

A note on Torsion-by-Nilpotent Groups.

TAREK ROUABHI (*) - NADIR TRABELSI (**)

ABSTRACT - In this note we prove that a finitely generated soluble-by-finite group G is torsion-by-nilpotent if and only if every infinite subset contains two distinct elements x, y such that $\langle x, x^y \rangle$ is torsion-by-nilpotent.

1. Introduction and results.

Let \mathcal{X} be a class of groups. Denote by (\mathcal{X}, ∞) (respectively, $(\mathcal{X}, \infty)^*$) the class of groups in which every infinite subset contains two distinct elements x, y such that $\langle x, y \rangle$ (respectively, $\langle x, x^y \rangle$) belongs to \mathcal{X} . The second author proves in [7] that a finitely generated soluble group in the class (\mathcal{TN}, ∞) (respectively, $(\mathcal{CN}, \infty)^*$) is torsion-by-nilpotent (respectively, finite-by-nilpotent), where \mathcal{N} (respectively, \mathcal{C}, \mathcal{T}) denotes the class of nilpotent (respectively, Chernikov, torsion) groups. In this note we extend the result obtained on (\mathcal{TN}, ∞) to the class $(\mathcal{TN}, \infty)^*$. Many results have been obtained on (\mathcal{X}, ∞) for various classes of groups \mathcal{X} , see for example the papers referred to in [7]. Recall that the origin of this type of problems is a result of B. H. Neumann [5] which asserts that a group is centre-by-finite if and only if every infinite subset contains a commuting pair of distinct elements. Our main result is as follows:

THEOREM 1.1. A finitely generated soluble-by-finite group in the class $(\mathcal{TN}, \infty)^*$ is torsion-by-nilpotent.

(*) Indirizzo dell'A.: Department of Mathematics, Faculty of Sciences, University of Setif, Algeria.

E-mail: rtarek5@yahoo.fr

(**) Department of Mathematics, Faculty of Sciences, University of Setif, Algeria.

E-mail: nadir_trabelsi@yahoo.fr

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Note that Theorem 1.1 is not true for arbitrary groups. Indeed, in [3, Corollary 4.2] it is proved that for every integer $d \geq 2$, there exists a non nilpotent residually nilpotent torsion-free group generated by d elements such that any $d - 1$ elements generate a nilpotent group.

Let k be a positive integer and let $\mathcal{E}_k(\infty)$ be the class of groups in which every infinite subset contains two distinct elements x, y such that $[x, {}_k y] = 1$. In [1] Abdollahi proved that a finitely generated metabelian group G is in the class $\mathcal{E}_k(\infty)$ if, and only if, $G/Z_k(G)$ is finite, and if G is a finitely generated soluble group in the class $\mathcal{E}_k(\infty)$, then there exists an integer $c = c(k)$, depending only on k , such that $G/Z_c(G)$ is finite. Note that $(\mathcal{N}_k, \infty)^*$ is contained in $\mathcal{E}_{k+1}(\infty)$, where \mathcal{N}_k stands for the class of nilpotent groups of class at most k . Combining the results of [1] and Theorem 1.1, we shall obtain the following consequences.

COROLLARY 1.2. Let k be a positive integer.

(i) If G is a finitely generated soluble-by-finite group in the class $(\mathcal{TN}_k, \infty)^*$, then there exists an integer $c = c(k)$, depending only on k , such that G belongs to \mathcal{TN}_c .

(ii) A finitely generated metabelian-by-finite group is in the class $(\mathcal{TN}_k, \infty)^*$ if, and only if, it belongs to \mathcal{TN}_{k+1} .

2. Proofs of the results.

In [4, Theorem 1] Longobardi and Maj proved that a finitely generated soluble group in the class $\mathcal{E}(\infty)$ is finite-by-nilpotent, where $\mathcal{E}(\infty)$ denotes the class of groups in which every infinite subset contains two distinct elements x, y such that $[x, {}_n y] = 1$ for some integer $n = n(x, y)$. To prove Theorem 1.1, we extend this result to finitely generated soluble-by-finite groups.

LEMMA 2.1. A finitely generated soluble-by-finite group in the class $\mathcal{E}(\infty)$ is finite-by-nilpotent.

PROOF. Let G be an infinite finitely generated soluble-by-finite group in the class $\mathcal{E}(\infty)$ and let H be a normal soluble subgroup of finite index. First of all note that by [4, Theorem 1] G is a finitely generated nilpotent-by-finite group, so it satisfies the maximal condition on subgroups and therefore all its subgroups are finitely generated. To prove

our lemma we proceed by induction on the derived length d of H . If $d = 1$, then G is a finitely generated abelian-by-finite group, so it has a normal torsion-free abelian subgroup A of finite index. Let $1 \neq a \in A$ and let $g \in G$, then the subset $\{a^i g : i > 0\}$ is infinite. So there are two distinct positive integers m, n such that $[a^m g, a^n g] = 1$ for some positive integer c . Since A is normal and abelian we obtain that $[a, c g]^{m-n} = 1$, and this gives $[a, c g] = 1$ as A is torsion-free. It follows that a is a right Engel element of G . Since G satisfies the maximal condition on subgroups, the set of its right Engel elements coincides with a term of the upper central series [6, Theorem 7.21]. Hence $A \leq Z_i(G)$, for some integer $i > 0$. So $G/Z_i(G)$ is finite, and this gives that G is finite-by-nilpotent by a result of Baer [6, Corollary 2 of Theorem 4.21]. Now assume that $d > 1$. Then by the inductive hypothesis $G/H^{(d-1)}$ is finite-by-nilpotent. So G belongs to the class $(\mathcal{A}\mathcal{F})\mathcal{N}$, where \mathcal{A} (respectively, \mathcal{F}) stands for the class of abelian (respectively, finite) groups. Hence G is in the class $(\mathcal{F}\mathcal{N})\mathcal{N}$ by the first part of the proof. It follows that G is a finitely generated finite-by-soluble group. Consequently by [4, Theorem 1] G is finite-by-nilpotent, as required. \square

LEMMA 2.2. Let G be a soluble-by-finite group in the class $(\mathcal{TN}, \infty)^*$. If G is abelian-by-torsion then it is torsion-by-abelian.

PROOF. Let G be a soluble-by-finite group in the class $(\mathcal{TN}, \infty)^*$ and suppose that G is abelian-by-torsion. Firstly, we show that G has a torsion subgroup. Let $x, y \in G$ be two elements of finite order. Then $H = \langle x, y \rangle$ is a finitely generated soluble-by-finite group which belongs to \mathcal{AT} , so it is abelian-by-finite. Clearly we may assume that H is infinite. Therefore H has a torsion-free normal abelian subgroup A of finite index. Let $1 \neq a \in A$ and let $h \in H$, then the subset $\{a^i h : i > 0\}$ is infinite. Therefore there are two distinct positive integers m, n such that $\langle a^m h, (a^n h)^{a^m h} \rangle$ is torsion-by-nilpotent, so $\langle [a^m h, a^n h], a^n h \rangle$ is also torsion-by-nilpotent. Hence $[[a^m h, a^n h], c a^n h]^d = 1$ for some positive integers c, d . Since A is abelian and normal in H , we obtain that $[a, c+1 h]^{(m-n)d} = 1$ and this gives that $[a, c+1 h] = 1$ as A is torsion-free. It follows that A is contained in $R(H)$ the set of right Engel elements of H . Since H satisfies the maximal condition on subgroups, $R(H)$ coincides with $Z_i(H)$ for some integer $i > 0$ [6, Theorem 7.21] and hence $H/Z_i(H)$ is finite. Thus by a result of Baer [6, Corollary 2 of Theorem 4.21] H is a finite-by-nilpotent group generated by two elements of finite order. So H is finite, which is a contradiction. Consequently H is a finite group and this means that G

has a torsion subgroup T , as claimed. Now G/T is a torsion-free group in the class $(\mathcal{TN}, \infty)^*$. Therefore G/T belongs to $(\mathcal{N}, \infty)^*$ which is contained in $\mathcal{E}(\infty)$. It follows that G/T is a soluble-by-finite group in the class $\mathcal{E}(\infty)$ and this gives, by Lemma 2.1, that G/T is locally finite-by-nilpotent. Therefore G/T is a locally nilpotent torsion-free group in the class \mathcal{AT} ; so by [6, Lemma 6.33] G/T is abelian. It follows that G is torsion-by-abelian, as claimed. \square

PROOF OF THEOREM 1.1. Let G be a finitely generated soluble-by-finite group in the class $(\mathcal{TN}, \infty)^*$ and let H be a normal soluble subgroup of finite index. We proceed by induction on the derived length d of H . From Lemma 2.2, this is true if $d = 1$, so we can assume that $d > 1$. By the inductive hypothesis we have that $G/H^{(d-1)}$ is torsion-by-nilpotent. Thus G is in the class $(\mathcal{AT})\mathcal{N}$, and by Lemma 2.2 it belongs to $(\mathcal{TA})\mathcal{N}$. Let N be a normal torsion-by-abelian subgroup of G such that G/N is nilpotent and let T be the torsion subgroup of N . Clearly, T is normal in G and G/T is (torsion-free abelian)-by-nilpotent. Since $(\mathcal{TN}, \infty)^*$ is a quotient closed class of groups, we may, therefore, suppose G (torsion-free abelian)-by-nilpotent. Let A be a normal torsion-free abelian subgroup of G such that G/A is nilpotent. Let $1 \neq a \in A$ and let $g \in G$, then the subset $\{a^i g : i > 0\}$ is infinite. Hence there are two distinct positive integers m, n such that $\langle a^n g, (a^n g)^{a^m g} \rangle$ is torsion-by-nilpotent. So $\langle [a^m g, a^n g], a^n g \rangle$ is also torsion-by-nilpotent. Thus $[[a^m g, a^n g]_c, a^n g]^d = 1$ for some positive integers c and d . Therefore $[a_{c+1} g]^{(m-n)d} = 1$ as A is abelian and normal in G , and this gives that $[a_{c+1} g] = 1$ as A is torsion-free. It follows that a is a right Engel element of G . Since G is a finitely generated soluble group, the set of its right Engel elements coincides with its hypercenter $\overline{Z}(G)$ [2, Theorem A]. Consequently $A \leq \overline{Z}(G)$, hence $G/\overline{Z}(G)$ is nilpotent from which we deduce that G is hypercentral. Therefore G is nilpotent since it is finitely generated. \square

PROOF OF COROLLARY 1.2. Let k be a positive integer.

(i) It suffices to factor G by its torsion subgroup and to apply [1, Theorem 3].

(ii) Let T be the torsion subgroup of G . Then by [1, Theorem 2] G/T is a torsion-free nilpotent group in the class $\mathcal{N}_{k+1}\mathcal{F}$. So by [6, Lemma 6.33] G/T is in \mathcal{N}_{k+1} . Hence G belongs to \mathcal{TN}_{k+1} . The converse follows from the fact that $\langle x, x^y \rangle$ is contained in $\langle x, \langle x, y \rangle' \rangle$. Therefore if $\langle x, y \rangle$ belongs to \mathcal{TN}_{k+1} , then $\langle x, x^y \rangle$ is in \mathcal{TN}_k . \square

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