak and the Mackey topologies on X with respect to the natural duality between X and its dual X'. X is called a *Mackey space* if its topology coincides with $\tau(X, X')$. X^* will denote the set of all linear forms on X, while $X^\times$ will denote the set of those linear forms on X that are bounded; so we have $X' \subseteq X^\times \subseteq X^*$.

The following proposition is well-known (see, for instance, A. Grothendieck [3, 2.16], or H. Jarchow [7, 8.6]).

PROPOSITION 2.1. *Let X, Y be Hausdorff locally convex spaces. For any linear mapping $\varphi : X \to Y$ the following three statements are equivalent:*

(I) *for each $y' \in Y'$ the map $y' \circ \varphi$ is continuous* [*i.e.* ${}^t\varphi(Y') \in X'$];
(II) *φ is weakly continuous* [*i.e. φ is continuous for the topologies $\sigma(X, X')$ on X and $\sigma(Y, Y')$ on Y*];
(III) *φ is Machey continuous* [*i.e. φ is continuous for the topologies $\tau(X, X')$ on X and $\tau(Y, Y')$ on Y*].

It is also well-known that *the bounded subsets of a Hausdorff locally convex space are the same for all locally convex Hausdorff topologies on X which are compatible with the natural duality between X and X'*. So, the following proposition holds.

PROPOSITION 2.2. *A subset B of a Hausdorff locally convex space X is bounded if and only if every element x' of X' is bounded on B* (*namely, if and only if B is bounded for the topology $\sigma(X, X')$*).

COROLLARY 2.3. *Let X, Y be locally convex spaces, with Y a Hausdorff space. A mapping $f: X \to Y$ is bounded [i.e., f maps each bounded subset of X to a bounded subset of Y] if and only if for each $y' \in Y'$, $y' \circ f$ is bounded.*

When X is a Hausdorff locally convex space, Y is a topological linear space and U is a subset of X, we say that a map $f: U \to Y$ is *Lipschitzian* (respectively, *locally Lipschitzian*) if for each absolutely convex, bounded subset B of X the set $\{(f(x_1) - f(x_2))/\|x_1 - x_2\|_B : x_1, x_2 \in U \cap X_B, x_1 \neq x_2\}$ is bounded in Y (respectively, locally bounded in Y). Here X_B denotes the linear subspace of X spanned by B, equipped with the norm $\|\cdot\|_B$ defined by $\|x\|_B = \inf\{\lambda > 0 : x \in \lambda B\}$. The set of all Lipschitzian maps from U into Y will be denoted by $Lip(U, Y)$.

Let us emphasize the following consequence of Proposition 2.2.

PROPOSITION 2.4. *Let X, Y be Hausdorff locally convex spaces, and let U be a subset of X. A map $f\colon U \to Y$ is Lipschitzian if and only if for each $y' \in Y'$ the (scalar) map $y' \circ f\colon U \to \mathbb{R}$ is Lipschitzian.*

PROOF. If $f\colon X \to Y$ is Lipschitzian and $y' \in Y'$ then, obviously, the composite $y' \circ f$ is Lipschitzian. Conversely, suppose that for each $y' \circ Y'$ the map $y' \circ f$ is Lipschitzian; then for each $y' \in Y'$ and each absolutely convex, bounded subset B of X there is a number $c > 0$ such that $|y'(f(x_1) - f(x_2))|/\|x_1 - x_2\|_B \le c \quad \forall x_1, x_2 \in U \cap X_B$ with $x_1 \ne x_2$, namely y' is bounded on the subset $\{f(x_1) - f(x_2)/\|x_1 - x_2\|_B : x_1, x_2 \in U \cap X_B, x_1 \ne x_2\}$ of Y. Then, in view of Proposition 2.2, such subset of Y is bounded, thus f is Lipschitzian. □

3. Surjectivity criteria.

THEOREM 3.1. *Let X be a set, let X_1, X_2 be Hausdorff locally convex spaces, and let $u_1\colon X \to X_1$, $u_2\colon X \to X_2$ be two maps. The following statements are equivalent:*

- (a') ${}^t u_1(X_1') \subseteq {}^t u_2(X_2')$, *(i.e. $\forall x_1' \in X_1' \ \exists x_2' \in X_2'$ such that $x_1' \circ u_1 = = x_2' \circ u_2$);*
- (b') *there is a unique weakly continuous, linear map $\varphi\colon \langle u_2(X)\rangle \to X_1$ such that $u_1 = \varphi \circ u_2$.*

Moreover, if the subspace $\langle u_2(X)\rangle$ of X_2, spanned by $u_2(X)$, is a Mackey space then the properties (a') and (b') are equivalent to

- (c') *there is a unique continuous, linear map $\varphi\colon \langle u_2(X)\rangle \to X_1$ such that $u_1 = \varphi \circ u_2$.*

PROOF. $(b') \Rightarrow (a')$ because if $\varphi\colon \langle u_2(X)\rangle \to X_1$ is a weakly continuous, linear map such thath $u_1 = \varphi \circ u_2$, then for each $x_1' \in V_1'$ then linear form $x_1' \circ \varphi$ on $\langle u_2(X)\rangle$ is continuous, and hence (by the Hahn-Banach theorem) it has a continuous, linear extension x_2' on X_2, which evidently is related to x_1' by the equality $x_1' \circ u_1 = x_2' \circ u_2$. Let us prove that $(a') \Rightarrow (b')$. Proceeding as in the proof of Remark 1.1 and using the Hahn-Banach theorem one shows that if (a') holds then there is a unique linear map $\varphi\colon \langle u_2(X) \to X_1$ such that $u_1 = \varphi \circ u_2$. Note that from (a') it follows also that for each $x_1' \in X_1'$ the linear form $x_1' \circ \varphi$ on $\langle u_2(X)\rangle$ is the restriction to $\langle u_2(X)\rangle$ of some element x_1'

of X_1', and so it is continuous; thus we have

$$x_1' \circ \varphi \in \langle u_2(X)\rangle' \quad \forall x_1' \in X_1',$$

which means, in view of Proposition 2.1, that φ is weakly continuous. It remains to prove that, if $\langle u_2(X)\rangle$ is a Mackey space, then (b') is equivalent to (c'). To do this it suffices to observe that if φ is continuous it is also weakly continuous, and that, in view of Proposition 2.1, φ is weakly continuous if and only if it is Mackey continuous. □

REMARK 3.2. *The fact that X_2 is a Mackey space does not imply that the subspace $\langle u_2(X)\rangle$ of X_2 is a Mackey space. However, the subspace $\langle u_2(X)\rangle$ of X_2 is a Mackey space provided one of the following conditions is satisfied:*

(i) *X_2 is metrizable;*
(ii) *X_2 is barrelled, and $\langle u_2(X)\rangle$ is a countable codimension linear subspace of X_2;*
(iii) *X_2 is bornological, and $\langle u_2(X)\rangle$ is a finite codimension linear subspace of X_2.*

PROOF. If X_2 is metrizable then $\langle u_2(X)\rangle$ is a Mackey space, because any metrizable locally convex space is a Mackey space. If (ii) holds then $\langle u_2(X)\rangle$ is a Mackey space, because a countable codimension subspace of a barreled space is barrelled (see J. C. Ferrando - M. Lopez Pellicer - L. M. Sanchez Rui [8, Prop. 1.1.15]), and so it is a Mackey space. Finally, (iii) implies that $\langle u_2(X)\rangle$ is bornological (see H. Jarchow [7, Theorem 13.5.2]), and hence it is a Mackey space. □

Taking, in Theorem 3.1, $X \subseteq X_2$ and $u =$ identity map, we obtain the following

COROLLARY 3.3 (Continuous, linear extension). *Let X_1, X_2 be Hausdorff locally convex spaces, and let X be a subset of X_2 such that the subspace $\langle X\rangle$ of X_2 generated by X is a Mackey space. A map $u\colon X \to X_1$ has a continuous, linear extension to $\langle X\rangle$ if and only if for each $x_1' \in X_1'$ the (scalar) map $x_1' \circ u$ has a continuous, linear extension to $\langle X\rangle$.*

In the case when X is a linear space and u_1, u_2 are linear, Theorem 3.1 yields

COROLLARY 3.4 (The linear case). *Let X be a linear space, let X_1, X_2 be Hausdorff locally convex spaces, let $u_1\colon X \to X_1$, $u_2\colon X \to X_2$ be linear maps, and let $\mathcal{P}_1$ be a family of seminorms on X defining the topology of X_1 and $\mathcal{P}_2$*

a filtering family of seminorms on X_2 defining the topology of X_2. If the subspace $\langle u_2(X)\rangle$ of X_2 is a Mackey space, then (a') is satisfied if and only if

(3.1) *for each $p_1 \in \mathcal{P}_1$ there is $p_2 \in \mathcal{P}_2$ and a number $c > 0$ such that $p_1(u_1(x)) \le c\,p_2(u_2(x))\ \forall x \in X$.*

PROOF. It suffices to observe that, when X is a linear space and u_1, u_2 are linear, the property (c') can be expressed in the form

> *Ker $u_2 \subseteq$ Ker u_1, and the map $u_2(x) \mapsto u_1(x)$, $x \in X$, from the subspace $u_2(X)$ of X_2 into X_1, is continuous,*

namely, in the from (3.1). □

We recall that the statement of Corollary 3.4 was proved by Valent [4]. It generalizes to the case of locally convex spaces a theorem proved by G. Fichera for Banach spaces (see G. Fichera [1], [2]). Note also that no completeness hypothesis on X_1 or X_2 is required in Corollary 3.3. (The proof by Fichera needs the completeness of X_1 and X_2, because it makes use of the closed graph theorem).

In the following theorem condition (a') is replaced by the (weaker) condition (a_H).

THEOREM 3.5. *Let X, X_1, X_2, u_1, u_2 be as in the statement of Theorem* 3.1, *and let H be a linear subspace of X_1. The following statements are equivalent*

- (a_H) ${}^t u_1(H^0) \subseteq {}^t u_2(X_2')$, *(i.e., for each $x_1' \in X_1'$ such that $x_1'(h) = 0$ $\forall h \in H$ there is $x_2' \in X_2'$ such that $x_1' \circ u_1 = x_2' \circ u_2$);*
- (b_H) *there is a unique weakly continuous linear map $\varphi_H : \langle u_2(X)\rangle \to$ $\to X_1/\overline{H}$ such that $\pi_H \circ u_1 = \varphi_H \circ u_2$, where π_H denotes the canonical projection of X_1 onto $X_1/\overline{H}$.*

Moreover, if the subspace $\langle u_2(X)\rangle$ of X_2 is a Mackey space, then (a_H) and (b_H) are equivalent to

- (c_H) *there is a unique continuous linear map $\varphi_H\colon \langle u_2(X)\rangle \to X_1/\overline{H}$ such that $\pi_H \circ u_1 = \varphi_H \circ u_2$.*

PROOF. It suffices to apply Thorem 3.1 with $\pi_H \circ u_1$ instead of u_1 and observe that

$$ {}^t(\pi_H \circ u_1)((X_1/\overline{H})') = {}^t u_1(H^0) $$

Remark 3.6. *A remarkable choice of H in the statement of Theorem 3.5 is*

$$H = \langle u_1(Ker\ u_2)\rangle.$$

For this choice of H the property (a_H) *becomes: for each* $x_1' \in X_1'$ *satisfying*

$$u_1(Ker\ u_2) \subseteq Ker\, x_1' \tag{3.2}$$

there is $x_2' \in X_2'$ *such that* $x_1' \circ u_1 = x_2' \circ u_2$. *Note that* (3.2) *is a necessary condition on* $x_1' \in X_1'$ *in order* (a') *to be satisfied.*

Theorem 3.7. *Let* X, X_1, X_2, u_1, u_2 *be as in the statement of Theorem* 3.1. *The following statements are equivalent:*

- $(a^\times)$ ${}^t u_1(X_1') \subseteq {}^t u_2(\langle u_2(X)\rangle^\times)$;
- $(b^\times)$ *there is a unique bounded, linear map* $\varphi\colon \langle u_2(X)\rangle \to X_1$ *such that* $u_1 = \varphi \circ u_2$.

Proof. Obviously $(b^\times)$ implies $(a^\times)$. In order to prove that $(a^\times)$ implies $(b^\times)$ one can essentially proceed as in the proof of Remark 1.1 in showing that if $(a^\times)$ holds then there is a unique linear map $\varphi\colon \langle u_2(X)\rangle \to X_1$ such that $u_1 = \varphi \circ u_2$. Then it suffices to observe that, by Corollary 2.3, $(a^\times)$ implies that the linear map φ is bounded. □

Remark 3.8. *From Theorem* 3.7 *a bounded, linear extension result like Corollary* 3.3 *can be deduced.*

4. Other surjectivity criteria. The Lipschitzian case.

Theorem 4.1. *Let* X, X_1, X_2, u_1, u_2 *be as in the statement of Theorem* 3.1. *The following statements are equivalent*

- (a_L) ${}^t u_1(X_1') \subseteq {}^t u_2(Lip\,(u_2(X), \mathbb{R}))$;
- (b_L) *there is a unique map* $f \in Lip\,(u_2(X), X_1)$ *such that* $u_1 = f \circ u_2$.

Proof. $(b_L) \Rightarrow (a_L)$ because $X_1' \subseteq Lip\,(X, \mathbb{R})$ and the composed of two Lipschitzian map is a Lipschitzian map. In order to prove that $(a_L) \Rightarrow (b_L)$ we first observe that (by the Hahn-Banach theorem) (a_L) implies that (1.1) holds and hence one can define a map $f\colon u_2(X) \to X_1$ by putting

$$f(u_2(x)) = u_1(x) \qquad \forall x \in X;$$

moreover, from (a_L) it follows that $x'_1 \circ f \in Lip\,(u_2(X), \mathbb{R})\ \forall x'_1 \in X'_1$, and by Proposition 2.4 this implies that *f* is Lipschitzian. □

A consequence of Theorem 4.1 is the following

COROLLARY 4.2. *Let X, X_1, X_2, u_1, u_2 be as in the statement of Theorem* 3.1, *and let $\mathcal{P}$ be a family of seminorms on X_1 defining its topology. Then the condition (a_L) in the statement of Theorem* 4.1 *is satisfied if and only if*

(4.1) *for each absolutely convex, bounded subset B of X_2 and each $p \in \mathcal{P}$ there is a number $c_{B,p} > 0$ such that*

$$p(u_1(x) - u_1(\xi)) \leq c_{B,p} \|u_2(x) - u_2(\xi)\|_B \qquad \forall x, \xi \in \overleftarrow{u_2}(X_{2,B}),$$

where $X_{2,B}$ is the linear subspace of X_2 spanned by B, equipped with the norm $\|\cdot\|_B$ defined by $\|x\|_B = inf\,\{\lambda > 0 \colon x \in \lambda B\}$.

PROOF. A subset of X_1 is bounded if and only if every element of $\mathcal{P}$ is bounded on it; then it is easy to see that the condition (b_L) in the statement of Theorem 4.1 is equivalent to (4.1). □

A result of type Theorem 3.5 for Lipschitzian maps is the following

THEOREM 4.3. *Let X, X_1, X_2, u_1, u_2, H, π_H be as in the statement of Theorem* 3.5. *The following statements are equivalent:*

(a_H) ${}^t u_1(H^0) \subseteq {}^t u_2(Lip\,(u_2(X), \mathbb{R}))$;

(b_H) *there is a unique Lipschitzian map $f_H \colon u_2(X) \to X_1/\overline{H}$ such that $\pi_H \circ u_1 = f \circ u_2$.*

Theorem 4.3 follows from Theorem 4.1 in the same way as Theorem 3.5 follows from Theorem 3.1.

REMARK 4.4. *The statements of Theorems* 4.1 *and* 4.3 *hold also when one replace Lipschitzian maps with locally Lipschitzian maps.*

5. Some particularizations of Theorems 3.5 and 4.1.

Let X, Y be Hausdorff locally convex spaces, and let $u \colon X \to Y$ be a weakly continuous linear map. So *Ker u* is weakly closed, namely closed in X. (Recall that the closure of a convex subset of X is the same for all

Hausdorff locally convex topologies on X which are compatible with the natural duality between X and X').

Let us denote by $\tilde{u}: X/Ker\,u \to u(X) \subseteq Y$ the natural bijection (associated with u) such that $u = \pi \circ \tilde{u}$ where π is the projection of X onto $X/Ker\,u$. Let $\tilde{\varphi}: u(X) \to X/Ker\,u$ be the inverse of $\tilde{u}$.

Clearly, u is open if and only if $\tilde{\varphi}$ is continuous; furthermore, u is weakly open if and only if $\tilde{\varphi}$ is weakly continuous. From Theorem 3.5 (applied to the case when $H = \langle Ker\,u \rangle$, u is the identity map $id: X \to X$ and $u_2 = u$) it follows that $\tilde{\varphi}$ is weakly continuous if and only if ${}^t id((id(Ker\,u))^0) \subseteq {}^t u(Y')$, i.e.,

$$(5.1) \qquad (Ker\,u)^0 \subseteq {}^t u(Y'),$$

where $(Ker\,u)^0 := \{x' \in X': Ker\,u \subseteq Ker\,x'\}$ is the polar of $Ker\ u$. As $(Ker\ u)^0$ coincides with the weak closure of ${}^t u(Y')$ in X', we get that u is weakly open if and only if ${}^t u(Y')$ is weakly closed in X'. From Theorem 3.5 it also follows that, if the subspace $u(X)$ of Y is a Mackey space, then $\tilde{\varphi}$ is continuous if and only if (5.1) is satisfied. Thus the following result holds.

COROLLARY 5.1. *A weakly continuous linear map $u : X \to Y$, with X, Y Hausdorff locally convex spaces, is weakly open if and only if ${}^t u(Y')$ is weakly closed in X'. If the subspace $u(X)$ of Y is a Mackey space, then u is open if and only if ${}^t u(Y')$ is weakly closed in X'.*

If X, Y are Fréchet spaces and u is continuous, then (in view of the open mapping theorem) u is open if and only if $u(X)$ is complete (namely closed) in Y. Therefore a consequence of Corollary 5.1 is

COROLLARY 5.2. *If X, Y are Fréchet spaces and $u: X \to Y$ is a continuous linear map, then $u(X)$ is closed in Y if and only if ${}^t u(Y')$ is weakly closed in X'.*

It is well-known that $u(X)$ is dense in Y if and only if the mapping ${}^t u: Y' \to X'$ is one-to-one. Then Corollary 5.2 yields immediately the following result which is known as the "theorem on the surjections of Fréchet spaces".

COROLLARY 5.3. *If X, Y are Fréchet spaces, and $u: X \to Y$ is a continuous linear map, then u is onto if and only if ${}^t u$ is one-to-one and ${}^t u(Y')$ is weakly closed in X'.*

Let us now consider a particularization of Theorem 4.1 (concerning the Lipschitzian case).

Corollary 5.4. *Let X, Y be Hausdorff locally convex spaces, and U a subset of X. For any map $u: X \to Y$ the following statements are equivalent:*

(j) *for each $x' \in X'$ there is $l \in Lip\,(u(U), \mathbb{R})$ such that $x' = l \circ u$;*
(jj) *u is one-to-one and its left inverse $u^{-1}: u(U) \to U$ is Lipschitzian.*

This corollary can be obtained by applying Theorem 4.1 to the case when $X_1 = X$, $X_2 = Y$, $u_1 =$ the identity map $id: U \to X$, and $u_2 = u$.

Let us now denote by $Lip_0(X, Y)$ [respectively $Lip_0(Y, X)$, and $Lip_0(Y, \mathbb{R})$] the set of elements f of $Lip\,(X, Y)$ [respectively $Lip\,(Y, X)$, and $Lip\,(Y, \mathbb{R})$] such that $f(0) = 0$.

Corollary 5.5. *Let $u \in Lip_0(X, Y)$, with X, Y Hausdorff locally convex spaces, and suppose that X is complete. The following statements are equivalent:*

$(j)_0$ *for each $x' \in X'$ there is a unique $l \in Lip_0(Y, \mathbb{R})$ such that $x' = l \circ u$;*
$(jj)_0$ *u is bijective, and $u^{-1} \in Lip_0(Y, X)$.*

Proof. Evidently $(jj)_0 \Rightarrow (j)_0$. Suppose that $(j)_0$ holds. Then, by Corollary 5.4, u is one-to-one and its left inverse $u^{-1}: u(X) \to X$ is Lipschitzian. It follows that $u(X)$ is complete and so it is closed in Y. It remains to prove that $u(X)$ is dense in Y. This is true, because if $y' \in Y'$ and $y' \circ u = 0$ then $y' = 0$ in view of $(j)_0$, and so $u(X)$ is dense in Y by the Hahn-Banach theorem. □

6. Other consequences of Theorem 3.5.

In this section we emphasize some consequences of Theorem 3.5 in the case when the subspace H of X_1 is the kernel of a continuous, linear map $v_1: X_1 \to Y$, with Y a Hausdorff locally convex space. Let us denote by $\overline{{}^t v_1(Y')}$ the closure of ${}^t v_1(Y')$ in $(X_1', \sigma(X_1', X_1))$.

Since

$$\overline{{}^t v_1(Y')} = (Ker\ v_1)^0, \tag{6.1}$$

from Theorem 3.5 it follows that, if the subspace $\langle u_2(X) \rangle$ of X_2 is a Mackey space, then the inclusion

(a_{v_1}) ${}^t u_1(\overline{{}^t v_1(Y')}) \subseteq {}^t u_2(X_2')$

is equivalent to the property

(c_{v_1}) *there is a unique continuos, linear map* $\varphi\colon \langle u_2(X)\rangle \to X_1/Ker\, v_1$ *such that* $\pi_1 \circ u_1 = \varphi \circ u_2$, *where* π_1 *denotes the canonical projection of* X_1 *onto* $X_1/Ker\, v_1$.

Observe that, if v_1 is one-to-one, then $\overline{{}^t v_1(Y')} = X_1'$; thus in this case the conditions (a_{v_1}) and (c_{v_1}) coincide with (a') and (c') respectively, and so they does not involve the map v_1.

THEOREM 6.1. *Let* X_1, X_2, Y *be Hausdorff locally convex spaces, let* X *be a set, let* $u_1\colon X \to X_1$ *and* $u_2\colon X \to X_2$ *be two maps, and let* $v_1\colon X_1 \to Y$ *be a continuous, linear map. If* X_1 *and* X_2 *are metrizable and complete, then* (a_{v_1}) *is satisfied if and only if*

(d_{v_1}) *there is a continuous linear map* $v_2\colon \overline{\langle u_2(X)\rangle} \to v_1(X_1)$ *such that* $v_1 \circ u_1 = v_2 \circ u_2$.

PROOF. Let X_1 and X_2 be metrizable and complete. Since the subspace $\langle u_2(X)\rangle$ of X_2 is metrizable and hence a Mackey space, from Theorem 6.2 it follows (as we have remarked above) that (a_{v_1}) is equivalent to (c_{v_1}). Then, in order to prove the equivalence of (a_{v_1}) and (d_{v_1}), we shall show that (c_{v_1}) is equivalent to (d_{v_1}). We have (c_{v_1}) $\Rightarrow$ (d_{v_1}) because (under our hypotheses) $X_1/Ker\, v_1$ is complete, and hence the continuous linear map $\varphi\colon \langle u_2(X)\rangle \to X_1/Ker\, v_1$ can be extended to a continuous linear map from $\overline{\langle u_2(X)\rangle}$ into $X_1/Ker\, v_1$. We now prove that (d_{v_1}) $\Rightarrow$ (c_{v_1}). Accordingly, we denote by $\tilde{v}_1\colon X/Ker\, v_1 \to v_1(X) \subseteq Y$ the bijection (associated with v_1) such that $v_1 = \pi_1 \circ \tilde{v}_1$, and consider the map

$$\tilde{v}_1^{-1} \circ v_1\colon \overline{\langle u_2(X)\rangle} \to X_1/Ker\, v_1 .$$

Note that the subspace $\overline{\langle u_2(X)\rangle}$ of X_2 and the quotient space $X_1/Ker\, v_1$ are both metrizable and complete. Moreover (using the fact that the map v_1 is continuous) it is easy to prove that the graph of the map $\tilde{v}_1^{-1} \circ v_1$ is closed. Thus, in view of the closed graph theorem, the map $\tilde{v}_1^{-1} \circ v_1$ is continuous. Then (c_{v_1}) is satisfied with φ the restriction on $\langle u_2(X)\rangle$ of the continuous linear map $\tilde{v}_1^{-1} \circ v_1$. □

Often, in concrete cases, two continuous linear maps $v_1\colon X_1 \to Y$ and $v_2\colon X_2 \to Y$ are assigned such that $v_1 \circ u_1 = v_2 \circ u_2$; thus condition ($d_{v_1}$) in the statement of Theorem 6.1 becames

$$v_2(\overline{\langle u_2(X)\rangle}) \subseteq v_1(X_1).$$

Therefore the following corollary of Theorem 6.1 can be useful. It fournishes another generalization of the "Theorem on the suriections of Fréchet spaces" (see Corollary 5.3).

Corollary 6.2. *Let X be a set, let X_1, X_2, Y be Hausdorff locally convex spaces, let $u_1: X \to X_1$, $u_2: X \to X_2$ be two maps, and let $v_1: X_1 \to Y$, $v_2: X_2 \to Y$ be continuous linear maps such that $v_1 \circ u_1 = v_2 \circ u_2$. If X_1, X_2 are metrizable and complete, then the following statements are equivalent:*

(1) ${}^t u_1(\overline{{}^t v_1(Y')}) \subseteq {}^t u_2(X_2')$;

(2) $v_2(\overline{\langle u_2(X)\rangle}) \subseteq v_1(X_1)$.

We observe that, if v_1 is one-to-one (i.e., if ${}^t v_1(Y')$ is weakly dense in X_1') and $\langle u_2(X)\rangle$ is dense in X_2, then (1) and (2) reduce respectively to

(3) ${}^t u_1(X_1') \subseteq {}^t u_2(X_2')$

and

(4) $v_2(X_2) \subseteq v_1(X_1)$.

We also remark that, taking in Corollary 6.2: $X = X_1$, $Y = X_2$, $u_1 = id_X$, $u_2 = v_1(= u)$, and $v_2 = id_Y$, one immediately obtains the statement of Corollary 5.2. Hence the so called "Theorem on the surjections of Fréchet spaces" is generalized by Corollary 6.2.

Corollary 6.3. *Let X, X_1, X_2, be Hausdorff locally convex spaces, let $v_1: X_1 \to Y$ be a continuous linear map, and let $u_2: X_1 \to X_2$ be a map. If X_1 and X_2 are metrizable and complete, then the following statements are equivalent:*

(5) $(Ker\ v_1)^0 \subseteq {}^t u_2(X_2')$, *(i.e., for each $x_1' \in X_1'$ such that $x_1'(h) = 0$ $\forall h \in Ker\ v_1$ there is $x_2' \in X_2'$ such that $x_1' = u \circ x_2'$);*

(6) *there is a continuous linear map $v_2: \overline{\langle u_2(X)\rangle} \to v_1(X_1)$ such that $v_1 = v_2 \circ u_2$.*

Proof. It suffices to take $X = X_1$ and $u_1 =$ identity map in the statement of Theorem 6.1, and recall (6.1). □

REFERENCES

[1] G. Fichera, *Alcuni recenti sviluppi della teoria dei problemi al contorno per le equazioni alle derivate parziali lineari*, "Atti del Convegno sulle Equazioni alle derivate parziali", Ed. Cremonese, Roma (1954), pp. 174–227.

[2] G. FICHERA, *Linear elliptic differential systems and eingevalue problems*, "Lecture Notes in Math.", 8 (1965), Springer-Verlag, Berlin-Heidelberg-New York.
[3] A. GROTHENDIECK, *Topological vector spaces* (1973), Gordon and Breach, New York, London, Paris.
[4] T. VALENT, *Su un principio di esistenza in analisi lineare*, Rend. Acc. Naz. dei Lincei, **66** (1979), pp. 331–337.
[5] G. ZAMPIERI, *Un'estensione del Teorema sulle suriezioni fra spazi di Fréchet. Qualche sua applicazione*, Rend. Sem. Mat. Univ. Padova, **61** (1979), pp. 145–153.
[6] G. ZAMPIERI, *On some conjectures by E. De Giorgi*, Rend. Sem. Mat. Univ. Padova, **62** (1980), pp. 115–128.
[7] H. JARCHOW, *Locally convex spaces* (1981), Teubner, Stuttgart.
[8] J. C. FERRANDO - M. LOPEZ PELLICER - L. M. SANCHEZ RUIZ, *Metrizable barreled spaces (1995)*, Longmann Group Limited England.

Manoscritto pervenuto in redazione il 29 novembre 2005