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On Locally Graded Non-Periodic Barely Transitive Groups.

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Dedicated to Prof A. O. Asar on his 65th birthday

ABSTRACT - We prove that both the Hirsch-Plotkin radical and the periodic radical of a point stabilizer in a simple locally graded non-periodic barely transitive group are trivial.

1. Introduction.

Let G be a permutation group on an infinite set Ω. If G acts transitively on Ω and every orbit of every proper subgroup of G is finite, then G is called a *barely transitive* group. This class of groups was considered for the first time by Hartley in a discussion of the Heineken-Mohamed groups.

Hartley mentioned in [10] that an infinite group G can be represented as a barely transitive group if and only if G has a subgroup H of infinite index such that $\bigcap_{x \in G} H^x = 1$ and $|K : K \cap H|$ is finite for every proper subgroup K of G.

Though there are many papers in the locally finite case (see [1]-[6]), not much research has appeared in the non-periodic case [11]. It is not known yet whether there exists a perfect locally finite barely transitive group (by [6, Theorem 1] if there exists one, it must be a locally finite p-group).

We denote the classes of barely transitive, torsion-free barely transitive, non-periodic barely transitive groups by BT, $TFBT$, $NPBT$ respectively.

In the present note we consider locally graded non-periodic barely transitive groups. Exploiting ideas in [9] (hence in [13]) we prove:

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THEOREM. *Let G be a locally graded simple NPBT-group with a point stabilizer H. Then*

a) H has no non-trivial periodic normal subgroup,
b) the Hirsch-Plotkin radical of H is trivial.

The theorem gives much information about H. For example H is neither soluble nor hypercentral. We also have that the Gruenberg radical (hence the Baer radical) of H is trivial by [15, 12.1.4] and (b) of the theorem.
It is also not known that whether there exists a non-periodic barely transitive group.

COROLLARY. *If G is a locally graded TFBT-group with a point stabilizer H, then the Hirsch-Plotkin radical of H is trivial.*

Of course (b) holds under the above hypothesis. Also in [11] it is proved that the FC-centre of a point stabilizer H in a *TFBT*-group is trivial.

2. Preliminary Lemmas.

In this section we give some facts in the general and non-periodic cases that are needed in the sequel and proved in [12] in the locally finite case.

LEMMA 2.1. *Let G be a locally graded BT-group with a point stabilizer H. If there exists a maximal subgroup M of G, then H is not contained in M.*

PROOF. (c.f. [12, Lemma 2.10] and [8, Lemma 5]) Suppose that $H \leq M$ and that $x \in G$ then $|M : M \cap M^x|$ and $|M^x : M \cap M^x|$ are finite. Put $K = M \cap M^x$, so that there exist finitely generated subgroups X and Y such that $M = XK = KX$ and $M^x = YK = KY$. Hence

$$\langle M^x, M\rangle = \langle X, Y, K\rangle = \langle X, Y\rangle K = K\langle X, Y\rangle.$$

Since G is locally graded, $\langle X, Y\rangle$ is proper in G, and hence

$$|\langle X, Y\rangle : \langle X, Y\rangle \cap K|$$

is finite. Thus $|\langle M^x, M\rangle : K|$ is finite. Since $|K : H \cap H^x|$ is finite, it follows that $|\langle M^x, M\rangle : H|$ is finite. Hence we have that M is a normal subgroup of G of finite index. But this is a contradiction. □

If G is a *TFBT*-group, then by [11, Proposition 1] G is simple. So the study of *TFBT*-groups restricts to the study of simple ones.

Let G be a BT-group. If G is not finitely generated, then G is locally graded. For, let F be a non-trivial finitely generated subgroup of G. Then $F \neq G$ by hypothesis. But by [12] every proper subgroup of G is residually finite and thus F has a proper subgroup of finite index, i.e., G is locally graded. This easy observation shows that non-finitely generated groups are our objects to study.

We also note that if G is an $NPBT$- group, then by [5, Proposition 2] G is perfect.

Arguing exactly as in [12, Lemma 2.13 (i)], we see that every proper subgroup of a BT-group is residually finite.

3. The proof of the theorem.

A subgroup H of a group G is called *inert* if $|H : H \cap H^g|$ is finite for all $g \in G$. In a barely transitive group the point stabilizers are inert and in the following proof it will be useful to use the results related to inert subgroups which are due to Belyaev.

A group G is called a *minimal non* **X**-*group* for a class of groups **X** if every proper subgroup of G is an **X**-group but G itself not. In the following proof we shall use minimal non FC-groups.

Proof of the Theorem. We first show that G is countable. If

$$y_1H, y_2H, \dots, y_iH, \dots$$

are distinct left cosets of H in G, then $G = \langle y_i : i \geq 1 \rangle$. Put $M_i = \langle y_1, \dots, y_i \rangle$ for all $i \geq 1$. Since for every $i \geq 1$; $y_i \in M_i$ and $M_i \leq M_{i+1}$, $G = \bigcup\limits_{i=1}^{\infty} M_i$. We also have that each M_i is countable, since it is finitely generated. Consequently, G is countable and hence we can write $G = \{x_i : i \geq 1\}$.

Now we argue as in the proof of [9, Lemma 4]. Put $Y_n = \langle x_1, \dots, x_n \rangle$, $X_n = \langle H, Y_n \rangle$, $R_n = Core_{X_n}(H)$ for $n \geq 1$. Then by Lemma 2.1, H is not a maximal subgroup of G. Following the proof of [12, Lemma 2.10] we see that $X_n \neq G$. Now $G = \bigcup\limits_{i=1}^{\infty} Y_i$ as above, $R_n \lhd X_n$, $|X_n/R_n|$ is finite for all $n \geq 1$, since $|X_n : H|$ is finite.

Let $\Re$ denote an inert-radical class of groups in a subgroup-closed class of groups (see [7, 3.1. Definition]). First suppose for the contrary that $\Re(H) \neq 1$ and in addition that $\Re(H)$ is infinite.

Assume that there exists n such that $\Re(R_n) = 1$, so that $\Re(H) \cap R_n = 1$. It follows that $\Re(H)$ is finite, a contradiction. Thus $\Re(R_n) \neq 1$ for all $n \geq 1$. Since $R_n \lhd X_n$, $\Re(X_n) \neq 1$ for all $n \geq 1$.

Let U be a finitely generated subgroup of G. By [9, Lemma 3] there is a perfect group P and a non-zero integer i such that $U \leq P \leq Y_i \leq X_i$. By [7, 4.3 Theorem], $X_i/\Re(X_i)$ is an FC-group for each $i \geq 1$. Then by [14, Theorem 4.32], $X_i{}'\Re(X_i)/\Re(X_i)$ is locally finite, i.e., $X_i{}'$ is $\Re$-by-locally finite. Also we have $U \leq X_i{}'$, since P is perfect. Consequently, U is $\Re$-by-finite, i.e., G is locally ($\Re$-by-finite). Hence for each $i \geq 1$, Y_i is $\Re$-by-finite.

Now assume that H has a non-trivial normal periodic subgroup, then $T(H) \neq 1$. Let us take $\Re(H) = T(H)$, where $T(H)$ is the periodic radical of H. Now we have that every finitely generated subgroup of G is periodic, i.e., G is periodic. But this is a contradiction. So $T(H)$ is finite and it follows that $FC_G(H) \neq 1$ and hence by [7, 2.2. Corollary], $FC_G(H) = G$. Following the proof of [11, Proposition 2] we see that G is a minimal non FC-group. Now since two torsion elements generate a finite group, the set of all torsion elements $T(G)$ is a subgroup of G. Hence $T(G) = 1$, i.e., G is torsion-free. It follows that $T(H) = 1$, a contradiction. So (a) is proved.

We continue the proof of (b) taking $S = \Re(H)$, the Hirsch-Plotkin radical of H. If S is infinite, then for each $i \geq 1$, Y_i is nilpotent-by-finite. Hence every subgroup of Y_i is finitely generated. Recall that H/S is an FC-group. Put $T/S = Z(H/S)$, then H/T is locally finite and T is locally nilpotent-by-abelian. By hypothesis $|Y_n : Y_n \cap H|$ is finite. Since $|H \cap Y_n : T \cap Y_n|$ is finite, $|Y_n : T_n|$ is finite, and T_n is nilpotent-by-abelian. Since Y_n is almost nilpotent, it has a normal nilpotent subgroup V_n such that $V_n \leq T_n$ and hence there exists $L_n \lhd Y_n$ with $L_n \leq V_n$, L_n torsion-free and $|Y_n : L_n|$ is finite. Next follow the proof of [9, Theorem 2] to reach a contradiction.

Now we have that S is finite. Arguing as in (a) we see that $S = 1$. This contradiction completes the proof of (b). □

The Proof of the Corollary. Since G is simple, the result follows by the theorem.

If G is a non-simple locally graded $NPBT$-group with a point stabilizer H, then since every proper normal subgroup of G is locally finite by [11, Proposition 1], G contains a unique maximal proper normal subgroup. As we shall see this maximal normal subgroup of G contains some non-trivial normal subgroups of H.

COROLLARY 3.1. *Let G be a locally graded NPBT-group with a point stabilizer H and let N be the maximal normal subgroup of G. Then the Hirsch-Plotkin radical of H is contained in N.*

PROOF. Suppose that the Hirsch-Plotkin radical S of H is non-trivial. We also have that G/N is a simple non-periodic group. Since G has no proper subgroup of finite index, HN/N is a proper subgroup of G/N of infinite index. Put $L/N = Core_{G/N}HN/N$. Since N is the unique maximal proper normal subgroup of G, $L \leq N$. Hence G/N is a BT-group with a point stabilizer HN/N. We will also show that G/N is locally graded. By the observation following the proof of Lemma 2.1, it is enough to show that G/N is not finitely generated. Now assume that G/N is finitely generated, then G has a finitely generated subgroup F such that $G = FN$. Arguing as in the proof of [11, Lemma 2.10] it can be shown that if $F \neq G$, then $G \neq FN$. Hence $G = F$, i.e., G is finitely generated. This contradicts the fact that G is locally graded. Hence G/N is a simple locally graded $NPBT$-group with point stabilizer HN/N. Then by Theorem Hirsch-Plotkin radical of HN/N is trivial. But SN/N is a normal locally nilpotent subgroup of HN/N. Hence $S \leq N$. □

PROPOSITION 3.2. *Let G be a non-simple NPBT-group with a point stabilizer H. Let N be the maximal normal subgroup of G. If N is infinite, then the periodic radical $T(H)$ is contained in N and $|N : T(H)|$ is finite.*

PROOF. Since N is infinite, $N \cap H$ is an infinite periodic normal subgroup of H, hence $T(H) \neq 1$. Since G/N is simple $NPBT$-group, by the Theorem $T(H) \leq N \cap H$. It follows that $N \cap H = T(H)$ and that $|N : T(H)|$ is finite. □

PROPOSITION 3.3. *Let G be a locally graded NPBT-group with a torsion-free point stabilizer. Then $G/Z(G)$ is simple.*

PROOF. Let H be a torsion-free point stabilizer of G and assume that N is the maximal non-trivial proper normal subgroup of G. Then N is locally finite and hence $N \cap H = 1$ by hypothesis. It follows that N is finite and hence N is contained in $Z(G)$. Now we have that $N = Z(G)$. Consequently, $G/Z(G)$ is simple. □

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