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Maximal Mordell-Weil lattices of fibred surfaces with $p_g = q = 0$.

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Dedicated to Professor Yoshihara Hisao on his sixtieth birthday

ABSTRACT - Slope inequalities are given for fibred regular surfaces with geometric genus zero according as the Clifford index of a general fibre. In the case where the slopes attain the minimums, such fibred surfaces are rational, whose constructions are described. Furthermore, the maximal Mordell-Weil lattices of the fibred rational surfaces are completely determined.

1. Introduction.

We shall work over the complex number field $\mathbb{C}$. Let X be a smooth projective surface with $p_g = q = 0$, where p_g and q respectively denote as usual the geometric genus and the irregularity of X. Let $f : X \to \mathbb{P}^1$ be a relatively minimal fibration whose general fibre F is a smooth projective curve of genus g and of Clifford index c. After Shioda ([13]) introduced and developed the theory of the Mordell-Weil lattice, several attempts have been made to clarify the Mordell-Weil lattices for higher genus fibrations. For example, the cases where $c = 0, 1$ and 2 are respectively studied in [12], [11] and [8], where the maximal Mordell-Weil lattices are completely determined (see also [7]). The present article is an extension of them.

THEOREM 1.1. *Keep the same notation as above. Assume that $c \geq 3$. Let r be the Mordell-Weil rank of $f : X \to \mathbb{P}^1$. Then the following hold*:

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(1) *When* $g > (c+2)(c+3)/2$,

$$r \leq 2(c+2)(g+c+1)/(c+1).$$

If the right-hand side is an integer, then there exists f *which attains the equality except when c is odd,* g *is a multiple of* $(c+1)$ *and* $g < (c+1)^2$.

(2) *When* $(c+1)(c+2)/2 \leq g \leq (c+2)(c+3)/2$,

$$r \leq 3g - (c+3)(c-4)/2.$$

For all g, *there exists* f *which attains the equality.*

Furthermore, in both cases, X *is a rational surface if* r *attains the above maximum.*

We consider fibred rational surfaces whose Mordell-Weil ranks attain the maximums as in Theorem 1.1. Then we have an explicit description of the Néron-Severi group $\mathrm{NS}(X)$. Thus, for such fibrations, the Mordell-Weil lattices are completely determined and the corresponding Dynkin diagrams are expressed in terms of c and g. In particular, for the case (1), they are certain extensions of E_8 (see Theorem 3.7).

In order to show Theorem 1.1, we apply the same method as in [8], where we take a reduction (Y, G) of (X, F) and give a lower bound for $(K_Y + G)^2$. In Theorem 2.3, we have two slope inequalities, which give upper bounds of the Mordell-Weil rank as in Theorem 1.1. If r attains the bound, then any fibres of f must be irreducible. In the first of §3, we show such fibrations are indeed constructed from (Y, G) with the minimal $(K_Y + G)^2$ under the suitable conditions.

When $g < (c+1)(c+2)/2$, which Theorem 1.1 does not cover, we have a loose bound $r \leq ((2c+8)g + 6c + 8)/(c+2)$ in the same way as in Theorem 1.1. The proof of Theorem 2.3 shows that the case where $g < (c+1)(c+2)/2$ is complicated. If $c \geq 4$ and $g = (c+1)(c+2)/2$, then there in fact exist two kinds of fibred rational surfaces with the minimal slopes whose structures are distinct for the plane curve models of F. One of them is a plane curve of degree $(c+4)$ having $(c+2)$ double points as its singularity. Another G is near to a multi-anti-canonical divisor on Y which is obtained by blowing $\mathbb{P}^2$ up at five points.

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2. Slope inequalities

Let X be a smooth projective surface with $p_g = q = 0$ and $f : X \to \mathbb{P}^1$ a relatively minimal fibration of curves of genus $g \geq 2$. We briefly review basic notation and results of such fibred surfaces while considering the adjoint bundle $K_X + F$ according to [8, §1]. It is known that $K_{X/\mathbb{P}^1} = K_X + 2F$ is nef and we have $(K_X + 2F).C = 0$ for an irreducible curve C on X if and only if C is a (-2)-curve contained in a fibre of f. Since $p_g = q = 0$, we have that $K_X + F = K_{X/\mathbb{P}^1} - F$ is also nef and $h^0(X, K_X + F) = g$. Remark that X is automatically a rational surface when $(K_X + F)^2 < 2g - 2$ (see [8, Lemma 1.1]).

Lemma 2.1 [cf. [8, Lemma 1.2]]. *Let C be an irreducible curve on X such that $(K_X + F).C = 0$. If $(K_X + F)^2 > 0$, then C is a smooth rational curve satisfying one of the following*:

(1) *C is a (-2)-curve contained in a fibre.*
(2) *C is a (-1)-section, i.e., a (-1)-curve with $F.C = 1$.*

Suppose that there exists a (-1)-curve E with $(K_X + F).E = 0$ and let $\mu_1 : X \to X_1$ be its contraction. Since $F.E = 1$, $F_1 := (\mu_1)_* F$ is smooth on X_1. Furthermore, we have $\mu_1^*(K_{X_1} + F_1) = K_X + F$. If there exists a (-1)-curve E_1 with $(K_{X_1} + F_1).E_1 = 0$, then, by contracting it, we get the pair (X_2, F_2) with F_2 smooth and $K_{X_2} + F_2$ pulls back to $K_X + F$. We can continue the procedure until we arrive at a pair (X_n, F_n) such that we cannot find a (-1)-curve E_n with $(K_{X_n} + F_n).E_n = 0$. We put $Y := X_n$ and $G := F_n$. If $\mu : X \to Y$ denotes the natural map, then $\mu^*(K_Y + G) = K_X + F$ and $G = \mu_* F$ is a smooth curve isomorphic to F. The original fibration $f : X \to \mathbb{P}^1$ corresponds to a pencil $\Lambda_f \subset |G|$ with at most simple (but not necessarily transversal) base points. When $(K_X + F)^2 > 0$, $K_X + F$ is nef and big. This implies that, Y is the minimal resolution of singularities of the surface $\mathrm{Proj}(R(X, K_X + F))$, which has at most rational double points by Lemma 2.1, where $R(X, K_X + F) = \bigoplus_{n \geq 0} H^0(X, n(K_X + F))$. Therefore, such a model is uniquely determined. We call the pair (Y, G) the *reduction* of (X, F).

We consider the rational map $\Phi_{|K_X+F|} : X \to \mathbb{P}^{g-1}$ defined by $|K_X + F|$. If $|K_X + F|$ is composed of a pencil, then $f : X \to \mathbb{P}^1$ is a hyperelliptic fibration (cf. [8, Lemma 1.3]). Furthermore, we have the following:

PROPOSITION 2.2 [cf. [8, Lemma 1.3 and Proposition 1.1]]. Assume that $0 < (K_X + F)^2 \leq 2g - 5$. Then $\Phi_{|K_X+F|}$ defines a birational morphism onto the image, which factors through the reduction of (X, F).

Before proceeding further, we recall here two important invariants of a smooth projective curve C of genus $g(C) \geq 4$. The gonality $\mathrm{gon}(C)$ of C is given as the minimum of the degrees of surjective morphisms of C to $\mathbb{P}^1$. The existence theorem on special divisors (e.g. [1]) implies that $\mathrm{gon}(C) \leq (g(C) + 3)/2$. The Clifford index of C is defined as

$$\mathrm{Cliff}(C) = \min\{\deg L - 2h^0(L) + 2 \mid L \in \mathrm{Pic}(C),\ h^0(L) > 1,\ h^1(L) > 1\}.$$

These two invariants are closely related to each other. It is known that $\mathrm{Cliff}(C) = \mathrm{gon}(C) - 2$ or $\mathrm{gon}(C) - 3$ and in the latter case we have infinitely many $g^1_{\mathrm{gon}(C)}$'s on C. For this and further properties, see [2].

We now return to the situation we are interested in. The slope of f is defined by $\lambda_f = K^2_{X/\mathbb{P}^1} / \deg f_* K_{X/\mathbb{P}^1}$. We have the following lower bounds:

THEOREM 2.3. *Let X be a smooth projective surface with $p_g = q = 0$ and $f : X \to \mathbb{P}^1$ a relatively minimal fibration of genus g and of Clifford index $c \geq 3$. Let (Y, G) denote the reduction of (X, F). Then the following hold*:

When $g > (c + 2)(c + 3)/2$,

$$\lambda_f \geq 5 + (c - 1)/(c + 1) - (2c + 4)/g. \tag{2.1}$$

If λ_f attains the equality in (2.1), then Y is the Hirzebruch surface Σ_d of degree d.

When $(c + 1)(c + 2)/2 \leq g \leq (c + 2)(c + 3)/2$,

$$\lambda_f \geq 5 + (c + 3)(c - 4)/(2g). \tag{2.2}$$

If λ_f attains the equality in (2.2), then there exists a blow-down $\upsilon_0 : (Y, G) \to (\mathbb{P}^2, G_0)$ such that G_0 is one of the following:

(i) *a curve of degree $(3c + 6)/2$ with four $(c + 2)/2$-ple points and a $c/2$-ple point, where c is even and $g = (c + 1)(c + 2)/2$,*

(ii) *a curve of degree $(3c + 5)/2$ with five $(c + 1)/2$-ple points, where c is odd, $c \geq 5$ and $g = (c + 1)(c + 2)/2$,*

(iii) *a curve of degree nine with four triple points, where $c = 4$ and $g = 16$,*

(iv) *a curve of degree $(c + 4)$ with $(n + 1)$ double points, where $g = (c + 1)(c + 4)/2 - n$ and $-1 \leq n \leq c + 1$.*

In particular, X is a rational surface if λ_f *attains the equality in* (2.1) *or* (2.2).

In order to show Theorem 2.3, from $g\lambda_f = (K_X + F)^2 + 4(g-1)$ and Proposition 2.2, we only have to prove the following:

PROPOSITION 2.4. *Let* Y *be a smooth rational surface and* G *a smooth irreducible curve of genus* g *and of Clifford index* $c \geq 3$ *on* Y. *Assume that* $(K_Y + G).E \geq 1$ *for any* (-1)*-curves* E *on* Y. *Then the following hold*:

When $g > (c+2)(c+3)/2$,

$$(K_Y + G)^2 \geq 2c(g-c-1)/(c+1).$$

If the equality sign holds, then $Y = \Sigma_d$.

When $(c+1)(c+2)/2 \leq g \leq (c+2)(c+3)/2$,

$$(K_Y + G)^2 \geq g + (c^2 - c - 4)/2\,.$$

If $(K_Y + G)^2$ *attains the equality, then there exists a blow-down* $\upsilon_0 : (Y, G) \to \to (\mathbb{P}^2, G_0)$ *such that* G_0 *is a curve with the same degree and singularity as in* (i)–(iv) *of* Theorem 2.3.

First of all, in the case $Y = \mathbb{P}^2$, it is well-known that G is a smooth plane curve of degree $(c+4)$. Therefore, we have $g = (c+2)(c+3)/2$ and $(K_Y + G)^2 = (c+1)^2 = g + (c^2 - c - 4)/2$. Next, in the case $Y = \Sigma_d$, we describe explicitly as follows:

LEMMA 2.5. *Let* G *be a smooth irreducible curve of genus* g *and of Clifford index* $c \geq 3$ *on* Σ_d. *Let* Δ_0 *be a section with* $\Delta_0^2 = -d$ *and* Γ *a fibre of* Σ_d. *Assume that* $\Delta_0.G \geq 2$ *when* $d = 1$ *and* $\Delta_0.G \geq \Gamma.G$ *when* $d = 0$. *Then*

$$G \sim (c+2)\Delta_0 + ((c+2)d/2 + 1 + g/(c+1))\Gamma,$$

where the symbol $\sim$ *means the linear equivalence of divisors. Furthermore,* $(K_{\Sigma_d} + G)^2 = 2c(g-c-1)/(c+1)$ *holds and* (c, g, d) *satisfies one of the following*:

(i) c *is even,* $g \geq (c+1)^2$, g *is divided by* $(c+1)$ *and* $0 \leq d \leq \leq 2(g+c+1)/((c+1)(c+2))$.

(ii) c *is odd,* $g \geq (c+1)^2$, g *is divided by* $(c+1)/2$ *and* $0 \leq d \leq \leq 2(g+c+1)/((c+1)(c+2))$ *with* $d \equiv 2g/(c+1) \bmod 2$.

(iii) c *is even,* $(c+1)(c+4)/2 \leq g \leq c(c+1)$, g *is divided by* $(c+1)$ *and* $d = 1$.

(iv) *c is odd,* $(c+1)(c+4)/2 \le g \le (c+1)(2c+1)/2$*, g is an odd multiple of* $(c+1)/2$ *and* $d=1$.

PROOF. Under the assumptions, $\Gamma.G = \mathrm{gon}(G) = c+2$ holds (see [9]). Remark that $\Delta_0.G \ge 0$ since G is irreducible. The rest of statements follow from a standard calculation. □

Let V be a smooth rational surface which is neither $\mathbb{P}^2$ nor Σ_d and D a smooth irreducible curve of genus $g \ge 7$ on V. Assume that $\mathrm{Cliff}(D) \ge 3$ and $(K_V + D).E \ge 1$ for any (-1)-curves E on V. Then we can find at least one base-point-free pencil of rational curves on V. We choose among them a pencil $|R|$ of rational curves with $R^2 = 0$ in such a way that $a := (K_V + D).R$ is minimal. We call a the *minimal ruling degree of* (V, D). Note that we have $D.R = a + 2 \ge 5$ since $K_V.R = -2$. Let $\psi : V \to \mathbb{P}^1$ be the morphism defined by $|R|$. We take a relatively minimal model of V with respect to ψ and consider the image of D. Then we perform a succession of elementary transformations ([5]) at singular points of the image curve to arrive at a particular relatively minimal model $(V^\#, D^\#)$, called a #-minimal model in [6], enjoying several nice properties which we collect below. The natural map $\upsilon : (V, D) \to (V^\#, D^\#)$ is a minimal succession of blowing-ups which resolves the singular points of $D^\#$. We assume that $V^\# \simeq \Sigma_d$ and $D^\# \sim (a+2)\Delta_0 + b\Gamma$. Let p_i, $1 \le i \le n$, be the singular points of $D^\#$ including infinitely near ones, and let m_i be the multiplicity of $D^\#$ at p_i. Assume for simplicity that $m_1 \ge m_2 \ge \cdots \ge m_n \ge 2$. Since $|R|$ is chosen so that $(K_V + D).R$ is minimal, we can assume that the following are satisfied (see [5] and [6]):

(#1) $b \ge (a+2)d$ when $d > 0$, and $b \ge a+2$ when $d = 0$,
(#2) $b \ge a + 2 + m_1$ when $d = 1$,
(#3) $m_1 \le (a+2)/2$ and $m_1 \le \min\{(a+2)/2,\ b-(a+2)\}$ when $d = 1$.

We say that $(V^\#, D^\#)$ is of *general type* if $2b - (a+2)d \ge 2(a+2)$. Otherwise, i.e., when $d = 1$ and $2b < 3(a+2)$, the pair is called of *special type*. If this is the case, by contracting the minimal section, we get a model of D which is a plane curve of degree b with a $(b-a-2)$-ple point and n other singular points of respective multiplicities m_i ($\le b-a-2$).

At first, consider the case where $(V^\#, D^\#)$ is of general type. Set $b' = b - (d+2)(a+2)/2$, where $b' \ge \max\{0, (d-2)(a+2)/2\}$, and where $b' \in \mathbb{Z}$ or $b' \in \mathbb{Z}[1/2]$ according as a is even or odd. Then we have

$$D^\# \sim (a+2)\Delta_0 + (b' + (d+2)(a+2)/2)\Gamma. \tag{2.3}$$

LEMMA 2.6. *Keep the notation and assumptions as above. Then the following hold*:

When $g \geq (a+1)^2$,

$$(K_V + D)^2 > a(g-a-1)/(a+1).$$

If a *is even and* $g < (a+1)^2$, *then*

$$(K_V + D)^2 \geq 2a(g-1)/(a+2).$$

When the equality sign holds here, $b' = 0$ *and* $m_n = (a+2)/2$.

If a *is odd and* $g < (a+1)^2$, *then*

$$(K_V + D)^2 \geq 2g(a-1)/(a+1) + 2,$$

When the equality sign holds, $b' = 0$ *and* $m_n = (a+1)/2$.

PROOF. From a standard calculation, we have $(K_V + D)^2 = = 2a(a+b') - \sum_{i=1}^{n}(m_i - 1)^2$ and

$$2g = 2(a+1)(a+1+b') - \sum_{i=1}^{n} m_i(m_i - 1). \tag{2.4}$$

Therefore, we have

$$(K_V + D)^2 - \frac{2(m_1 - 1)}{m_1}(g - (a+1)(a+1+b')) - 2a(a+b') = \sum_{i=1}^{n}(m_i - 1)\left(1 - \frac{m_i}{m_1}\right),$$

which is non-negative. Hence

$$(K_V + D)^2 \geq \frac{2(m_1 - 1)}{m_1}(g - (a+1)(a+1+b')) + 2a(a+b'). \tag{2.5}$$

If the equality sign holds here, then $m_1 = \cdots = m_n$. On the other hand,

$$\frac{2(m_1 - 1)}{m_1}(g - (a+1)(a+1+b')) + 2a(a+b') - \left(\frac{2a}{a+1}g - 2a\right) = \frac{2(m_1 - a - 1)(g - (a+1)(a+1+b'))}{m_1(a+1)}$$

is positive from (#3) and (2.4). Thus $(K_V + D)^2 > 2a(g-a-1)/(a+1)$ holds.

In the rest of the proof, we assume that $g < (a+1)^2$. We restrict ourselves to the case where a is even, since the other case is quite similar. Then (#3) and (2.4) imply that

$$\frac{2(m_1-1)}{m_1}(g-(a+1)(a+1+b')) + 2a(a+b') - \frac{2a}{a+2}(g+b'-1)$$
$$= \frac{2(2m_1-a-2)(g-(a+1)(a+1+b'))}{m_1(a+2)}$$

is non-negative. Therefore we have

$$(K_V+D)^2 \geq 2a(g+b'-1)/(a+2) \tag{2.6}$$

from (2.5). Thus $(K_V+D)^2 \geq 2a(g-1)/(a+2)$. If the equality sign holds, then $b'=0$ and $m_1 = \cdots = m_n = (a+2)/2$. □

Next, we consider the case where $(V^{\#}, D^{\#})$ is of special type. Let $m_0 = b-(a+2)$. Then we have

$$D^{\#} \sim (a+2)\Delta_0 + (a+2+m_0)\Gamma, \quad 2 \leq m_0 < (a+2)/2. \tag{2.7}$$

LEMMA 2.7. *Keep the notation and assumptions as above. Then the following hold*:

When $(a+1)(a+4)/2 \leq g \leq a(a+1)$ *or* $(a+1)(2a+1)/2$ *according as* a *is even or odd,*

$$(K_V+D)^2 > 2a(g-a-1)/(a+1).$$

If $a(a+3)/2 \leq g < (a+1)(a+4)/2$ *and* a *is even, then*

$$(K_V+D)^2 \geq g + (a^2-a-4)/2.$$

When the equality sign holds, $(m_0, n) = (2, (a+1)(a+4)/2 - g)$ *or* $(a, n, m_5) = (6, 5, 3)$.

If $g < a(a+3)/2$ *and* a *is even, then*

$$(K_V+D)^2 \geq 2g(a-2)/a + 4.$$

When the equality sign holds, $m_n = a/2$ *and* $n > 4 + 4/(a-2)$.

If $(a+1)(a+2)/2 \leq g < (a+1)(a+4)/2$ *and* a *is odd, then*

$$(K_V+D)^2 \geq g + (a^2-a-4)/2.$$

When the equality sign holds, $(m_0, n) = (2, (a+1)(a+4)/2 - g)$ *or* $(n, m_4) = (4, (a+1)/2)$.

If $g < (a+1)(a+2)/2$ and a is odd, then

$$(K_V + D)^2 \geq 2g(a-1)/(a+1) + 1.$$

When the equality sign holds, $m_n = (a+1)/2$ and $n > 4$.

PROOF. From a standard calculation, we have

$$\begin{aligned}(K_V + D)^2 &= a^2 + 2(m_0 - 1)a - \sum_{i=1}^{n} (m_i - 1)^2, \\ 2g &= (a+1)(a+2m_0) - \sum_{i=1}^{n} m_i(m_i - 1).\end{aligned} \tag{2.8}$$

In the same way of (2.5), we have

$$(K_V + D)^2 \geq \frac{2(m_0 - 1)}{m_0} g + \frac{a^2 + (1 - m_0)a - 2m_0^2 + 2m_0}{m_0}. \tag{2.9}$$

If the equality sign holds here, then $m_0 = m_1 = \cdots = m_n$. By an argument similar to Lemma 2.6, we have $(K_V + D)^2 > 2a(g - a - 1)/(a+1)$.

When $g \geq (a^2 + a + 4m_0)/2$,

$$\frac{2(m_0 - 1)}{m_0} g + \frac{a^2 + (1 - m_0)a - 2m_0^2 + 2m_0}{m_0} - \left(g + \frac{a^2 - a - 4}{2}\right) = \frac{(2g - a^2 - a - 4m_0)(m_0 - 2)}{2m_0} \tag{2.10}$$

is non-negative. Remark that $(a^2 + a + 4m_0)/2$ increases as m_0 grows. Consider the case where $g \geq a(a+3)/2$ or $(a+1)(a+2)/2$ according as a is even or odd. From (2.10), (#3) and (2.9), we have $(K_V + D)^2 \geq g + (a^2 - a - 4)/2$. Furthermore, if $m_0 = 2$, then $(K_V + D)^2 = g + (a^2 - a - 4)/2$ and $n = (a+1)(a+4)/2 - g$. When $m_0 \neq 2$ and $(K_V + D)^2 = g + (a^2 - a - 4)/2$ hold, $m_0 = m_1 = \cdots = m_n$ and $2g = = a^2 + a + 4m_0$. Hence, then, we have $m_0 = \cdots = m_n = a/2$ or $(a+1)/2$ since $g \geq a(a+3)/2$ or $(a+1)(a+2)/2$ according to the parity of a. These and (2.8) imply $n = 4 + 4/(a-2)$ or 4 according as a is even or odd.

In the rest of the proof, we may consider the case where $g < a(a+3)/2$ and a is even, since the other case is similar. Recall that $m_0 \leq a/2$. When $g \leq a(a + 1 + m_0)/2$,

$$\frac{2(m_0 - 1)}{m_0} g + \frac{a^2 + (1 - m_0)a - 2m_0^2 + 2m_0}{m_0} - \left(\frac{2a - 4}{a} g + 4\right) = \frac{(2g - a(a + 1 + m_0))(2m_0 - a)}{am_0}$$

is non-negative. Since $a(a+3)/2 \leq a(a+1+m_0)/2$ for any m_0, it follows from (2.9) that $(K_V+D)^2 \geq 2g(a-2)/a+4$. If $(K_V+D)^2$ attains the equality, then $m_0 = \cdots = m_n$ and $2m_0 - a = 0$. □

As a corollary of Lemmas 2.6 and 2.7, we have the following.

LEMMA 2.8. *Let V be a smooth rational surface which is neither $\mathbb{P}^2$ nor Σ_d and D a smooth irreducible curve of genus $g \geq 7$ on V. Assume that $(K_V+D).E \geq 1$ for any (-1)-curves E on V and the minimal ruling degree of (V,D) is $a \geq 3$. Then the following hold*:

When $g \geq (a+1)(a+4)/2$,

$$(K_V+D)^2 > 2a(g-a-1)/(a+1).$$

When $(a+1)(a+2)/2 \leq g < (a+1)(a+4)/2$,

$$(K_V+D)^2 \geq g + (a^2-a-4)/2.$$

If $(K_V+D)^2$ attains the equality, then there exists a blow-down $\upsilon_0 : V \to \mathbb{P}^2$ contracting (-1)-curves E_i such that D is one of the following:

(i) $D \sim ((3a+6)/2)\upsilon_0^*\mathcal{O}_{\mathbb{P}^2}(1) - ((a+2)/2)\sum_{i=0}^{3} E_i - (a/2)E_4$, *where a is even and $g = (a+1)(a+2)/2$,*

(ii) $D \sim ((3a+5)/2)\upsilon_0^*\mathcal{O}_{\mathbb{P}^2}(1) - ((a+1)/2)\sum_{i=0}^{4} E_i$, *where a is odd, $a \geq 5$ and $g = (a+1)(a+2)/2$,*

(iii) $D \sim 9\upsilon_0^*\mathcal{O}_{\mathbb{P}^2}(1) - 3\sum_{i=0}^{3} E_i$, *where $a = 4$ and $g = 16$,*

(iv) $D \sim (a+4)\upsilon_0^*\mathcal{O}_{\mathbb{P}^2}(1) - 2\sum_{i=0}^{n} E_i$, *where $n = (a+1)(a+4)/2 - g)$ and $(a+1)(a+2)/2 \leq g < (a+1)(a+4)/2$.*

Furthermore, Cliff$(D) = a$ *if (V,D) is one of them.*

When $g < (a+1)(a+2)/2$,

$$(2.11) \qquad (K_V+D)^2 \geq \begin{cases} 2a(g-1)/(a+2) & \text{if } a \text{ is even}, \\ 2g(a-1)/(a+1)+1 & \text{otherwise}. \end{cases}$$

PROOF. When $g \geq (a+1)(a+4)/2$, from Lemmas 2.6 and 2.7, we have $(K_V+D)^2 > 2a(g-a-1)/(a+1)$. When $g < (a+1)(a+2)/2$, the inequalities (2.11) also follow immediately. Suppose that (V,D) satisfies one of (i)–(iv). Then we have Cliff$(D) = a$ from [8, Proposition 2.2], since $a \geq 3$ and $D^2 > (a+2)^2$.

For the case where $(a+1)(a+2)/2 \le g < (a+1)(a+4)/2$, we may restrict ourselves to the case where a is even, since the other case is similar and simpler. When $g \ge (a^2+3a+4)/2$, we have

$$2a(g-1)/(a+2) \ge g + (a^2-a-4)/2,$$

where the equality sign holds if and only if $g = (a^2+3a+4)/2$. Thus Lemmas 2.6 and 2.7 imply the statement for $g > (a^2+3a+4)/2$. Furthermore, if

$$(g, (K_V+D)^2) = ((a^2+3a+4)/2, a^2+a), \quad ((a+1)(a+2)/2, a^2+a-1)$$

and $(V^{\#}, D^{\#})$ is of special type, then $(m_0, n) = (2, (a+1)(a+4)/2 - g)$ from Lemma 2.7.

In what follows, we consider the case where $(V^{\#}, D^{\#})$ is of general type. At first, we concentrate on the case where $g = (a^2+3a+4)/2$. Then we only have to determine the pair (V, D) satisfying $(K_V+D)^2 = a^2+a$. It follows from Lemma 2.6 that such a pair satisfies $b' = 0$ and $m_n = (a+2)/2$. Furthermore we have $n = 4 - 4/a$ from (2.4). Thus $a = 4$ and $n = 3$ follow. Therefore $D \sim 6\upsilon^*\Delta_0 + (6+3d)\upsilon^*\Gamma - 3\sum_{i=1}^{3} E_i$ and we have $d \le 2$ because $\upsilon^*\Delta_0.D = 6 - 3d \ge 0$. Remark that $D^{\#}$ has no singular points on the minimal section Δ_0 of $V^{\#}$ in the case where $d = 2$. Hence the case where $d = 1$ covers the cases where $d = 0, 2$.

We next consider the case where $g = (a+1)(a+2)/2$. It follows from (2.6) that

$$(K_V+D)^2 \ge a^2 + a + 2a(b'-1)/(a+2) > a^2+a-1$$

when $b' \ge 1$. Hence we can assume $b' = 0$. If $D^{\#}$ has at least four $(a+2)/2$-ple points, then (2.4) shows $g < (a+1)(a+2)/2$. In the same way as in Lemma 2.6, we have $(K_V+D)^2 \ge a^2+a+2-n'$ for (V, D) whose $D^{\#}$ has just n' $(a+2)/2$-ple points with $n' = 0, 1, 2, 3$. If $(K_V+D)^2 = a^2+a+2-n'$ holds, then $m_{n'+1} = \cdots = m_n = a/2$. Furthermore, $(K_V+D)^2$ attains the minimum a^2+a-1 at $n' = 3$. In fact, (2.4) and $(g, (K_V+D)^2) = ((a+1)(a+2)/2, a^2+a-1)$ imply that $D^{\#}$ has exactly three $(a+2)/2$-ple points and a $a/2$-ple point as its singularity. Thus

$$D \sim (a+2)\upsilon^*\Delta_0 + \frac{(d+2)(a+2)}{2}\upsilon^*\Gamma - \frac{a+2}{2}\sum_{i=1}^{3} E_i - \frac{a}{2}E_4$$

and we have $d \le 2$ since $\upsilon^*\Delta_0.D = (a+2)(1-d/2) \ge 0$. We can show that the case where $d=1$ covers the cases where $d=0,2$ similarly as in the previous case. □

Keep the situation as in Lemma 2.8. Furthermore, we consider the case where $4 \le c+1 \le a \le 2c-2$. When $g \ge (c+1)(c+2)/2$, we have $(K_V+D)^2 > g+(c^2-c-4)/2$ from Lemma 2.8. We also have $(K_V+D)^2 > 2c(g-c-1)/(c+1)$ when

$$g < (a+1)(a+4)/2 - (a-c)(2c-2-a)/(c-1)$$

or $g > (c+1)(a+1)$. For (V,D) satisfying

$$(K_V+D)^2 \le 2c(g-c-1)/(c+1) \tag{2.12}$$

and

$$(a+1)(a+4)/2 - (a-c)(2c-2-a)/(c-1) \le g \le (c+1)(a+1), \tag{2.13}$$

we have the following:

LEMMA 2.9. *Keep the notation and assumptions as above. Then* $\mathrm{Cliff}(D) = a$.

PROOF. Assume that (V,D) satisfies (2.12). Then it follows from Lemma 2.6 that (V,D) has a #-minimal model of special type. We recall (2.7). First of all, we show that (V,D) attains maximum of n at $m_0 = 2$ as follows: Take a smooth irreducible curve $B \in |D + \sum_{i=1}^{n}(m_i-2)E_i|$ for a fixed (V,D). We consider (V,B) and its #-minimal model $(V^{\#},B^{\#})$. Remark that $B^{\#}$ has just n double points as its singularity. Then $(K_V+B)^2 = (K_V+D)^2 + \sum_{i=1}^{n} m_i(m_i-2)$ and $2g(B) = 2g + \sum_{i=1}^{n}(m_i+1)(m_i-2)$. These and (2.12) imply that

$$\frac{2c}{c+1}g(B) - 2c - (K_V+B)^2 \ge \frac{1}{c+1}\sum_{i=1}^{n}(c-m_i)(m_i-2).$$

Since $c \ge (a+2)/2 > m_0 \ge m_1 \ge \cdots \ge m_n \ge 2$, the right-hand side is non-negative. Hence (V,B) also satisfies $(K_V+B)^2 \le 2c(g(B)-c-1)/(c+1)$. From (2.12) and (2.8) with $m_1 = 2$, we have $n \le (a-c)(2c-2-a+2(2-m_0))/(c-1)$. In the end, at $m_0 = 2$,

$$n \le (a-c)(2c-2-a)/(c-1). \tag{2.14}$$

We now return to the proof of Lemma 2.9. We have $D^2 \geq 4 + 2c + 2g - 2g/(c+1) - n$ from (2.12) and $(K_V + D)^2 = 2(K_V + D).D + K_V^2 - D^2 = 4g - 4 + 8 - n - D^2$. Therefore (2.13) and (2.14) imply that

$$D^2 - (a+2)^2 \geq 4(a + 2 - (a-c)(2c - 2 - a)/(c-1)) > 0.$$

Hence Lemma 2.9 follows from [8, Proposition 2.2]. □

PROOF OF PROPOSITION 2.4. By Lemma 2.5 and the argument just before Lemma 2.5, we can assume that Y is a smooth rational surface which is neither $\mathbb{P}^2$ nor Σ_d. Now the assumption implies that the minimal ruling degree a of (Y, G) is greater than or equal to c. From Lemma 2.9, in the case where $a > c$, we have $(K_Y + G)^2 > \max\{g + (c^2 - c - 4)/2, 2c(g - c - 1)/(c+1)\}$ when $g \geq (c+1)(c+2)/2$. For the case where $a = c$, Lemma 2.8 implies Proposition 2.4. □

3. Mordell-Weil lattices

We briefly review the theory of the Mordell-Weil lattice due to Shioda. Let X be a smooth projective surface with $p_g = q = 0$ and $f : X \to \mathbb{P}^1$ a relatively minimal fibration of genus g. We assume that it has a section. Via f, we can regard X as a smooth projective curve of genus g defined over the rational function field $\mathbb{K} = \mathbb{C}(\mathbb{P}^1)$. Let $\mathscr{J}_{\mathscr{F}}$ be the Jacobian variety of a generic fibre $\mathscr{F}$ of f. The Mordell-Weil group of f is the group of $\mathbb{K}$-rational points $\mathscr{J}_{\mathscr{F}}(\mathbb{K})$. It is a finitely generated abelian group, since X is a regular surface. The rank of the group, which we denote by r, is called the *Mordell-Weil rank*. It follows from [13, Theorem 3] that r is given by

$$r = \rho(X) - 2 - \sum_{t \in \mathbb{P}^1} (v_t - 1),$$

where $\rho(X)$ denotes the Picard number, that is, the rank of NS(X), and v_t denotes the number of irreducible components of the fibre $f^{-1}(t)$. In particular, we have $r = \rho(X) - 2$ if f has irreducible fibres only.

There exists a natural one-to-one correspondence between the set of $\mathbb{K}$-rational points $\mathscr{F}(\mathbb{K})$ and the set of sections of f. For $P \in \mathscr{F}(\mathbb{K})$ we denote by (P) the section corresponding to P which is regarded as a curve on X. We specify a section (O) corresponding to the origin of $\mathscr{J}_{\mathscr{F}}(\mathbb{K})$ and call it the *zero section*. Shioda's main idea in [13] is to regard $\mathscr{J}_{\mathscr{F}}(\mathbb{K})$ as a Euclidean lattice endowed with a natural pairing induced by the intersection form on $H^2(X)$. Let T be the subgroup of NS(X) generated by (O) and all the irreducible components of fibres of f. With respect to the intersection pairing, the sublattice T is called the *trivial sublattice* and its orthogonal complement $L = T^{\perp} \subset \mathrm{NS}(X)$ is called the *essential sublattice*. Via the natural isomorphism of groups $\mathscr{J}_{\mathscr{F}}(\mathbb{K}) \simeq \mathrm{NS}(X)/T$ in [13, Theorem 3], we obtain a symmetric bilinear form $\langle , \rangle$ on $\mathscr{J}_{\mathscr{F}}(\mathbb{K})$ which induces the structure of a positive-definite lattice on $\mathscr{J}_{\mathscr{F}}(\mathbb{K})/\mathscr{J}_{\mathscr{F}}(\mathbb{K})_{\mathrm{tor}}$ (see [13, Theorem 7]). The lattice $(\mathscr{J}_{\mathscr{F}}(\mathbb{K})/\mathscr{J}_{\mathscr{F}}(\mathbb{K})_{\mathrm{tor}}, \langle , \rangle)$ is called the *Mordell-Weil lattice* of the fibration $f : X \to \mathbb{P}^1$. Shioda shows that if all the fibres of f are irreducible, then the Mordell-Weil lattice of f is isomorphic to L^-, where the opposite lattice L^- is defined from L by putting the minus sign on the intersection pairing on L (see [13, Theorems 3 and 8]).

As to fibred surfaces with $p_g = q = 0$ of Clifford index c, we can bound the Mordell-Weil rank by Theorem 2.3 in the same way as in [8, Lemma 3.1] (see also [12, Theorem 2.8], [11, Proposition 2.2] and [7, Theorem 3.4]):

Lemma 3.1. *Let X be a smooth projective surface with $p_g = q = 0$, $f : X \to \mathbb{P}^1$ a relatively minimal fibration of genus g and of Clifford index $c \geq 3$. Let r be the Mordell-Weil rank of f. Then the following hold*:

(1) $r \leq 2(c+2)(g+c+1)/(c+1)$ *when* $g > (c+2)(c+3)/2$.
(2) $r \leq 3g-(c+3)(c-4)/2$ *when* $(c+1)(c+2)/2 \leq g \leq (c+2)(c+3)/2$.

If r attains the maximum, then all fibres of f are irreducible and the reduction of (X, F) is obtained by $\Phi_{|K_X+F|}$. In particular, X is a rational surface when r attains the maximum.

In what follows, we restrict ourselves to the case when r attains the maximum. Namely, we let $f : X \to \mathbb{P}^1$ be a fibred rational surface of genus g and of Clifford index $c \geq 3$ whose Mordell-Weil rank attains the maximum value given in Lemma 3.1. Suppose that we are given such a fibration $f : X \to \mathbb{P}^1$ and let (Y, G) be the reduction of (X, F). Then f gives us a pencil $\Lambda_f \subset |G|$ all of whose members must be irreducible. Note also that all the base points of Λ_f must be transversal ones since $K_{X/\mathbb{P}^1}$ is ample. Conversely, if we are given a pencil $\Lambda \subset |G|$ which has only transversal base points and all fo whose members are irreducible, then we get a fibration with the desired properties by blowing up BsΛ.

Let the situation be as in (1) of Lemma 3.1. Then $Y = \Sigma_d$ from Theorem 2.3. We show that the case $(c+1)(c+2)d = 2(g+c+1)$ is impossible. This can be seen as follows: If $(c+1)(c+2)d = 2(g+c+1)$, then $\Delta_0.G = 0$. Hence the pull-back to X of Δ_0 is a rational curve contained a fibre of f, which is inadequate in our situation. Therefore, we have $(c+1)(c+2)d < 2(g+c+1)$.

Let (Σ_d, G) be as in Lemma 2.5 with $g > (c+2)(c+3)/2$ and $(c+1)(c+2)d < 2(g+c+1)$. Then G is very ample. Hence we can find a pencil $\Lambda \subset |G|$ whose members are all irreducible and which has exactly G^2 transversal base points. In fact, we can take it as a Lefschetz pencil (see [4]). Then the fibration $f : X \to \mathbb{P}^1$ obtained by Λ has Mordell-Weil rank which attains the equality as in (1) of Lemma 3.1.

DEFINITION 3.2. Let (c, g, d) be as in (i)–(iv) of Lemma 2.5 with $g > (c+2)(c+3)/2$ and $d < 2(g+c+1)/((c+1)(c+2))$. A fibration $f : X \to \mathbb{P}^1$ of genus g and of Clifford index $c \geq 3$ obtained by blowing up Σ_d as above is called a fibration of type $(c, g, d, 0)$.

Let the situation be as in (2) of Lemma 3.1. From Theorem 2.3, there exists the blow-down $\upsilon_0 : (Y, G) \to (\mathbb{P}^2, G_0)$ constracting $(n+1)$ (-1)-curves $E_0, E_1, \ldots, E_n$ such that G_0 is as in (i)–(iv) of Theorem 2.3, where $n = 4$ for (i) or (ii), and where $n = 3$ for (iii). Put $p_j = \upsilon_0(E_j)$. If p_1 is an infinitely near point of p_0, then the pull-back to X of $E_0 - E_1$ is a rational curve contained in a fibre of f from $(E_0 - E_1).G = 0$, which contradicts the irreducibility of f. Furthermore, in the quite same argument, we have the following:

PROPOSITION 3.3. *Keep the notation and assumptions as above. If $n \geq 1$, then a configuration of $(n+1)$ points $p_0, \ldots, p_n$ satisfies two conditions as follows: The first condition is that $p_0, \ldots, p_n$ expect the $c/2$-ple point p_4 as in* (i) *of Theorem* 2.3 *are not infinitely near points. For each case of* (i)–(iv)*, the second condition is the following*:

(i) *Any three points of $p_0, \ldots, p_3$ are not colinear, though p_4 may be an infinitely near point.*

(ii) *Any four points of $p_0, \ldots, p_4$ are not colinear.*

(iii) *$p_0, \ldots, p_3$ are in general position, that is, any three points of them are not colinear.*

(iv) *Any $(c+4)/2$ or $(c+5)/2$ points of $p_0, \ldots, p_n$ are not colinear according as c is even or odd.*

We use the following theorem in order to show the next lemma.

THEOREM 3.4 ([10]). *Let $S \subset \mathbb{P}^{\deg(S)-g(S)+1}$ be a nondegenerate, linearly normal, smooth regular surface of degree* $\deg(S)$ *and of sectional genus* $g(S)$. *Let* $x_0, \ldots, x_k$ *be distinct* $(k+1)$ *points of* S *for* $k \le \deg(S) - 2g(S) - 2$. *Let* σ *be a blowing up at* $x_0, \ldots, x_k$ *of* S *and* $E_j = \sigma^{-1}(x_j)$. *Set* $H = \sigma^* \mathcal{O}_S(1) - \sum_{i=0}^{k} E_i$. *Then* H *is very ample if and only if for all* j *with* $1 \le j \le k+1$, *any distinct* j *points of* $\{x_0, \ldots, x_k\}$ *do not lie on any irreducible reduced curve of degree* j *on* S.

LEMMA 3.5. *Let* D_0 *be a plane curve with the same degree and singularity as in* (i)–(iv) *of Theorem* 2.3. *Assume that a configuration of singular points* $p_0, \ldots, p_n$ *of* D_0 *satisfies the two conditions as in Proposition* 3.3. *Let* $\upsilon_0 : (V, D) \to (\mathbb{P}^2, D_0)$ *denote the composite of blow-ups at* $n+1$ *points* $p_0, \ldots, p_n$. *Then* D *is very ample.*

PROOF. Put $E_j = \upsilon_0^{-1}(p_j)$. At first, we consider the case (i). We remark that $|-K_V|$ is free from base points, because $p_0, \ldots, p_4$ are in almost general position (see [3, Theorem 1 of III in p. 39]). Consider the surface obtained by blowing $\mathbb{P}^2$ up at $p_0, \ldots, p_3$ in general position. Then the anti-bicanonical map is an embedding to $\mathbb{P}^{15}$. Let S be the embedded surface to $\mathbb{P}^{15}$ and $\sigma : V \to S$ the blow-down contracting E_4. Since no line is on S,

$$H = \sigma^*(-2K_S) - E_4 = 6\upsilon_0^* \mathcal{O}_{\mathbb{P}^2}(1) - 2\sum_{i=0}^{3} E_i - E_4$$

is very ample from Theorem 3.4. Therefore $D = H + (c/2 - 1)(-K_V)$ is also. We can treat the case (ii) in the same way by setting $H = 4\upsilon_0^* \mathcal{O}_{\mathbb{P}^2}(1) - \sum_{i=0}^{4} E_i$. For the case (iii), it is obvious.

Let the situation be as in (iv). Since the case where c is even is similar and simpler, we restrict ourselves to the case where c is odd. Let S be $\mathbb{P}^2$ embedded by $|\mathcal{O}_{\mathbb{P}^2}((c+5)/2)|$. Then irreducible reduced curves whose degree is at most $(n+1)$ come from only lines on $\mathbb{P}^2$. Therefore Theorem 3.4 implies that $H = \upsilon_0^* \mathcal{O}_{\mathbb{P}^2}((c+5)/2) - \sum_{i=0}^{n} E_i$ is very ample. Hence we only have to show that

$$|D - H| = \left| \frac{c+3}{2} \upsilon_0^* \mathcal{O}_{\mathbb{P}^2}(1) - \sum_{i=0}^{n} E_i \right|$$

is free from base points. If any $(c+3)/2$ points of $p_0, \ldots, p_n$ are not colinear, then we can see similarly that $D - H$ is also very ample by applying

Theorem 3.4. Thus we suppose that $p_0, \dots, p_{(c+1)/2}$ lie on a line $l_{0,\dots,(c+1)/2}$. Furthermore, we concentrate on the case where $n = c+1$, since the other cases are similar and simpler. Remark that any p_i, $(c+3)/2 \le i \le c+1$ are not lie on $l_{0,\dots,(c+1)/2}$ from the second condition (iv) in Proposition 3.3. Let l_i be a line through p_i. In particular, we denote the line through p_i and $p_{i+(c+1)/2}$ by $l_{i,i+(c+1)/2}$, $1 \le i \le (c+1)/2$. Consider the strict transform of $\left(l_0 + \sum_{i=1}^{(c+1)/2} l_{i,i+(c+1)/2}\right)$ by υ_0. It is disjoint from the strict transform of $l_{0,\dots,(c+1)/2}$ if l_0 is distinct from $l_{0,\dots,(c+1)/2}$. Furthermore, $|D-H|$ is free from base points on E_0 since we can choose l_0 for any directions at p_0. In the same way, $|D-H|$ is also on the whole of ${\upsilon_0}^* l_{0,\dots,(c+1)/2}$. Next, consider $l_{0,\dots,(c+1)/2} + \sum_{i=(c+3)/2}^{c+1} l_i$. Then $|D-H|$ is free from base points on $V \setminus {\upsilon_0}^* l_{0,\dots,(c+1)/2}$ since we can choose l_i for any directions at p_i. □

From Lemma 3.5, for any cases as in Proposition 3.3, we can show the existence of the fibrations similarly as in the previous case. Hence we can take a pencil $\Lambda \subset |D|$ enjoying the desired properties, as a Lefschetz pencil for example. In particular, Λ has $(3g - (c+3)(c-4)/2 - n)$ transversal base points.

Definition 3.6. Let (V, D) be as in Lemma 3.5. If V is obtained by blowing up $n+1$ points of $\mathbb{P}^2$, the corresponding fibration $f : X \to \mathbb{P}^1$ of genus g and of Clifford index c as above is called a fibration of type $(c, g, 1, n)$.

3.1 – *Type* $(c, g, d, 0)$

Here we determine the Mordell-Weil lattices for fibrations $f : X \to \mathbb{P}^1$ of type $(c, g, d, 0)$. For this purpose, we use the following notation. We denote the pull-backs to X of Δ_0 and Γ by the same symbols. Furthermore, we denote by $e_1, e_2, \dots, e_r$ the disjoint (-1)-sections of f coming from the base points of Λ_f, where $r = 2(c+2)(g+c+1)/(c+1)$. Then we have $\mathrm{NS}(X) \simeq \mathbb{Z}\Delta_0 \oplus \mathbb{Z}\Gamma \oplus \bigoplus_{i=1}^{r} \mathbb{Z}e_i$ and

$$F = (c+2)\Delta_0 + \left(\frac{(c+2)d}{2} + 1 + \frac{g}{c+1}\right)\Gamma - \sum_{i=1}^{r} e_i. \tag{3.15}$$

We take e_r as the zero section (O). The sublattice $T_{(c,g,d,0)} \subset \mathrm{NS}(X)$ generated by e_r and F is the trivial sublattice. Let $L_{(c,g,d,0)}$ be the orthogonal complement of $T_{(c,g,d,0)}$. Then the Mordell-Weil lattice $(\mathscr{I}_{\mathscr{F}}(\mathbb{K}), \langle , \rangle)$ is isomorphic to $L_{(c,g,d,0)}{}^{-}$.

Recall then that the degree d of the Hirzebruch surface is an invariant of the fibration.

THEOREM 3.7. *For a fibration of type* $(c, g, d, 0)$, *the lattice* $L_{(c,g,d,0)}{}^{-}$ *is isomorphic to a positive-definite unimodular lattice of rank* $r = 2(c+2)(g+c+1)/(c+1)$ *whose Dynkin diagram is given by the following*:

Figure 1 *in the case when* $(c+2)d/2 - g/(c+1) \equiv 0 (\mathrm{mod}\, c+2)$
Figure 2 *in the case when* $(c+2)d/2 - g/(c+1) \equiv 1 (\mathrm{mod}\, c+2)$
Figure 3 *in the case when* $(c+2)d/2 - g/(c+1) \equiv 1 + \ell (\mathrm{mod}\, c+2)$ *with* $1 \leq \ell \leq c-1$,
Figure 4 *in the case when* $(c+2)d/2 - g/(c+1) \equiv c+1 (\mathrm{mod}\, c+2)$

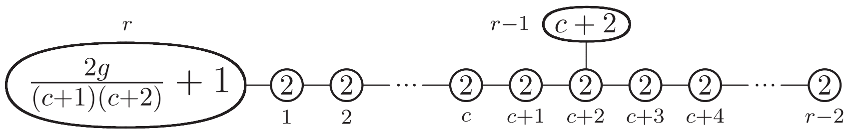

Fig. 1

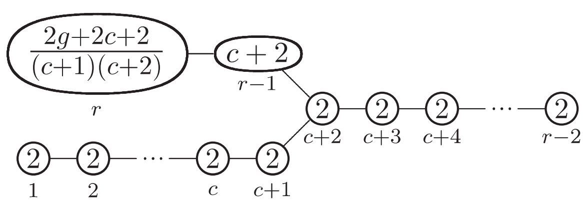

Fig. 2

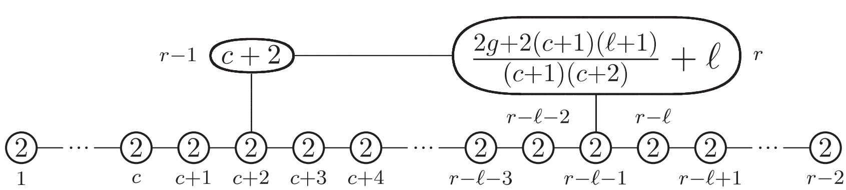

Fig. 3

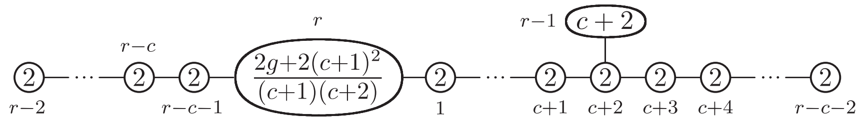

Fig. 4

In particular, $L_{(c,g,d,0)}{}^{-}$ *depends on only* $2g/(c+1) \bmod (c+2)$ *in the case when c is odd, and on the combination of* $g/(c+1) \bmod (c+2)$ *and the parity of d in the case when c is even. Here the numbers in the circles denote the self-parings of elements, and a line between two circles shows that the paring of the corresponding two elements is equal to* (-1)*. Furthermore,* $L_{(c,g,d,0)}{}^{-}$ *is an odd lattice in the case when c is odd (see Table* 1*) and the parity of the lattice is the same as that of* $(d+g+1)$ *or of* $(g+1)$ *respectively in the case when* $c \equiv 2 \bmod 4$ *(see Tables* 2 *and* 3*) or* $c \equiv 0 \bmod 4$ *(see Tables* 4 *and* 5*). In particular, even and odd lattices both occur for a fixed* $g \geq (c+1)^2$ *in the case when* $c \equiv 2 \bmod 4$.

Table 1. – $L_{(c,g,d,0)}{}^{-}$ is an odd lattice in the case when $c \equiv 1 \bmod 2$.

$2g/(c+1) \bmod c+2$		Dynkin diagram
0		Figure 1
1		Figure 3 with $\ell = (c-1)/2$
2		Figure 4
$2i+1$	$(i = 1, 2, \ldots, (c-3)/2)$	Figure 3 with $\ell = (c-1)/2 - i$
$2j$	$(j = 2, 3, \ldots, (c-1)/2)$	Figure 3 with $\ell = c+1-j$
c		Figure 2
$c+1$		Figure with $\ell = (c+1)/2$

Table 2. – $L_{(c,g,d,0)}{}^{-}$ is an even lattice in the case when $c \equiv 2 \bmod 4$.

$g/(c+1) \bmod c+2$		d	Dynkin diagram
0		odd	Figure 3 with $\ell = c/2$
1		even	Figure 4
h	$(h = 2, 4, \ldots, c/2 - 1)$	odd	Figure 3 with $\ell = c/2 - h$
i	$(i = 3, 5, \ldots, c/2)$	even	Figure 3 with $\ell = c+1-i$
$c/2+1$		odd	Figure 1
$c/2+1+j$	$(j = 1, 3, \ldots, c/2 - 2)$	even	Figure 3 with $\ell = c/2 - j$
$c/2+1+k$	$(k = 2, 4, \ldots, c/2 - 1)$	odd	Figure 3 with $\ell = c+1-k$
$c+1$		even	Figure 2

Table 3. – $L_{(c,g,d,0)}{}^{-}$ is an odd lattice in the case when $c \equiv 2 \bmod 4$.

$g/(c+1) \bmod c+2$		d	Dynkin diagram
0		even	Figure 1
h	$(h = 1, 3, \ldots, c/2-2)$	odd	Figure 3 with $\ell = c/2 - h$
i	$(i = 2, 4, \ldots, c/2-1)$	even	Figure 3 with $\ell = c+1-i$
$c/2$		odd	Figure 2
$c/2+1$		even	Figure 3 with $\ell = c/2$
$c/2+2$		odd	Figure 4
$c/2+2+j$	$(j = 1, 3, \ldots, c/2-2)$	even	Figure 3 with $\ell = c/2-1-j$
$c/2+2+k$	$(k = 2, 4, \ldots, c/2-1)$	odd	Figure 3 with $\ell = c-k$

Table 4. – $L_{(c,g,d,0)}{}^{-}$ is an even lattice in the case when $c \equiv 0 \bmod 4$.

$g/(c+1) \bmod c+2$		d	Dynkin diagram
1		even	Figure 4
h	$(h = 1, 3, \ldots, c/2-1)$	odd	Figure 3 with $\ell = c/2 - h$
i	$(i = 3, 5, \ldots, c/2+1)$	even	Figure 3 with $\ell = c+1-i$
$c/2+1$		odd	Figure 1
$c/2+1+j$	$(j = 2, 4, \ldots, c/2-2)$	even	Figure 3 with $\ell = c/2-j$
$c/2+1+k$	$(k = 2, 4, \ldots, c/2)$	odd	Figure 3 with $\ell = c+1-k$
$c+1$		even	Figure 2

Table 5. – $L_{(c,g,d,0)}{}^{-}$ is an odd lattice in the case when $c \equiv 0 \bmod 4$.

$g/(c+1) \bmod c+2$		d	Dynkin diagram
0		even	Figure 1
h	$(h = 0, 2, \ldots, c/2-2)$	odd	Figure 3 with $\ell = c/2 - h$
i	$(i = 2, 4, \ldots, c/2)$	even	Figure 3 with $\ell = c+1-i$
$c/2$		odd	Figure 2
$c/2+2$		even	Figure 3 with $\ell = c/2-1$
$c/2+2$		odd	Figure 4
$c/2+2+j$	$(j = 2, 4, \ldots, c/2-2)$	even	Figure 3 with $\ell = c/2-1-j$
$c/2+2+k$	$(k = 2, 4, \ldots, c/2-2)$	odd	Figure 3 with $\ell = c-k$

PROOF. Let us keep the notation as above. In particular, F is given by (3.15) and $(O) = e_r$. Take the following elements from $L_{(c,g,d,0)}$:

$$\xi_{r-1} = \Gamma - \sum_{i=1}^{c+2} e_i, \qquad \xi_i = e_i - e_{i+1} \ \ (1 \le i \le r-2, \ \ i \ne r-c-1).$$

We take ξ_r and ξ_{r-c-1} from $L_{(c,g,d,0)}$ according to the following rule:

(1) If $c_1 = d/2 - g/((c+1)(c+2)) \in \mathbb{Z}$, then put

$$\xi_r = \Delta_0 + c_1\Gamma - e_1, \quad \xi_{r-c-1} = e_{r-c-1} - e_{r-c}.$$

(2) If $c_2 = d/2 - g/((c+1)(c+2)) - 1/(c+2) \in \mathbb{Z}$, then put

$$\xi_r = \Delta_0 + c_2\Gamma, \quad \xi_{r-c-1} = e_{r-c-1} - e_{r-c}.$$

(3) If $c_3 = d/2 - g/((c+1)(c+2)) - (1+\ell)/(c+2) \in \mathbb{Z}$, then put

$$\xi_r = \Delta_0 + c_3\Gamma + \sum_{i=r-\ell}^{r-1} e_i, \quad \xi_{r-c-1} = e_{r-c-1} - e_{r-c}.$$

(4) If $c_4 = d/2 - g/((c+1)(c+2)) + 1/(c+2) \in \mathbb{Z}$, then put

$$\xi_r = \Delta_0 + c_4\Gamma - e_1 - (F + (O)), \quad \xi_{r-c-1} = F + (O) - e_{r-c}.$$

Here the numbering of the ξ_i's corresponds to that of the vertices in Figures 1, 2, 3, 4 according to $2g/(c+1) \bmod (c+2)$ with $d \equiv 2g/(c+1) \bmod 2$ in the case when c is odd and to the combination of $g/(c+1) \bmod (c+2)$ and the parity of d in the case when c is even.

Then these together with F, (O) clearly form a basis for $\mathrm{NS}(X)$ over $\mathbb{Q}$ in either case. While we see that $\{\xi_1, \xi_2, \dots, \xi_r\}$ forms a $\mathbb{Z}$-basis for $L_{(c,g,d,0)}{}^{-}$, we divide our argument between the case (4) and the other cases. At first, we restrict ourselves to the case (3), since the cases (1) and (2) are quite similar. Consider the matrix representing the base change from $(\Delta_0, \Gamma, e_1, e_2, \dots, e_r)$ to $(\xi_r, \xi_{r-1}, \xi_1, \xi_2, \dots, \xi_{r-2}, F, (O))$. Then it is easy to see that, off the $(r+1)$-th row, it is an integral triangular matrix all of whose diagonal entries are equal to one, and we have

$$F = e_{r-1} + (c+2)\xi_r + \left(\frac{2g}{c+1} + 2 + \ell\right)\xi_{r-1} + \left(\frac{2g}{c+1} + 1 + \ell\right)\sum_{k=1}^{c+2} k\xi_k$$
$$+ \sum_{k=c+3}^{r-\ell-1} (r + \ell(c+2) - k)\xi_k + \sum_{k=r-\ell}^{r-2} ((c+3)(r-k) - c - 2)\xi_k - (O).$$

For (4), we only have to note that the matrix representing the base change from $(\Delta_0, \Gamma, e_1, e_2, \ldots, e_r)$ to $(\xi_r + F + (O), \xi_{r-1}, \xi_1, \xi_2, \ldots, \xi_{r-c-2}, e_{r-c-1}, e_{r-c}, \ldots, e_{r-1}, (O))$ is an integral triangular matrix, all of whose diagonal entries are equal to one, and we have

$$F = -e_{r-c-1} + (c+2)(\xi_r + F + (O))$$
$$+\frac{2g}{c+1}\xi_{r-1} + \sum_{k=1}^{c+2}\left(\frac{2gk}{c+1} + c + 2 - k\right)\xi_k + \sum_{k=c+3}^{r-c-3}(r-c-2-k)\xi_k - \sum_{k=r-c}^{r-1} e_k - (O),$$
$$\xi_{r-c-1} - F - (O) = -e_{r-c}, \quad \xi_i = -e_{i+1} + e_i \quad (i = r-c, \ldots, r-2).$$

Hence in either case $\{\xi_1, \xi_2, \ldots, \xi_r\}$ forms a $\mathbb{Z}$-basis for ${L_{(c,g,d,0)}}^-$ and we obtain the corresponding Dynkin diagrams.

As to the last statement, we consider the case when $c \equiv 2 \bmod 4$ only, since the other cases are similar. We note that the self-pairing numbers of ξ_i's are always even except for ξ_r, while the parity of that of ξ_r varies even if we fix $g \geq (c+1)^2$. Consider for example the case when g is divisible by $(c+1)(c+2)$. If d is even then ${\xi_r}^2 = 2g/((c+1)(c+2)) + 1$ is odd, and if d is odd then ${\xi_r}^2 = 2g/((c+1)(c+2)) + (c+2)/2$ is even with $\ell = c/2$. $\square$

3.2 – *Type* $(c, g, 1, n)$

We shall determine the Mordell-Weil lattices for fibrations $f : X \to \mathbb{P}^1$ of type $(c, g, 1, n)$ with $(c+1)(c+2)/2 \leq g \leq (c+2)(c+3)/2$. When $n \geq 0$, the reduction Y is obtained by blowing $\mathbb{P}^2$ up at $p_0, \ldots, p_n$ corresponding to singular points of G_0 as in Theorem 2.3. Furthermore, the configuration is as in Proposition 3.3. Let e_i be the inverse image on X of p_i $(i = 0, \ldots, n)$. We denote by l the pull-back to X of a line on $\mathbb{P}^2$ and let e_i, $n+1 \leq i \leq r$, be the disjoint (-1)-sections of f coming from base points of Λ_f. Then $\mathrm{NS}(X) \simeq \mathbb{Z}l \oplus \bigoplus_{i=0}^{r} \mathbb{Z}e_i$. We take the last (-1)-section e_r as the zero section (O). Then (O) and F generate the trivial sublattice $T_{(c,g,1,n)} \subset \mathrm{NS}(X)$. Let $L_{(c,g,1,n)}$ be the essential sublattice obtained as the orthogonal complement of $T_{(c,g,1,n)}$.

We first consider the case where G_0 is as in (iv) of Theorem 2.3. Then

$$F = (c+4)l - 2\sum_{i=0}^{n} e_i - \sum_{i=n+1}^{r} e_i.$$

THEOREM 3.8. *Let* $(c+1)(c+2)/2 \leq g \leq (c+2)(c+3)/2$. *For a fibration of type* $(c, g, 1, n)$ *with* $n = (c+1)(c+4)/2 - g$, *the lattice* ${L_{(c,g,1,n)}}^-$ *is* *isomorphic to a positive-definite odd unimodular lattice of rank* $r = 3g - (c+3)(c-4)/2$ *whose Dynkin diagram is as in Figure* 5.

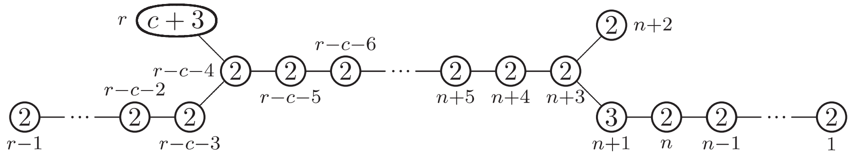

Fig. 5

PROOF. Take the following elements from $L_{(c,g,1,n)}$, whose numbering corresponds to that of the vertices in Figure 5:

$$\xi_i = e_{i-1} - e_i \quad (1 \le i \le r-1,\ i \ne n+1),$$

$$\xi_{n+1} = e_n - e_{n+1} - e_{n+2}, \qquad \xi_r = -l + \sum_{i=r-c-4}^{r-1} e_i.$$

Then we can show that the above elements form a $\mathbb{Z}$-basis for $L_{(c,g,1,n)}$ in the same way as in the case (3) in the proof of Theorem 3.7. □

Secondly, we shall determine the Mordell-Weil lattices for fibrations of type $(c, (c+1)(c+2)/2, 1, 4)$. If c is even, then

$$F = \frac{3c+6}{2} l - \frac{c+2}{2} \sum_{i=0}^{3} e_i - \frac{c}{2} e_4 - \sum_{i=5}^{c^2+5c+9} e_i, \quad (O) = e_{c^2+5c+9}.$$

THEOREM 3.9. *For a fibration of type* $(c, (c+1)(c+2)/2, 1, 4)$ *with an even number* c, *the lattice* $L_{(c,(c+1)(c+2)/2,1,4)}^{-}$ *is isomorphic to a positive-definite odd unimodular lattice of rank* $c^2 + 5c + 9$ *whose Dynkin diagram is as in Figure* 6.

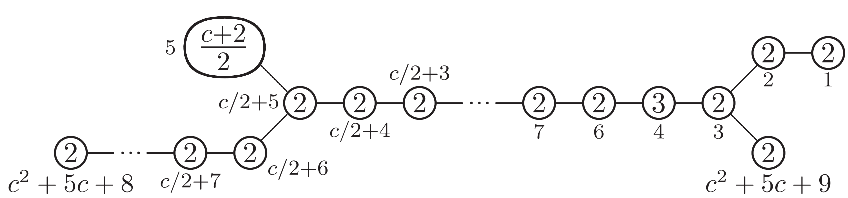

Fig. 6

PROOF. Take the following elements from $L_{(c,(c+1)(c+2)/2,1,4)}$, whose numbering corresponds to that of the vertices in Figure 6:

$$\xi_i = e_{i-1} - e_i \quad (i = 1, 2, 3, 6, 7, \dots, c^2+5c+8),$$

$$\xi_4 = e_3 - e_4 - e_5, \qquad \xi_5 = e_4 - \sum_{i=5}^{c/2+4} e_i, \qquad \xi_{c^2+5c+9} = l - e_0 - e_1 - e_2.$$

Then we can show that the above elements form a $\mathbb{Z}$-basis for $L_{(c,(c+1)(c+2)/2,1,4)}$ in the same way as in the case (3) in the proof of Theorem 3.7. □

Next, we consider the case where c is odd, $g = (c+1)(c+2)/2$, $d = 1$ and $n = 4$. Then

$$F = \frac{3c+5}{2}l - \frac{c+1}{2}\sum_{i=0}^{4} e_i - \sum_{i=5}^{c^2+5c+9} e_i, \quad (O) = e_{c^2+5c+9}.$$

THEOREM 3.10. *For a fibration of type* $(c, (c+1)(c+2)/2, 1, 4)$ *with an odd number* c*, the lattice* $L_{(c,(c+1)(c+2)/2,1,4)}{}^{-}$ *is isomorphic to a positive-definite odd unimodular lattice of rank* c^2+5c+9 *whose Dynkin diagram is as in Figure* 7.

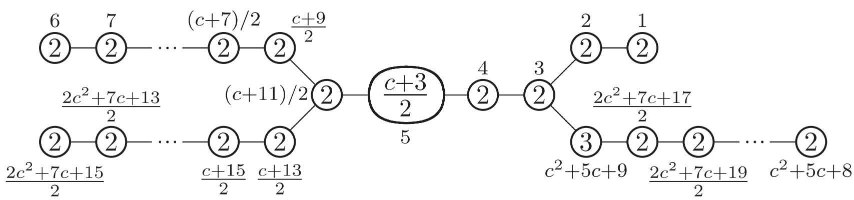

Fig. 7

PROOF. Take the following elements from $L_{(c,(c+1)(c+2)/2,1,4)}$, whose numbering corresponds to that of the vertices in Figure 7:

$$\xi_i = e_{i-1} - e_i \ \ (1 \le i \le c^2+5c+8, \ \ i \ne 5, \ \ (2c^2+7c+15)/2),$$

$$\xi_5 = e_4 - \sum_{i=5}^{(c+9)/2} e_i, \quad \xi_{(2c^2+7c+15)/2} = e_{(2c^2+7c+13)/2} - (F + (O)),$$

$$\xi_{c^2+5c+9} = l - e_0 - e_1 - e_2 - e_{(2c^2+7c+15)/2}.$$

Then we can show that the above elements form a $\mathbb{Z}$-basis for $L_{(c,(c+1)(c+2)/2,1,4)}$ in the same way as in the case (4) in the proof of Theorem 3.7 □

In the last, we shall determine the Mordell-Weil lattices for fibrations of type $(4, 16, 1, 3)$. Then

$$F = 9l - 3\sum_{i=0}^{3} e_i - \sum_{i=4}^{48} e_i, \quad (O) = e_{48}.$$

THEOREM 3.11. *For a fibration of type* $(4, 16, 1, 3)$*, the lattice* $L_{(4,16,1,3)}{}^{-}$

is isomorphic to a positive-definite even unimodular lattice of rank 48 *whose Dynkin diagram is as in Figure 8.*

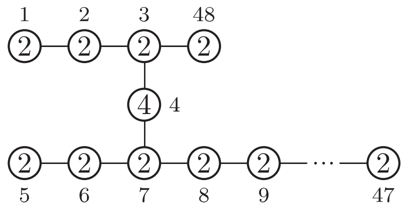

Fig. 8

PROOF. Take the following elements from $L_{(4,16,1,3)}$, whose numbering corresponds to that of the vertices in Figure 8:

$$\xi_i = e_{i-1} - e_i \quad (i = 1, 2, 3, 5, 6, \ldots, 47), \qquad \xi_4 = e_3 - e_4 - e_5 - e_6,$$
$$\xi_{48} = l - e_0 - e_1 - e_2.$$

Then we can show that the above elements form a $\mathbb{Z}$-basis for $L_{(4,16,1,3)}$ in the same way as in the case (3) in the proof of Theorem 3.7. □

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