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(S_3, S_6)-Amalgams V.

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Introduction.

In part IV of this present series the commuting case for (S_3, S_6)-amalgams was examined when $\alpha \in O(S_6)$ for (α, α') a critical pair. This paper and the succeeding two parts are devoted to the commuting case when, for (α, α') a critical pair, $\alpha \in O(S_3)$. In fact the bulk of our work is concerned with the situation $\eta(G_\beta, V_\beta) = 1$, where $\beta \in O(S_6)$. Unlike Part IV, for this situation, there appear to be no subamalgams which can be exploited early on. Although a very precise description of V_β is obtained in Theorem 12.1 for our subsequent analysis we need to consider five subcases. This subdivision, given in Section 12, is done according to the size of $\mathrm{core}_{G_\alpha} V_\beta$ $(\alpha \in \Delta(\beta))$ and also whether V_β/Z_β is an orthogonal module or not. Of the five possibilities three, as it were, are the "mainline" cases - Cases 1, 3 and 4. We briefly discuss and compare each of these cases.

Case 3 is concerned with the smallest possibility for $\mathrm{core}_{G_\alpha} V_\beta$, namely $\mathrm{core}_{G_\alpha} V_\beta = Z_\alpha$. This case has the most complicated possibilities for V_β/Z_β, with V_β/Z_β being a quotient of $\binom{4}{1} \oplus 1$. For this case the core argument (Lemma 9.9) is especially valuable as it often enables us to restrict the size of certain commutators. Our scrutiny of Case 3 takes place in Part VI - the end result being that this case cannot arise! For all the other cases we obtain bounds on the parameter b which are pursued further in [LPR2]. At the other end of the scale we have Case 1 with $[V_\beta : \mathrm{core}_{G_\alpha} V_\beta] = 2$ (and then V_β/Z_β is a natural module). This, from the outset, is a very tight configuration (for example, V_β acts as a central transvection on $V_{\alpha'}/Z_{\alpha'}$ for

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$(\alpha, \alpha') \in \mathscr{C}$), which nevertheless requires a delicate analysis. By contrast with Case 3 the core argument is no use here whatsoever. Case 4 (when $[V_\beta : \mathrm{core}_{G_\alpha} V_\beta] = 2^2$ and V_β/Z_β is a natural module) lies somewhere between Cases 1 and 3. The core argument is of little use and, initally, the configuration has a greater degree of freedom.

We remark that central transvections make their presence felt in a big way in Parts V, VI and VII. Indeed, without such transvections Cases 1, 2 and 3 would be virtually non-existent.

For ease of reference we continue the section numbering started in [LPR1]. We make the most use of material in Sections 1 and 2, and it is there we refer the reader to for notation and background results. Briefly the contents of this paper are as follows. Section 11 is, mostly, a warm-up for the work in Section 12 where the above mentioned structure of V_β is determined. Cases 1 and 2 are the subject of Section 13, the main conclusions being given in Theorems 13.1 and 13.11.

11. Some preliminary results.

For this paper and Parts V and VI we assume the following hypothesis.

HYPOTHESIS 11.0. *If* $(\alpha, \alpha') \in \mathscr{C}$, *then* $[Z_\alpha, Z_{\alpha'}] = 1$ *and* $\alpha \in O(S_3)$.

Some elementary observations on this hypothesis are gathered in our first result of this section.

LEMMA 11.1. *Let* $\mu \in O(S_3)$ *and* $\lambda \in \Delta(\mu)$.

(i) $[G_{\lambda\mu}, Z_\mu] = Z_\lambda = \Omega_1(Z(G_{\lambda\mu})) = \Omega_1(Z(G_\lambda))$.

(ii) $C_{Z_\mu}(G_\mu) = 1$, $G_{G_\mu}(Z_\mu) = Q_\mu$ *and* $Z_\mu = Z_{\lambda_1} \times Z_{\lambda_2}$ *whenever* $\lambda_1, \lambda_2 \in \Delta(\mu)$ *with* $\lambda_1 \neq \lambda_2$.

(iii) $b \equiv 1(2)$ *and* $b > 1$.

(iv) *If* $X \leq G_{\lambda\mu}$ *and* $X \not\leq Q_\mu$, *then* $[X, Z_\mu] = Z_\lambda = C_{Z_\mu}(X)$.

(v) *If* $X \leq Q_\lambda$ *is such that* $[X, V_\lambda] \neq 1$, *then* $[X, V_\lambda] = Z_\lambda$.

(vi) $[Q_\lambda, V_\lambda] \leq Z_\lambda$.

(vii) *G is transitive on paths* $(\lambda_1, \lambda_2, \lambda_3)$ *of length* 2 *where* $\lambda_1 \in O(S_6)$.

PROOF. Let $(\alpha, \alpha') \in \mathscr{C}$. Then $[Z_\alpha, Z_{\alpha'}] = 1$ implies $Z_{\alpha'} \leq Z(G_{\alpha'})$ and hence $\Omega_1(Z(G_{\alpha'-1\alpha'})) = Z_{\alpha'} \leq Z_\xi$ for all $\xi \in \Delta(\alpha')$. Therefore $b \equiv 1(2)$. Since $G_{\alpha\beta}$ acts as an involution on Z_α, $[G_{\alpha\beta}, Z_\alpha] \leq C_{Z_\alpha}(G_{\alpha\beta}) = \Omega_1(Z(G_{\alpha\beta})) = Z_\beta$. Likewise $[G_{\alpha\alpha-1}, Z_\alpha] \leq Z_{\alpha-1}$. Hence $Z_\alpha = Z_\beta Z_{\alpha-1} = [G_{\alpha\beta}, Z_\alpha] Z_{\alpha-1}$. By

Lemma 11.1(ii) $C_{Z_\alpha}(G_\alpha) = 1$ and thus $Z_\beta \cap Z_{\alpha-1} = 1$. Hence, by orders, $Z_\beta = [G_{\alpha\beta}, Z_\alpha]$. Since G acts edge-transitively upon Γ, we have verified (i) and (ii).

Assume $b = 1$ holds. Since, by part (ii), $[Z_\alpha, Q_\beta] \leq Z_\beta$ this then gives $\eta(G_\beta, Q_\beta) = 0$, contrary to the hypothesis $C_{G_\beta}(Q_\beta) \leq Q_\beta$. Therefore $b > 1$.

For part (iv) we have $G_{\lambda\mu} = XQ_\mu$ and so (iv) follows easily from (i).

Because $[X, V_\lambda] \neq 1$ there exists $\xi \in \Delta(\lambda)$ such that $[X, Z_\xi] \neq 1$. Thus $X \not\leq Q_\xi$ by (ii) and now (iv) implies (v). Part (vi) is an easy consequence of (v). While (vii) follows from G being edge-transitive and, as $\lambda_2 \in O(S_3)$, $G_{\lambda_1\lambda_2}$ being transitive on $\Delta(\lambda_2)\backslash\{\lambda_1\}$.

Our next lemma will be put to use after we have established Theorem 12.1.

LEMMA 11.2. *If V_β/Z_β contains either a natural or orthogonal S_6-module, then $Q_\alpha Q_\beta = G_{\alpha\beta}$ and $[V_\beta, Q_\beta] = Z_\beta$.*

PROOF. Let $V_\beta \geq X \geq Z_\beta$ be such that X/Z_β is a natural or orthogonal S_6-module, and suppose $Q_\alpha Q_\beta \neq G_{\alpha\beta}$. Then we have $Q_\alpha \geq Q_\beta$ and so $Q_\xi \geq Q_\beta$ for all $\xi \in \Delta(\beta)$ by Lemma 1.1(i), and hence $V_\beta \leq Z(Q_\beta)$. If $[V_\beta, Q_\beta] \neq Z_\beta$, then, by Lemma 11.1(v), we also have $V_\beta \leq Z(Q_\beta)$. So, if the lemma is false, we may view V_β as a $GF(2)$ (G_β/Q_β)-module. Appealing to Lemma 2.2(iv) gives $X = Y \oplus C_X(G_\beta)$ with $Y \cong 4$ or $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$. In the former case $C_X(G_\beta) = Z_\beta$ and in the latter $[Z_\beta : C_X(G_\beta)] = 2$. In either case (using Proposition 2.5(i) for $Y \cong \begin{pmatrix} 4 \\ 1 \end{pmatrix}$) we obtain $\Omega_1(Z(G_{\alpha\beta})) > Z_\beta$, against Lemma 11.1(i). Therefore $Q_\alpha Q_\beta = G_{\alpha\beta}$ and $[V_\beta, Q_\beta] = Z_\beta$, as required.

LEMMA 11.3. *Let $(\alpha, \alpha') \in \mathscr{C}$ and put $C = C_{V_\beta}(O^2(G_\beta))$. Then $Z_\alpha \cap C = Z_\beta$.*

PROOF. Since $C_{Z_\alpha}(G_\alpha) = 1$, Z_α is a direct sum of 2-dimensional irreducible G_α/Q_α-modules. If $Z_\alpha \cap C > Z_\beta$ were to hold, then $Z_\alpha \cap C$ would contain a non-zero G_α/Q_α-submodule of Z_α, against Lemma 1.1(ii).

LEMMA 11.4. *Let $(\delta, \delta') \in \mathscr{C}$. If $\eta(G_{\delta+1}, V_{\delta+1}) > 1$, then $[V_{\delta+1} \cap \cap Q_{\delta'}, V_{\delta'}] \neq 1$.*

PROOF. Suppose the result is false and, without loss of generality, we take $\alpha = \delta$ and $\alpha' = \delta'$. So $\eta(G_\beta, V_\beta) > 1$ and $[V_\beta \cap Q_{\alpha'}, V_{\alpha'}] = 1$. Since $[V_\beta : V_\beta \cap Q_{\alpha'}] \leq 2^3$, $[V_\beta : C_{V_\beta}(V_{\alpha'})] \leq 2^3$. If $[V_{\alpha'} : V_{\alpha'} \cap Q_\beta] \geq 2^2$, then $\eta(G_\beta, V_\beta) \leq 1$. So $[V_{\alpha'} : V_{\alpha'} \cap Q_\beta] \leq 2$. Moreover, $V_{\alpha'} \cap Q_\beta \leq Q_\alpha$ implies that

$[V_{\alpha'} : C_{V_{\alpha'}}(Z_\alpha)] \leq 2$, contrary to $\eta(G_{\alpha'}, V_{\alpha'}) > 1$. Therefore $V_{\alpha'} \cap Q_\beta \not\leq Q_\alpha$ and, similarly, $V_{\alpha'} \not\leq Q_\beta$.

(11.4.1) $$|Z_{\alpha-1}| = 2$$

By Lemma 11.1(iv) $C_{Z_\alpha}(V_{\alpha'} \cap Q_\beta) = Z_\beta$. Hence Lemma 11.1(ii) implies $C_{Z_{\alpha-1}}(V_{\alpha'} \cap Q_\beta) = 1$. So

$$1 = Z_{\alpha-1} \cap V_\beta \cap Q_{\alpha'} = Z_{\alpha-1} \cap Q_{\alpha'} .$$

Since $[V_{\alpha'} : V_{\alpha'} \cap Q_\beta] \leq 2$, we have $[V_{\alpha'} : C_{V_{\alpha'}}(Z_{\alpha-1})] \leq 2^2$. Therefore, as $\eta(G_{\alpha'}, V_{\alpha'}) > 1$, we get $|Z_{\alpha-1}Q_{\alpha'}/Q_{\alpha'}| = 2$, which yields (11.4.1).

(11.4.2) $$\text{For } x \in V_\beta, \quad [V_{\alpha'} : C_{V_{\alpha'}}(x)] \leq 2^2 .$$

Let $x \in V_\beta$. From $V_{\alpha'} \cap Q_\beta \not\leq Q_\alpha$ and Lemma 11.1(v) $[V_{\alpha'} \cap Q_\beta, V_\beta] = Z_\beta$. Thus $|[V_{\alpha'} \cap Q_\beta, x]| \leq 2$ by (11.4.1) from which, as $[V_{\alpha'} : V_{\alpha'} \cap Q_\beta] \leq 2$, (11.4.2) follows.

If $|V_\beta Q_{\alpha'}/Q_{\alpha'}| \geq 2^2$, then V_β contains an element y, $y \notin Q_{\alpha'}$, which is not a transvection on some non central $G_{\alpha'}$-chief factor in $V_{\alpha'}$ whence $\eta(G_{\alpha'}, V_{\alpha'}) \leq 1$ by (11.4.2). Thus $[V_\beta : V_\beta \cap Q_{\alpha'}] = 2$ and so $[V_\beta : C_{V_\beta}(V_{\alpha'})] \leq 2$. But then since $V_{\alpha'} \not\leq Q_\beta$, this gives $\eta(G_\beta, V_\beta) \leq 1$, a contradiction. So the lemma holds.

Corollary 11.5. *Let $(\delta, \delta') \in \mathscr{C}$ and assume $\eta(G_{\delta+1}, V_{\delta+1}) > 1$. Then*

(i) $[V_{\delta+1} \cap Q_{\delta'}, V_{\delta'}] = Z_{\delta'}$, $[V_{\delta'} \cap Q_{\delta+1}, V_{\delta+1}] = Z_{\delta+1}$; *and*
(ii) $V_{\delta'} \not\leq Q_{\delta+1}$.

Proof. Lemmas 11.1(v) and 11.4 yield $[V_{\delta+1} \cap Q_{\delta'}, V_{\delta'}] = Z_{\delta'}$. Suppose $V_{\delta'} \leq Q_{\delta+1}$. Using Lemma 11.1(vi) gives

$$Z_{\delta'} = [V_{\delta+1} \cap Q_{\delta'}, V_{\delta'}] \leq [V_{\delta+1}, V_{\delta'}] \leq Z_{\delta+1}$$

and thus $[V_{\delta+1}, V_{\delta'}] = Z_{\delta'}$ by orders. Then $\eta(G_{\delta'}, V_{\delta'}) = 0$ which is impossible. Therefore $V_{\delta'} \not\leq Q_{\delta+1}$. Hence we may find $\rho \in \Delta(\alpha')$ such that $(\rho, \delta+1) \in \mathscr{C}$, and Lemmas 11.1(v) and 11.4 give the remaining part of (i).

Lemma 11.6. *Let $(\alpha, \alpha') \in \mathscr{C}$ and assume that $\eta(G_\beta, V_\beta) > 1$. If $|V_{\alpha'}Q_\beta/Q_\beta| \leq 2^2$, then $|V_\beta Q_{\alpha'}/Q_{\alpha'}| = 2 = |V_{\alpha'}Q_\beta/Q_\beta|$.*

Proof. By Corollary 11.5 we may find $\lambda \in \Delta(\beta)$ such that $[Z_\lambda, V_{\alpha'} \cap Q_\beta] \neq 1$. Lemma 11.1(iv) then gives $C_{Z_\lambda}(V_{\alpha'} \cap Q_\beta) = Z_\beta$ and so $C_{Z_\lambda \cap Q_{\alpha'}}(V_{\alpha'} \cap Q_\beta) = Z_\beta$. Since $[V_{\alpha'} : C_{V_{\alpha'}}(Z_\lambda)] \leq 2[V_{\alpha'} : V_{\alpha'} \cap Q_\beta] \leq 2^3$ and

$\eta(G_{\alpha'}, V_{\alpha'}) > 1$, we infer that $[Z_\lambda : Z_\lambda \cap Q_{\alpha'}] \leq 2$. Now $V_{\alpha'} \cap Q_\beta$ acts as an involution on $Z_\lambda \cap Q_{\alpha'}$ and hence

$$|[Z_\lambda \cap Q_{\alpha'}, V_{\alpha'} \cap Q_\beta]| = [Z_\lambda \cap Q_{\alpha'} : Z_\beta] \geq |Z_\beta|/2.$$

Because $[Z_\lambda \cap Q_{\alpha'}, V_{\alpha'} \cap Q_\beta] \leq Z_{\alpha'} \cap Z_\beta$ we see that $[Z_\beta : Z_{\alpha'} \cap Z_\beta] \leq 2$. Let $x \in V_\beta$ and put $\overline{V}_{\alpha'} = V_{\alpha'}/Z_{\alpha'}$. Then $|[\overline{V_{\alpha'} \cap Q_\beta}, x]| \leq |\overline{Z}_\beta| \leq 2$ (note that $Z_\beta \leq V_{\alpha'}$). Therefore $[V_{\alpha'} : C_{V_{\alpha'}}(x)] \leq 2[V_{\alpha'} : V_{\alpha'} \cap Q_\beta] \leq 2^3$ which, as $\eta(G_{\alpha'}, V_{\alpha'}) > 1$, forces $|V_\beta Q_{\alpha'}/Q_{\alpha'}| \leq 2$. From Corollary 11.5 there exists $\rho \in \Delta(\alpha')$ such that $(\rho, \beta) \in \mathscr{C}$ and so, as $|V_\beta Q_{\alpha'}/Q_{\alpha'}| = 2$, we may repeat the above argument to obtain $|V_{\alpha'} Q_\beta / Q_\beta| = 2$. This proves the lemma.

LEMMA 11.7. *Suppose that $b > 3$, $(\alpha, \alpha') \in \mathscr{C}$ and $\eta(G_\beta, V_\beta) > 1$. Then $U_\alpha \leq G_{\alpha'}$.*

PROOF. If $U_\alpha \not\leq Q_{\alpha'-2}$, then there exists $\alpha - 2 \in \Delta^{[2]}(\alpha)$ such that $(\alpha - 2, \alpha' - 2) \in \mathscr{C}$. Applying Corollary 11.5 to $(\alpha - 2, \alpha' - 2)$ gives

$$Z_{\alpha-1} = [V_{\alpha'-2} \cap Q_{\alpha-1}, V_{\alpha-1}] \leq V_{\alpha'-2} \leq Q_{\alpha'} ,$$

as $b > 3$. Then $Z_\alpha = Z_{\alpha-1} Z_\beta \leq Q_{\alpha'}$, a contradiction. Therefore $U_\alpha \leq Q_{\alpha'-2}$. Now, by Corollary 11.5, $Z_{\alpha'} = [V_\beta \cap Q_{\alpha'}, V_{\alpha'}] \leq V_\beta$ and so $[U_\alpha, Z_{\alpha'}] = 1$. Therefore

$$U_\alpha \leq C_{G_{\alpha'-1}}(Z_{\alpha'}) = G_{\alpha'-1\alpha'} ,$$

as required.

LEMMA 11.8. *Let $(\alpha, \alpha') \in \mathscr{C}$ and suppose $b > 3$ and $\eta(G_\beta, V_\beta) > 1$. Then $[U_\alpha, V_{\alpha'}] \not\leq U_\alpha$.*

PROOF. Supposing $[U_\alpha, V_{\alpha'}] \leq U_\alpha$ we seek to uncover a contradiction. Setting $P_\beta = \langle G_{\alpha\beta}, V_{\alpha'} \rangle$ and $\widehat{V}_\beta = V_\beta / C_{V_\beta}(O^2(G_\beta))$ we first show that

(11.8.1) (i) $P_\beta / Q_\beta \cong S_4 \times \mathbb{Z}_2$
(ii) $\eta(G_\beta, V_\beta) = 2$ and $\widehat{V}_\beta$ is isomorphic to a direct sum of two isomorphic natural S_6-modules
(iii) $|Z_\alpha| = 2^4$ and $|Z_\beta| = |Z_{\alpha'}| = 2^2$.
(iv) $|Z_\alpha Q_{\alpha'}/Q_{\alpha'}| = 2$ and $|[Z_\alpha, V_{\alpha'}/Z_{\alpha'}]| = 2^2$
(v) $C_{Z_\alpha}(V_{\alpha'} \cap G_{\alpha\beta}) = Z_\beta$

From $[U_\alpha, V_{\alpha'}] \leq U_\alpha$ we have that P_β normalizes U_α and hence $P_\beta \neq G_\beta$ by Lemma 1.1 (ii). By the parabolic argument (Lemma 3.10)

$[V_{\alpha'} : V_{\alpha'} \cap G_{\alpha\beta}] \leq 2$, and so $[V_{\alpha'} : V_{\alpha'} \cap Q_\alpha] \leq 2^2$. Thus $[V_{\alpha'} : C_{V_{\alpha'}}(Z_{\alpha'})] \leq 2^2$. Therefore $\eta(G_{\alpha'}, V_{\alpha'}) = 2$ with $|Z_\alpha Q_{\alpha'}/Q_{\alpha'}| = 2$, $[V_{\alpha'} : C_{V_{\alpha'}}(Z_\alpha)] = 2^2$ and both non central chief factors of $V_{\alpha'}$ are isomorphic natural S_6-modules. Clearly $V_{\alpha'} \not\leq G_{\alpha\beta}$ and so we conclude that $P_\beta/Q_\beta \cong S_4 \times \mathbb{Z}_2$. Combining Lemmas 2.16 and 11.1(i), (vi) gives (ii) for $\widehat{V}_{\alpha'}$, so proving part (ii). Because $\widehat{V}_\beta = \langle \widehat{Z}_\alpha^{G_\beta} \rangle$ it follows from part (ii) that $|\widehat{Z}_\alpha| = 2^2$. Hence $|Z_\alpha| = 2^4$ and $|Z_\beta| = 2^2$ by Lemmas 11.1(ii) and 11.3, and we have (iii). Also, by Lemma 11.1(vi), Z_α acts as an involution upon $V_{\alpha'}/Z_{\alpha'}$. By Lemma 11.1(i) $\eta(G_{\alpha'}, V_{\alpha'}/Z_{\alpha'}) = 2$ and so $[V_{\alpha'}/Z_{\alpha'} : C_{V_{\alpha'}/Z_{\alpha'}}(Z_\alpha)] = 2^2$. Hence $|[Z_\alpha, V_{\alpha'}/Z_{\alpha'}]| = 2^2$, so establishing (iv). Finally we prove (v). If $V_{\alpha'} \cap G_{\alpha\beta} \leq Q_\alpha$, then $[V_{\alpha'} : C_{V_{\alpha'}}(Z_\alpha)] \leq 2$ which is impossible. So $V_{\alpha'} \cap G_{\alpha\beta} \not\leq Q_\alpha$ and thus $Z_\beta = C_{Z_\alpha}(V_{\alpha'} \cap G_{\alpha\beta})$.

(11.8.2) P_β/Q_β is the parabolic with restriction $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$ on the non-central chief factors in $\widehat{V}_\beta$.

Suppose that (11.8.2) is false. Then, by (11.8.1)(i),(ii), P_β/Q_β is the parabolic with restriction $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ on all the (isomorphic) natural S_6-modules in $\widehat{V}_\beta$. From Lemma 11.1(i) $\widehat{Z}_\alpha \leq C_{\widehat{V}_\beta}(G_{\alpha\beta})$. Let d be an element of order 3 in P_β. Since d centralizes $C_{\widehat{V}_\beta}(G_{\alpha\beta})$, we infer that $Z_\alpha \leq C_{V_\beta}(d)$. Thus

$$Z_\alpha \trianglelefteq \langle G_{\alpha\beta}, d \rangle = P_\beta\,.$$

In particular, $V_{\alpha'}$ normalizes $Z_{\alpha'}$ and so $[V_{\alpha'}, Z_\alpha] \leq Z_\alpha$. From (11.8.1)(iv),(v) $|Z_\alpha \cap Q_{\alpha'}| = 2^3$ and $C_{Z_\alpha}(V_{\alpha'} \cap G_{\alpha\beta}) = Z_\beta$ and consequently $[Z_\alpha \cap Q_{\alpha'}, V_{\alpha'} \cap \cap G_{\alpha\beta}] \neq 1$. So $[Z_\alpha \cap Q_{\alpha'}, V_{\alpha'}] \neq 1$ and an appeal to Lemma 11.1(ii) gives $[Z_\alpha \cap Q_{\alpha'}, V_{\alpha'}] = Z_{\alpha'}$. Combining this with $|[Z_\alpha, V_{\alpha'}/Z_{\alpha'}]| = 2^2$ (since $|Z_{\alpha'}| = |Z_\beta|$) yields $|[Z_\alpha, V_{\alpha'}]| = 2^4$. But then

$$Z_\alpha = [Z_\alpha, V_{\alpha'}] \leq V_{\alpha'} \leq Q_{\alpha'}\,,$$

contrary to $(\alpha, \alpha') \in \mathscr{C}$. With this contradiction we have proved (11.8.2).

(11.8.3) $$|V_\beta Q_{\alpha'}/Q_{\alpha'}| \geq 2^2.$$

If $|V_\beta Q_{\alpha'}/Q_{\alpha'}| \not\geq 2^2$, then $V_\beta Q_{\alpha'} = Z_\alpha Q_{\alpha'}$ whence $[V_\beta, V_{\alpha'}/Z_{\alpha'}] = = [Z_\alpha, V_{\alpha'}/Z_{\alpha'}]$. Since $Z_{\alpha'} \leq [V_\beta, V_{\alpha'}]$ by Corollary 11.5, using (11.8.1)(iii),(iv) we see that $|[V_\beta, V_{\alpha'}]| = 2^4$. Hence, as $|Z_\beta| = 2^2$, $|[V_{\alpha'}, V_\beta/Z_\beta]| = 2^2$. Consequently $V_{\alpha'}$ acts as a transvection upon each of the non-central chief factors in $\widehat{V}_\beta$. Together (11.8.2) and Proposition 2.5(viii) imply that $O_2(P_\beta)/Q_\beta$ is the unique quadratically acting (upon each of the non-central chief factors in $\widehat{V}_\beta$) $E(2^3)$-subgroup of each Sylow 2-subgroup of P_β/Q_β. Then Proposition 2.5(iii) forces $V_{\alpha'} \leq O_2(P_\beta) \leq G_{\alpha\beta}$, contradicting (11.8.1)(i). Thus we conclude that $|V_\beta Q_{\alpha'}/Q_{\alpha'}| \geq 2^2$.

(11.8.4) A contradiction

By Corollary 11.5 there exists $\rho \in \Delta(\alpha')$ such that $(\rho, \beta) \in \mathscr{C}$. If $|V_{\alpha'} Q_\beta/Q_\beta| \leq 2^2$, then applying Lemma 11.6 to (ρ, β) yields $|V_\beta Q_{\alpha'}/Q_{\alpha'}| = 2$, against (11.8.3). Therefore, as $V_{\alpha'}$ is elementary abelian, $|V_{\alpha'} Q_\beta/Q_\beta| = 2^3$, and, of course, $V_{\alpha'}$ acts quadratically on V_β. Again, using (11.8.2) and Proposition 2.5(viii), we deduce the untenable $V_{\alpha'} \leq O_2(P_\beta) \leq G_{\alpha\beta}$. This is the desired contradiction which completes the proof of Lemma 11.8.

12. The structure of V_β/Z_β.

THEOREM 12.1. *Let $(\alpha, \alpha') \in \mathscr{C}$. Then*

(i) $\eta(G_\beta, V_\beta) = 1$ with V_β/Z_β isomorphic to a quotient of $\begin{pmatrix} 4 \\ 1 \end{pmatrix} \oplus 1$;

and

(ii) $|Z_\alpha| = 2^2$, $|Z_\beta| = 2$.

PROOF. First we suppose that $b > 3$ and $\eta(G_\beta, V_\beta) > 1$ and argue for a contradiction. So Lemmas 11.7 and 11.8 are available to give $U_\alpha \leq G_{\alpha'}$ and $[U_\alpha, V_{\alpha'}] \not\leq U_\alpha$. Also, from Corollary 11.5, $[V_\beta, V_{\alpha'}] \geq Z_{\alpha'}$. So if $U_\alpha Q_{\alpha'} = = V_\beta Q_{\alpha'}$, we then have $U_\alpha = V_\beta(U_\alpha \cap Q_{\alpha'})$ whence, using Lemma 11.1(vi), $[U_\alpha, V_{\alpha'}] = [V_\beta, V_{\alpha'}] \leq V_\beta \leq U_\alpha$, a contradiction. Therefore $U_\alpha Q_{\alpha'} \neq V_\beta Q_{\alpha'}$. Since U_α is elementary abelian we infer that $|V_\beta Q_{\alpha'}/Q_{\alpha'}| \leq 2^2$. Now Lemma 11.6 and Corollary 11.5(ii) give

(12.1.1) $$|V_\beta Q_{\alpha'}/Q_{\alpha'}| = |V_{\alpha'} Q_\beta/Q_\beta| = 2.$$

Put $\overline{V}_\beta = V_\beta/Z_\beta$.

(12.1.2) (i) $(V_{\alpha'} \cap Q_\beta)Q_\alpha = G_{\alpha\beta}$.

(ii) $\eta(G_\beta, V_\beta) = 2$ and there exists $X_\beta \trianglelefteq G_\beta$ with $Z_\beta \leq X_\beta \leq V_\beta$ such that $\overline{V}_\beta = \overline{X}_\beta C_{\overline{V}_\beta}(G_\beta)$ and $\overline{X}_\beta$ is isomorphic to a quotient of $\begin{pmatrix} 4 \\ 1 \end{pmatrix} \oplus \begin{pmatrix} 4 \\ 1 \end{pmatrix}$. Moreover, the two 4's in $\overline{X}_\beta$ are isomorphic natural S_6-modules.

(iii) $|Z_\alpha| = 2^4$, $|Z_\beta| = |Z_{\alpha'}| = 2^2$ and $\eta(G_\alpha, Z_\alpha) = 2$.

By (12.1.1) $[V_{\alpha'} : V_{\alpha'} \cap Q_\beta] = 2$ and so, as $\eta(G_{\alpha'}, V_{\alpha'}) > 1$, $V_{\alpha'} \cap Q_\beta \not\leq Q_\alpha$. Thus (i) holds, and $[V_{\alpha'} : C_{V_{\alpha'}}(Z_\alpha)] = 2^2$ with $\eta(G_{\alpha'}, V_{\alpha'}) = 2$. Applying

Lemmas 2.16 and 2.2(iv) to $\overline{V}_{\alpha'} = V_{\alpha'}/Z_{\alpha'}$ yields (ii) for $\overline{V}_{\alpha'}$, so proving (ii). Using Lemmas 11.1(ii), 11.3, part (ii) and the fact that $Z(G_\alpha) = 1$ gives (iii).

Set $\Delta(\alpha) = \{\beta, \alpha - 1, \lambda\}$.

(12.1.3) $$[V_{\alpha'} : V_{\alpha'} \cap Q_{\alpha-1}] \geq 2^4 .$$

From (12.1.1) we have $Z_\alpha Q_{\alpha'} = V_\beta Q_{\alpha'}$ and hence $V_{\alpha-1}Q_{\alpha'} \geq V_\beta Q_{\alpha'}$. By (12.1.2), $U_\alpha Q_{\alpha'} = V_{\alpha-1}Q_{\alpha'}$. Since $U_\alpha Q_{\alpha'} \neq V_\beta Q_{\alpha'}$, $|V_{\alpha-1}Q_{\alpha'}/Q_{\alpha'}| \geq 2^2$ and hence, by (12.1.2)(ii) and Proposition 2.5(iii), $[X_{\alpha'} : C_{X_{\alpha'}}(V_{\alpha'-1})] \geq 2^4$. If $[X_{\alpha'} \cap Q_{\alpha-1}, V_{\alpha-1}] \neq 1$, then by Lemma 11.1(v)

$$Z_{\alpha-1} = [X_{\alpha'} \cap Q_{\alpha-1}, V_{\alpha-1}] \leq V_{\alpha'} \leq Q_{\alpha'}$$

whence $Z_\alpha = Z_{\alpha-1}Z_\beta \leq Q_{\alpha'}$. Hence $[X_{\alpha'} \cap Q_{\alpha-1}, V_{\alpha-1}] = 1$ from which (12.1.3) follows.

(12.1.4) $$V_{\alpha'-2} \leq Q_{\alpha-1} \quad \text{and} \quad [U_\alpha, V_{\alpha'-2}] = 1.$$

Suppose $V_{\alpha'-2} \not\leq Q_{\alpha-1}$. Then there exists $\rho \in \Delta(\alpha'-2)$ such that $(\rho, \alpha-1) \in \mathscr{C}$. Applying Corollary 11.5 to $(\rho, \alpha-1)$ gives $Z_{\alpha-1} \leq$ $\leq [V_{\alpha'-2}, V_{\alpha-1}] \leq V_{\alpha'-2} \leq Q_{\alpha'}$, since $b > 3$. This is against $(\alpha, \alpha') \in \mathscr{C}$, and therefore $V_{\alpha'-2} \leq Q_{\alpha-1}$. If $[V_{\alpha'-2}, V_{\alpha-1}] \neq 1$, then Lemma 11.1(v) gives $Z_{\alpha-1} = [V_{\alpha'-2}, V_{\alpha-1}]$ which again contradicts $(\alpha, \alpha') \in \mathscr{C}$. Thus $[V_{\alpha'-2}, V_{\alpha-1}] = 1$ and likewise $[V_{\alpha'-2}, V_\lambda] = 1$, so we have (12.1.4).

(12.1.5) $$[V_{\alpha'} : V_{\alpha'} \cap V_{\alpha'-2}] \geq 2^4.$$

This follows from (12.1.3) and (12.1.4).

(12.1.6) $|[U_\alpha, V_{\alpha'} \cap Q_\beta]| \geq 2^4[V_{\alpha'} : V_{\alpha'} \cap V_{\alpha'-2}]$ and (hence) $|[U_\alpha, V_{\alpha'} \cap Q_\beta]| \geq 2^8$.

In view of (12.1.3) $V_{\alpha'} \cap Q_\beta \cap Q_\alpha$ must act as at least a fours group on $V_{\alpha-1}/Z_{\alpha-1}$, and thus $|[V_{\alpha'} \cap Q_\beta \cap Q_\alpha, V_{\alpha-1}]| \geq 2^4$ by (12.1.2)(ii) and Proposition 2.5(iii). Further, we observe, as $[V_{\alpha'} \cap Q_\beta \cap Q_\alpha, V_{\alpha-1}] \leq V_{\alpha'}$ and $V_{\alpha'} \cap Q_\beta$ interchanges λ and $\alpha - 1$, that

$$[V_{\alpha'} \cap Q_\beta \cap Q_\alpha, V_{\alpha-1}] = [V_{\alpha'} \cap Q_\beta \cap Q_\alpha, V_\lambda] \leq V_{\alpha-1} \cap V_\lambda .$$

Let $t \in V_{\alpha'} \cap Q_\beta$ be such that $V_{\alpha'} \cap Q_\beta = (V_{\alpha'} \cap Q_\beta \cap Q_\alpha)\langle t \rangle$. Then, as t normalizes $V_{\alpha-1}V_\lambda$ and $V_{\alpha-1} \cap V_\lambda$, we have $|[V_{\alpha-1}V_\lambda/V_{\alpha-1} \cap V_\lambda, t]| =$ $= [V_{\alpha-1} : V_{\alpha-1} \cap V_\lambda]$. Now

$$[V_{\alpha'} \cap Q_\beta, V_{\alpha-1}V_\lambda] = [V_{\alpha'} \cap Q_\beta \cap Q_\alpha, V_{\alpha-1}][t, V_{\alpha-1}V_\lambda]$$

from which we deduce that

$$|[V_{\alpha'} \cap Q_\beta, V_{\alpha-1}V_\lambda]| \geq 2^4[V_{\alpha-1} : V_{\alpha-1} \cap V_\lambda].$$

So, by Lemma 11.1(vii),

$$|[V_{\alpha'} \cap Q_\beta, U_\alpha]| \geq 2^4[V_{\alpha'} : V_{\alpha'} \cap V_{\alpha'-2}].$$

This, together with (12.1.5), yields $|[V_{\alpha'} \cap Q_\beta, U_\alpha]| \geq 2^8$.

(12.1.7) U_α doesn't act quadratically upon $X_{\alpha'}/Z_{\alpha'}$.

Suppose (12.1.7) is false. Then, combining (12.1.2)(ii), (iii) and Proposition 2.5(ii), we have $|[U_\alpha, X_{\alpha'}]| \leq 2^8$. Noting that $[U_\alpha, X_{\alpha'}] = [U_\alpha, V_{\alpha'}]$, (12.1.6) forces

$$[U_\alpha, V_{\alpha'}] = [U_\alpha, V_{\alpha'} \cap Q_\beta] \leq U_\alpha\,,$$

contradicting Lemma 11.8. Thus (12.1.7) holds.

(12.1.8) $\eta(G_\alpha, U_\alpha) \geq 4$.

Set $U_\alpha{}^{(i)} = [U_\alpha, Q_\alpha; i]$ and $V_\beta{}^{(i)} = [V_\beta, Q_\alpha; i]$. From (12.1.2)(ii) and Lemma 11.2, $G_{\alpha\beta} = Q_\alpha Q_\beta$. Hence $V_\beta{}^{(3)} \neq 1$ by Proposition 2.5(i). Our aim is to show that $U_\alpha^{(1)} \neq V_\beta^{(1)}$ and $U_\alpha{}^{(2)} \neq V_\beta^{(2)}$. As a consequences (see Lemma 1.2(v)) $U_\alpha/U_\alpha{}^{(1)}$, $U_\alpha{}^{(1)}/U_\alpha{}^{(2)}$, $U_\alpha{}^{(2)}/U_\alpha{}^{(3)}$ all contain at least one non-central chief factor for G_α. Because $G_{\alpha\beta} \trianglerighteq V_\beta{}^{(3)} \neq 1$, $V_\beta{}^{(3)} \cap Z_\beta \neq 1$ and so $\eta(G_\alpha, U_\alpha{}^{(3)}) \neq 0$ by Lemma 1.1(ii), we then obtain (12.1.8).

Suppose $U_\alpha{}^{(1)} = V_\beta{}^{(1)}$. Then $[V_{\alpha'}, Q_{\alpha'-1}] = [V_{\alpha'-2}, Q_{\alpha'-1}]$ and so, by (12.1.4), U_α centralizes $[V_{\alpha'}, Q_{\alpha'-1}] = [X_{\alpha'}, Q_{\alpha'-1}]$, contrary to (12.1.7). If $U_\alpha{}^{(2)} = V_\beta{}^{(2)}$, then likewise U_α centralizes $[X_{\alpha'}, Q_{\alpha'-1}, Q_{\alpha'-1}]$ which again contradicts (12.1.7). Therefore $U_\alpha{}^{(1)} \neq V_\beta{}^{(1)}$ and $U_\alpha{}^{(2)} \neq V_\beta{}^{(2)}$, as desired.

(12.1.9) $U_\alpha Q_{\alpha'}/Q_{\alpha'}$ is the non-quadratic $E(2^3)$-subgroup of $G_{\alpha'-1\alpha'}/Q_{\alpha'}$ acting on $X_{\alpha'}/Z_{\alpha'}$.

Since $[Z_\alpha : Z_\alpha \cap Q_{\alpha'}] = 2$, (12.1.2)(i), (iii) imply that $[Z_\alpha \cap Q_{\alpha'}, V_{\alpha'} \cap \cap Q_\beta] \neq 1$. So there exists $\rho \in \Delta(\alpha')$ for which $[Z_\alpha \cap Q_{\alpha'}, Z_\rho] \neq 1$. Thus $Z_\alpha \cap Q_{\alpha'} \not\leq Q_\rho$. Note that $[Z_\rho : Z_\rho \cap Q_\beta] \leq 2$ by (12.1.1). Hence $Z_\rho \cap Q_\beta \not\leq Q_\alpha$ for $Z_\rho \cap Q_\beta \leq Q_\alpha$ would imply $[Z_\rho : C_{Z_\rho}(Z_\alpha \cap Q_{\alpha'})] \leq 2$ and thence $\eta(G_\rho, Z_\rho) \leq 1$, contrary to (12.1.2)(iii).

Now

$$[U_\alpha : C_{U_\alpha}(Z_\rho \cap Q_\beta)] \leq [U_\alpha : U_\alpha \cap Q_\rho] \leq 2[U_\alpha : U_\alpha \cap Q_{\alpha'}]$$

and therefore (12.1.8) forces $[U_\alpha : U_\alpha \cap Q_{\alpha'}] = 2^3$. Thus, by (12.1.7), we have established (12.1.9).

We are now in a position to deduce the desired contradiction. Combining (12.1.9), (12.1.2)(ii) with Proposition 2.5(viii) we get

$[X_{\alpha'} : C_{X_{\alpha'}}(U_\alpha)] \geq 2^6$. Therefore $[V_{\alpha'} : V_{\alpha'} \cap V_{\alpha'-2}] \geq 2^6$ by (12.1.4). Hence (12.1.6) forces $|[U_\alpha, V_{\alpha'} \cap Q_\beta]| \geq 2^{10}$. Since $|Z_{\alpha'}| = 2^2$ and $Z_{\alpha'} \leq [U_\alpha, V_{\alpha'}]$, (12.1.2)(ii) implies $|[U_\alpha, V_{\alpha'}]| \leq 2^{10}$. Consequently

$$[U_\alpha, V_{\alpha'}] = [U_\alpha, V_{\alpha'} \cap Q_\beta] \leq U_\alpha ,$$

again contradicting Lemma 11.8. This completes the proof that $\eta(G_\beta, V_\beta) = 1$ when $b > 3$. Now we examine the situation $\eta(G_\beta, V_\beta) > 1$ when $b = 3$.

(12.1.10) (i) $Z_{\alpha+2} \leq [V_\beta, V_{\alpha'}]$;
(ii) either $|V_\beta Q_{\alpha'}/Q_{\alpha'}| = |V_{\alpha'} Q_\beta/Q_\beta| = 2$ or $|V_\beta Q_{\alpha'}/Q_{\alpha'}| = |V_{\alpha'} Q_\beta/Q_\beta| = 2^3$;
(iii) $\eta(G_\beta, V_\beta) = 2$ and, for $i = 1, 2$, there exists $X_\beta^{(i)} \trianglelefteq G_\beta$ with $Z_\beta \leq X_\beta^{(i)}$ such that $V_\beta = X_\beta^{(1)} X_\beta^{(2)}$ and $X_\beta^{(i)}/Z_\beta \cong 4$ or $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$. Further the 4's in $X_\beta^{(1)}$ and $X_\beta^{(2)}$ are isomorphic.

Part (i) follows from Corollary 11.5(i) while Corollary 11.5(ii) and Lemma 11.6 imply (ii). For (iii) we first note that (i) gives $[V_\beta, G_\beta] = V_\beta$. If $|V_\beta Q_{\alpha'}/Q_{\alpha'}| = |V_{\alpha'} Q_\beta/Q_\beta| = 2$, then $[V_{\alpha'} : C_{V_{\alpha'}}(Z_\alpha)] = 2^2$ and so $\eta(G_{\alpha'}, V_{\alpha'}) = 2$ with both non-central chief factors being isomorphic natural modules. Using Lemma 2.16 and $[V_{\alpha'}, G_{\alpha'}] = V_{\alpha'}$ yields the desired structure of V_β in this case. Turning to the latter possibility given in (ii) we deduce, just as in Lemma 11.6, that for all $x \in V_\beta$ $[V_{\alpha'} : C_{V_{\alpha'}}(x)] \leq 2^4$. Hence $\eta(G_{\alpha'}, V_{\alpha'}) = 2$ and, as V_β acts as a quadratic $E(2^3)$-group on $V_{\alpha'}$, both non central chief factors are isomorphic natural modules. Appealing to Lemma 2.4 and Proposition 2.7(ii) completes the proof of (iii).

By (12.1.10)(iii) V_β/Z_β is a quotient of $\begin{pmatrix} 4 \\ 1 \end{pmatrix} \oplus \begin{pmatrix} 4 \\ 1 \end{pmatrix}$. Put $Y_\beta = C_{V_\beta}(O^2(G_\beta))$ and, for $i = 1, 2$, let $Y_\beta^{(i)}$ be such that $Y_\beta \leq Y_\beta^{(i)}$, $Y_\beta^{(i)} \trianglelefteq G_\beta$ and $Y_\beta^{(i)}/Y_\beta \cong 4$ with $V_\beta/Y_\beta = Y_\beta^{(1)}/Y_\beta \times Y_\beta^{(2)}/Y_\beta$. Let $Y_\beta^{(3)}$ denote the inverse image in V_β of the diagonal submodule. Clearly $V_\beta = Y_\beta^{(i)} Y_\beta^{(j)}$ for $i \neq j \in \{1, 2, 3\}$ so, without loss of generality, we may assume that $Y_{\alpha'}^{(1)} \not\leq Q_\beta$ and $Y_{\alpha'}^{(2)} \not\leq Q_\beta$.

Assume for the moment that $|V_\beta Q_{\alpha'}/Q_{\alpha'}| = |V_{\alpha'} Q_\beta/Q_\beta| = 2$. So $Y_{\alpha'}^{(1)}$ acts as a transvection upon each of the non-central chief factors within V_β. If $[Y_{\alpha'}^{(1)} \cap Q_\beta, V_\beta] \neq 1$, then $E(2^2) \cong Z_\beta \leq [Y_{\alpha'}^{(1)} \cap Q_\beta, V_\beta]$ whence $[Y_{\alpha'}^{(1)}, V_\beta] \cong E(2^4)$. However considering V_β acting upon $Y_{\alpha'}^{(1)}$ yields, as V_β acts as a transvection upon $Y_{\alpha'}^{(1)}/Z_{\alpha'}$, that $[V_\beta, Y_{\alpha'}^{(1)}] \lesssim E(2^3)$. Thus we conclude that $[Y_{\alpha'}^{(1)} \cap Q_\beta, V_\beta] = 1$ and, similarly, $[Y_{\alpha'}^{(2)} \cap Q_\beta, V_\beta] = 1$. So $Y_{\alpha'}^{(i)} \cap Q_\beta = C_{Y_{\alpha'}^{(i)}}(V_\beta)$ for $i = 1, 2$ and, in particular, $Y_{\alpha'}^{(i)} \cap Q_\beta \geq Y_\beta$. Since

$[Y_{\alpha'}^{(i)} : Y_{\alpha'}^{(i)} \cap Q_\beta] = 2 = [V_{\alpha'} : V_{\alpha'} \cap Q_\beta]$ and, for $i = 1, 2$, $Y_{\alpha'}^{(i)} \not\leq Q_\beta$, we see that $V_{\alpha'} \cap Q_\beta = (Y_{\alpha'}^{(1)} \cap Q_\beta)(Y_{\alpha'}^{(2)} \cap Q_\beta)Y_{\alpha'}^{(3)}$. In particular $Y_{\alpha'}^{(3)} \leq Q_\beta$ and, since $\eta(G_{\alpha'}, Y_{\alpha'}^{(3)}) = 1$, $Y_{\alpha'}^{(3)} \not\leq Q_\alpha$. Therefore $Z_\beta = [Z_\alpha, Y_{\alpha'}^{(3)}] \leq Y_{\alpha'}^{(3)}$. But then $V_{\alpha'} = Y_{\alpha'}^{(3)}$ which is impossible. So now we only need consider the possibility $|V_\beta Q_{\alpha'}/Q_{\alpha'}| = |V_{\alpha'} Q_\beta / Q_\beta| = 2^3$. Recalling that $Z_{\alpha'} \leq [V_\beta, V_{\alpha'}]$, Propositio-n 2.5(ii) implies that

$$[V_{\alpha'}, V_\beta] \geq Z_{\alpha'} \left[X_{\alpha'}^{(1)}, G_{\alpha'\alpha'-1}; 2\right] \left[X_{\alpha'}^{(2)}, G_{\alpha'\alpha'-1}; 2\right].$$

Since $[V_{\alpha'}, V_\beta] \leq C_{V_{\alpha'}}(V_\beta)$ we see that

$$[V_{\alpha'}, V_\beta] = Z_{\alpha'} \left[X_{\alpha'}^{(1)}, G_{\alpha'\alpha'-1}; 2\right] \left[X_{\alpha'}^{(2)}, G_{\alpha'\alpha'-1}; 2\right].$$

Because $[V_{\alpha'} : V_{\alpha'} \cap Q_\beta] = 2^3$ we have $V_{\alpha'} \cap Q_\beta = \langle x \rangle [V_{\alpha'}, V_\beta]$ where $x = x_1 x_2$ with $x_i \in X_{\alpha'}^{(i)}$. Since $[V_{\alpha'} \cap Q_\beta, V_\beta] = Z_\beta \leq Z_{\alpha+2}$, $[x, V_\beta] \leq Z_{\alpha+2}$.

Set $\overline{V}_{\alpha'} = V_{\alpha'}/Y_{\alpha'}$. Then $\overline{V}_{\alpha'} = \overline{X_{\alpha'}^{(1)}} \times \overline{X_{\alpha'}^{(2)}} \cong 4 \oplus 4$ and by Lemma 11.3 $\overline{Z}_{\alpha+2} = (\overline{Z_{\alpha+2} \cap X_{\alpha'}^{(1)}}) \times (\overline{Z_{\alpha+2} \cap X_{\alpha'}^{(2)}}) \cong E(2^2)$. Since $\overline{X_{\alpha'}^{(i)}} \trianglelefteq G_{\alpha'}$, we infer that $[\overline{x}_i, V_\beta] \leq Z_{\alpha+2} \cap X_{\alpha'}^{(i)}$ $(i = 1, 2)$. Without loss of generality we may suppose $\overline{x}_1 \not\in [\overline{X_{\alpha'}^{(1)}}, G_{\alpha'\alpha'-1}; 2]$ (because $V_{\alpha'} \cap Q_\beta \neq [V_{\alpha'}, V_\beta]$). So $\langle \overline{x}_1 \rangle [\overline{X_{\alpha-1}^{(1)}}, G_{\alpha'\alpha'-1}; 2] \cong$ $\cong E(2^3)$ with

$$\left[V_\beta, \langle \overline{x}_1 \rangle \left[\overline{X_{\alpha'}^{(1)}}, G_{\alpha'\alpha'-1}; 2\right]\right] \leq \overline{Z_{\alpha+2} \cap X_{\alpha'}^{(1)}}.$$

But Proposition 2.5(ix) shows this to be impossible and so we have ruled out the possibility $\eta(G_\beta, V_\beta) > 1$ and $b = 3$. Therefore we have shown that $\eta(G_\beta, V_\beta) = 1$. The remainder of the theorem follows from Proposition 2.9(i) and Lemmas 2.2(iv), 11.3.

LEMMA 12.2. *Let* $(\alpha, \alpha') \in \mathscr{C}$ *and* $\Delta(\alpha) = \{\lambda_1, \lambda_2, \lambda_3\}$.

(i) $\Omega_1(Z(G_{\alpha\beta})) = Z_\beta \cong \mathbb{Z}_2$ *and* $\Omega_1(Z(Q_\alpha)) = Z_\alpha = Z_{\lambda_i} \times Z_{\lambda_j} \cong E(2^2)$, $1 \leq i < j \leq 3$.

(ii) $G_{\alpha\beta} = Q_\alpha Q_\beta$ *and* $[V_\beta, Q_\beta] = Z_\beta$.

(iii) $\mathrm{core}_{G_\alpha} V_\beta = V_{\lambda_i} \cap V_{\lambda_j}$, $i \neq j$.

(iv) $Z_\beta \leq \Omega_1(Z(Q_\beta)) \lesssim \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

PROOF. (i) is mostly a restatement of Theorem 12.1; $\Omega_1(Z(Q_\alpha)) = Z_\alpha$ follows since Z_α is a projective G_α/Q_α-module and $\Omega_1(Z(Q_{\alpha\beta})) = Z_\beta \leq Z_\alpha$. Part (ii) follows from Theorem 12.1 and Lemma 11.2.

If $\{i, j, k\} = \{1, 2, 3\}$, then clearly $G_{\alpha\lambda_k}$ normalizes $V_{\lambda_i} \cap V_{\lambda_j}$. Also,

$$[V_{\lambda_i} \cap V_{\lambda_j}, Q_{\lambda_i}] \leq [V_{\lambda_i}, Q_{\lambda_i}] \leq Z_{\lambda_i} \leq Z_\alpha \leq V_{\lambda_i} \cap V_{\lambda_j}.$$

Using part (ii) we now see that $V_{\lambda_i} \cap V_{\lambda_j} \trianglelefteq G_\alpha$, so proving (iii).

Turning to (iv), we set $M = \Omega_1(Z(Q_\beta))$. Observing that M is a G_β/Q_β-module with $\mathrm{soc}(M) = Z_\beta$ we have $M \hookrightarrow P(1)$. Since $|\Omega_1(Z(G_{\alpha\beta}))| = 2$, Proposition 2.5(i) implies that M doesn't contain submodules of type $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$. Hence $M \lessapprox \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ by Lemma 2.4.

We will make frequent use of Lemma 12.2, often without specific reference.

LEMMA 12.3. *Let* $(\alpha, \alpha') \in \mathscr{C}$ *and suppose that* $\mathrm{core}_{G_\alpha} V_\beta > Z_{\alpha'}$. *Then* V_β/Z_β *is isomorphic to* 4 *or* $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$.

PROOF. Since $\mathrm{core}_{G_\alpha} V_\beta > Z_\alpha$, $[\mathrm{core}_{G_\alpha} V_\beta, Q_\alpha] \neq 1$ by Lemma 12.2(i). Clearly $[\mathrm{core}_{G_\alpha} V_\beta, Q_\alpha] \trianglelefteq G_\alpha$ and so Lemma 12.2(i) implies that $Z_\alpha \leq$ $\leq [\mathrm{core}_{G_\alpha} V_\beta, Q_\alpha]$. Hence

$$Z_\alpha \leq [\mathrm{core}_{G_\alpha} V_\beta, Q_\alpha] \leq [V_\beta, G_\beta]$$

and therefore $V_\beta = [V_\beta, G_\beta]$. Now the lemma is a consequence of Theorem 12.1.

LEMMA 12.4. *Let* $(\alpha, \alpha') \in \mathscr{C}$. *Then*

(i) $[Q_\beta, O^2(G_\beta)] \not\leq Q_\alpha$; *in particular* $Q_\alpha \cap Q_\alpha \ntrianglelefteq G_\beta$.
(ii) $G_\beta = O^2(G_\beta) Q_\alpha$.

PROOF. (i) Put $R_\beta := [Q_\beta, O^2(G_\beta)]$ and assume that $R_\beta \leq Q_\alpha$. Clearly, $R_\beta \leq Q := Q_\alpha \cap Q_\beta$ and so $Q \trianglelefteq G_\beta$ with $Q = C_{Q_\beta}(Z_\alpha) = C_{Q_\beta}(V_\beta)$. Since $[Q_\beta : Q] = 2$ we get $2^2 \geq |\Omega_1(Z(Q_\beta))| \geq |C_{V_\beta}(Q_\beta)| \geq \sqrt{|V_\beta|}$ and hence $|V_\beta| \leq 2^4$, a contradiction. This proves (i).

(ii) If $L_\beta := O^2(G_\beta) Q_\alpha \neq G_\beta$, then $[G_\beta : L_\beta] = 2$ with $Q_\alpha \in SyI_2 L_\beta$; so $Q_\alpha \cap Q_\beta = O_2(L_\beta) \trianglelefteq G_\beta$, but this contradicts (i).

Let $(\alpha, \alpha') \in \mathscr{C}$ and put $Y_\beta = C_{V_\beta}(O^2(G_\beta))$. Assume that $|Y_\beta| = 2^2$. We now define

$$F_\alpha = \langle Y_\lambda \mid \lambda \in \Delta(\alpha) \rangle = \left\langle {Y_\beta}^{G_\alpha} \right\rangle$$

and

$$H_\beta = \langle F_\mu \mid \mu \in \Delta(\beta) \rangle = \left\langle {F_\alpha}^{G_\beta} \right\rangle.$$

Clearly $F_\alpha \trianglelefteq G_\alpha$ and $H_\beta \trianglelefteq G_\beta$. Some less obvious properties of these groups are given in the next lemma.

LEMMA 12.5.

(i) $\eta(G_\alpha, F_\alpha) = 2$; *and*
(ii) $\eta(G_\beta, H_\beta) \geq 2$.

PROOF. (i) First we observe that $Z_\alpha \leq F_\alpha$ and, as $|Y_\beta| = 2^2$, that $\eta(G_\alpha, F_\alpha) \leq 2$. If $\eta(G_\alpha, F_\alpha) = 1$, then $F_\alpha = Y_\beta Z_\alpha$ and hence

$$[Q_\alpha, F_\alpha] = [Q_\alpha, Y_\beta Z_\alpha] = [Q_\alpha, Y_\beta] \leq Z_\beta < Z_\alpha .$$

Since $[Q_\alpha, F_\alpha] \trianglelefteq G_\alpha$, we then get $F_\alpha \leq \Omega_1(Z(Q_\alpha)) = Z_\alpha$, which is impossible. Thus $\eta(G_\alpha, F_\alpha) = 2$.

(ii) Since $Z_\alpha \leq F_\alpha$, clearly $V_\beta \leq H_\beta$. So if (ii) is false, then $H_\beta = F_\alpha V_\beta$. Hence $[F_\alpha V_\beta, Q_\beta] \trianglelefteq G_\beta$ and

$$[F_\alpha V_\beta, Q_\beta] = [F_\alpha, Q_\beta][V_\beta, Q_\beta] = [F_\alpha, Q_\beta]Z_\beta .$$

By $G_{\alpha\beta} = Q_\alpha Q_\beta$ and (i) $|[F_\alpha, Q_\beta]| \geq 2^2$ and so $[F_\alpha V_\beta, Q_\beta] = [F_\alpha, Q_\beta]$. Because $F_\alpha/Z_\alpha \cong 2$ or $2 \oplus 1$, we see that $|[F_\alpha, Q_\beta]| \leq 2^3$ and therefore $[F_\alpha, Q_\beta]$ is centralized by $O^2(G_\beta)$. Consequently $Z_\alpha \nleq [F_\alpha, Q_\beta]$ and so $Z_\alpha \cap [F_\alpha, Q_\beta] = Z_\beta$. Next we consider

$$[F_\alpha, Q_\alpha \cap Q_\beta] \leq [F_\alpha, Q_\alpha] \cap [F_\alpha, Q_\beta] = Z_\alpha \cap [F_\alpha, Q_\beta] = Z_\beta .$$

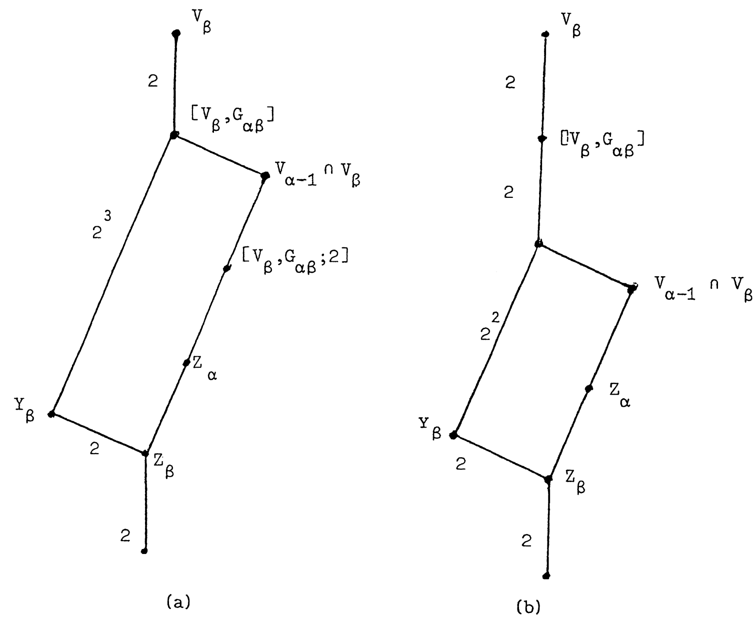
Therefore $[V_\beta F_\alpha, Q_\alpha \cap Q_\beta] \leq Z_\beta$. So $Q_\alpha \cap Q_\beta \leq C_{Q_\beta}(V_\beta F_\alpha / Z_\beta)$. Recalling that $[F_\alpha V_\beta, Q_\beta] = [F_\alpha, Q_\beta]$ has order at least 2^2 we deduce that $Q_\alpha \cap Q_\beta = C_{Q_\beta}(V_\beta F_\alpha / Z_\beta)$. But then $Q_\alpha \cap Q_\beta \trianglelefteq G_\beta$, contrary to Lemma 12.4(i). This completes the verification of (ii).

The groups F_α and H_β will be important in later arguments in, for example, Section 13 and Part VI. Now we return to examine further the situation in Lemma 12.3 in our next lemma, where F_α makes a brief appearance.

LEMMA 12.6. *Let* $(\alpha, \alpha') \in \mathscr{C}$, *and assume that* $\mathrm{core}_{G_\alpha} V_\beta = V_{\alpha-1} \cap V_\beta > Z_\alpha$.

(i) *If* $V_\beta / Z_\beta \cong 4$, *then* $V_{\alpha-1} \cap V_\beta = [V_\beta, G_{\alpha\beta}]$ *or* $V_{\alpha-1} \cap V_\beta = [V_\beta, G_{\alpha\beta}, G_{\alpha\beta}]$.

(ii) *Assume that* $V_\beta / Z_\beta \cong \begin{pmatrix} 4 \\ 1 \end{pmatrix}$, *and set* $Y_\beta = C_{V_\beta}(O^2(G_\beta))$. *Then* $Y_\beta \nleq V_{\alpha-1} \cap V_\beta$ *and* $Y_\beta(V_{\alpha-1} \cap V_\beta) \neq V_\beta$. *Hence one of the following holds:*



Furthermore, in both cases, $[V_\beta, G_{\alpha\beta}; 3] = Z_\alpha$.

PROOF. Part (i) follows from $V_{\alpha-1} \cap V_\beta \trianglelefteq G_{\alpha\beta}$ and the structure of a natural S_6-module.

(ii) Suppose $Y_\beta(V_{\alpha-1} \cap V_\beta) = V_\beta$. Then, employing Lemma 11.1(vii), $Y_{\alpha'}(V_{\alpha'-2} \cap V_{\alpha'}) = V_{\alpha'}$. Since $[Z_\alpha, V_{\alpha'-2}] = 1$, this gives $[V_{\alpha'}, Z_\alpha] \leq Y_{\alpha'}$, contrary to $\eta(G_{\alpha'}, V_{\alpha'}) = 1$. Therefore $Y_\beta(V_{\alpha-1} \cap V_\beta) \neq V_\beta$. If $Y_\beta \leq V_{\alpha-1} \cap V_\beta$, then $V_{\alpha-1} \cap V_\beta \trianglelefteq G_\alpha$ implies that $F_\alpha = \langle Y_\beta^{G_\alpha} \rangle \leq V_{\alpha-1} \cap V_\beta$. In particular, $[F_\alpha, Q_\beta] \leq [V_\beta, Q_\beta] = Z_\beta$ which, as $G_{\alpha\beta} = Q_\alpha Q_\beta$, yields that $\eta(G_\alpha, F_\alpha/Z_\alpha) = 0$, against Lemma 12.5(i). So we conclude that $Y_\beta \not\leq V_{\alpha-1} \cap V_\beta$. Since $V_{\alpha-1} \cap V_\beta > Z_\alpha$ and $Y_\beta(V_{\alpha-1} \cap V_\beta) \neq V_\beta$, we obtain the two indicated possibilities.

We now subdivide into the following cases:

Case 1. $V_\beta/Z_\beta \cong 4$ and $V_{\alpha-1} \cap V_\beta = [V_\beta, G_{\alpha\beta}]$.

Case 2. $V_\beta/Z_\beta \cong \begin{pmatrix} 4 \\ 1 \end{pmatrix}$ and Lemma 12.6(ii)(a) holds.

Case 3. $\mathrm{core}_{G_\alpha} V_\beta = V_{\alpha-1} \cap V_\beta = Z_\alpha$.

Case 4. $V_\beta/Z_\beta \cong 4$ and $V_{\alpha-1} \cap V_\beta = [V_\beta, G_{\alpha\beta}, G_{\alpha\beta}]$.

Case 5. $V_\beta/Z_\beta \cong \begin{pmatrix} 4 \\ 1 \end{pmatrix}$ and Lemma 12.6(ii)(b) holds.

Consulting Theorem 12.1 and Lemmas 12.3, 12.6 we see that Cases 1 - 5 exhaust all the possibilities.

Before confronting Case 1 in the next section, we give one further result. This result is particularly useful in dealing with Case 3.

LEMMA 12.7. *Let* $(\alpha, \alpha') \in \mathscr{C}$, *and put* $\lambda = \alpha' - 2$, $R = [V_\beta, V_{\alpha'}]$, $W_\lambda{}^* = [W_\lambda, Q_\lambda]V_\lambda$, $L_\lambda = O^2(G_\lambda)$ *and* $P_\lambda = \langle W_\beta, Q_{\alpha'-1}\rangle$. *Then the following statements hold.*

(i) *If* L_λ *normalizes* $V_\lambda R$, *then* $R \le \mathrm{core}_{G_{\alpha'-1}} V_{\alpha'} = V_\lambda \cap V_{\alpha'}$.

(ii) $V_\lambda R$ *is normalized by* L_λ *if any one of the following conditions is satisfied.*

(a) $\eta(G_\lambda, W_\lambda{}^*/V_\lambda) = 0$.

(b) $b \le 5$, $Z_{\alpha'}R$ *is normal in* $G_{\alpha'-1\alpha'}$ *and* $P_\lambda \ge L_\lambda$.

(c) $b \le 5$, $Z_{\alpha'-1}R$ *is normal in* $G_{\alpha'-1\alpha'}$ *and* $P_\lambda \ge L_\lambda$.

PROOF. (i) Put $X = V_\lambda R$, $Q_\lambda{}^* = [Q_\lambda, L_\lambda]$, $Y_\lambda = C_{V_\lambda}(L_\lambda)$ and $N = Z_\lambda[X, Q_\lambda{}^*]$. Let $\Delta(\alpha'-1) = \{\lambda, \rho, \alpha'\}$. Note that $N = Z_\lambda[R, Q_\lambda{}^*]$.

(12.7.1) $$|R(V_\lambda \cap V_{\alpha'})/V_\lambda \cap V_{\alpha'}| \le 2^2$$

If $|V_\beta Q_{\alpha'}/Q_{\alpha'}| \ge 2^2$, then Theorem 12.1 and Proposition 2.5(ii) gives $|RZ_{\alpha'-1}/Z_{\alpha'-1}| \le 2^2$ and, if $|V_\beta Q_{\alpha'}/Q_{\alpha'}| = 2$, then $|RZ_{\alpha'}/Z_{\alpha'}| \le 2^2$ so also giving $|RZ_{\alpha'-1}/Z_{\alpha'-1}| \le 2^2$. Hence we have (12.7.1).

Since L_λ normalizes X by hypothesis and by (12.7.1), $[X : V_\lambda] \le 2^2$, we deduce that $[X, L_\lambda] \le V_\lambda$. Hence $N \le V_\lambda$. We next investigate the location and order of $[R, Q_\lambda{}^*]$. By Lemma 12.4(i) $Q_\lambda{}^* = \langle t\rangle(Q_\lambda{}^* \cap Q_{\alpha'-1})$. Clearly

$$[Q_\lambda{}^* \cap Q_{\alpha'-1}, R] \le [Q_\lambda{}^* \cap Q_{\alpha'-1}, V_{\alpha'}] \le V_{\alpha'}\ ,$$

and so

$$[Q_\lambda{}^* \cap Q_{\alpha'-1}, R] \le N \cap V_{\alpha'} \le V_\lambda \cap V_{\alpha'} = \mathrm{core}_{G_{\alpha'-1}} V_{\alpha'}\ .$$

Therefore $[R, Q_\lambda{}^*] \le [t, R]\mathrm{core}_{G_{\alpha'-1}} V_{\alpha'}$. Since t acts upon $V_\rho V_{\alpha'}/V_\rho \cap V_{\alpha'}$, which is abelian, we note, using (12.7.1), that $|[t, R]\mathrm{core}_{G_{\alpha'-1}} V_{\alpha'}/\mathrm{core}_{G_{\alpha'-1}} V_{\alpha'}| \le$ $\le 2^2$.

Now assume that $\eta(L_\lambda, N) \ne 0$. So, by Theorem 12.1(i), $V_\lambda = Y_\lambda N =$ $= Z_{\alpha'-1}N = Z_{\alpha'-1}[R, Q_\lambda{}^*] \le [t, R]\mathrm{core}_{G_{\alpha'-1}} V_{\alpha'}$. Therefore $[V_\lambda : V_\lambda \cap V_{\alpha'}] \le$

$\leq 2^2$. So $V_{\alpha'}/Z_{\alpha'} \cong 4$ or $\binom{4}{1}$ by Lemma 12.3. If $V_{\alpha'}/Z_{\alpha'} \cong 4$, then by Proposition 2.5(ii),(iii), we have (i). While if $V_{\alpha'}/Z_{\alpha'} \cong \binom{4}{1}$, then we are in case (a) of Lemma 12.6(ii), whence $|RZ_{\alpha'}/Z_{\alpha'}| = 2$ and then $[V_\lambda : V_\lambda \cap V_{\alpha'}] \leq 2$, a contradiction. So we may suppose that $\eta(L_\lambda, N) = 0$. Hence $Z_\lambda \leq N \leq Y_\lambda$, and so $1 = [N, L_\lambda] = [R, {Q_\lambda}^*, L_\lambda]$. Since $[{Q_\lambda}^*, L_\lambda] = {Q_\lambda}^*$, the 3-subgroup lemma yields

$$[{Q_\lambda}^*, X] = [{Q_\lambda}^*, L_\lambda, X] = [L_\lambda, X, {Q_\lambda}^*] \leq [V_\lambda, {Q_\lambda}^*] \leq Z_\lambda \,.$$

Hence $[{Q_\lambda}^*, R] \leq Z_\lambda \leq R$, and so ${Q_\lambda}^*$ normalizes R. Therefore $R \leq V_\rho \cap V_{\alpha'}$ by Lemma 12.4(i), giving (i).

(ii) Suppose (a) holds. Since $V_\beta \leq Q_\beta$ and $V_{\alpha'} \leq W_\lambda$, $R \leq [W_\lambda, Q_\lambda] \leq {W_\lambda}^*$ and hence $V_\lambda R$ is normalized by L_λ. Parts (b) and (c) follow from the fact that $[W_\beta, R] = 1$.

13. Cases 1 and 2.

First we consider Case 1, our main conclusion being contained in

THEOREM 13.1. *Suppose that* $(\alpha, \alpha') \in \mathscr{C}$, $V_\beta/Z_\beta \cong 4$ *and* $V_{\alpha-1} \cap V_\beta = [V_\beta, G_{\alpha\beta}]$. *Then* $b \in \{3, 5\}$.

The description of W_δ for $\delta \in O(S_6)$ given in Lemma 13.4 plays a significant role in establishing Theorem 13.1. The proof of Lemma 13.4 is itself heavily dependent upon Lemmas 13.3 and 2.16. Use of Lemma 13.4 then enables us to show that $[V_{\alpha+3}, W_{\alpha'}] \leq Z_{\alpha'}$ for $(\alpha, \alpha') \in \mathscr{C}$. This turns out to be a telling blow. Our next step is to rule out the possibility that $[V_{\alpha+3}, W_{\alpha'}] = 1$. Then, knowing that $[V_{\alpha+3}, W_{\alpha'}] = Z_{\alpha'}$, we readily conclude the proof of the theorem.

Let (α, α') be a fixed critical pair of Γ. Until the end of Lemma 13.10 we assume the following

HYPOTHESIS.

(i) $V_\beta/Z_\beta \cong 4$;
(ii) $V_{\alpha-1} \cap V_\beta = [V_\beta, G_{\alpha\beta}]$; *and*
(iii) $b \geq 7$.

Note that $V_\beta \cap V_{\alpha-1} = [V_\beta, Q_\alpha] = [V_{\alpha-1}, Q_\alpha]$ and $[V_\beta : V_\beta \cap V_{\alpha-1}] = 2$.

Further, by Lemma 11.1(vii), we have that $[V_\tau : V_\tau \cap V_\delta] = 2$ for all τ, $\delta \in O(S_6)$ with $d(\tau, \delta) = 2$.

LEMMA 13.2. *Let (δ, λ, τ) be a path in Γ of length 2 with $\delta \in O(S_6)$. Then $(Q_\lambda \cap Q_\tau)Q_\delta/Q_\delta$ is contained in the non-quadratic $E(2^3)$-subgroup of $G_{\delta\lambda}/Q_\delta$ acting on V_δ/Z_δ.*

PROOF. Since $[Q_\tau, V_\tau] = Z_\tau$,

$$[[V_\delta, Q_\lambda], Q_\lambda \cap Q_\tau] = [V_\delta \cap V_\tau, Q_\lambda \cap Q_\tau] \leq Z_\tau .$$

Thus

$$[[V_\delta, G_{\delta\lambda}]/Z_\delta, Q_\lambda \cap Q_\tau] \leq Z_\tau Z_\delta/Z_\delta = Z_\lambda/Z_\delta = C_{V_\delta/Z_\delta}(G_{\delta\lambda}) ,$$

and the result is now a consequence of Proposition 2.5(vii).

LEMMA 13.3. *There exists an involution $y \in G_{\alpha\beta}\backslash Q_\beta$ such that $[W_\beta/V_\beta : C_{W_\beta/V_\beta}(y)] \leq 2^2$.*

PROOF. Suppose that statement is false. So $[W_\beta/V_\beta : C_{W_\beta/V_\beta}(y)] \geq 2^3$ for all $y \in G_{\alpha\beta}\backslash Q_\beta$ with $y^2 = 1$ and hence, as $\eta(G_\beta, V_\beta) = 1$, $[W_\beta/V_\beta : C_{W_\beta/V_\beta}(y)] \geq 2^4$ for all $y \in G_{\alpha\beta}\backslash Q_\beta$ with $y^2 = 1$. In particular we have

(13.3.1) $$[W_{\alpha'} : C_{W_{\alpha'}}(x)] \geq 2^4 \text{ for any } x \in V_\beta\backslash Q_{\alpha'} .$$

We first show that

(13.3.2) $$W_{\alpha'} \not\leq G_{\alpha+2} .$$

Supposing that $W_{\alpha'} \leq G_{\alpha+2}$ we argue for a contradiction. Now we have either

(a) $W_{\alpha'} \cap Q_\beta \not\leq Q_\alpha$ or
(b) $W_{\alpha'} \cap Q_\beta \leq Q_\alpha$.

Assume that Case (a) holds. Then $Z_\beta = [Z_\alpha, W_{\alpha'} \cap Q_\beta] \leq W_{\alpha'}$ and hence, as $b \geq 7$, $W_{\alpha'}$ centralizes $Z_\beta Z_{\alpha+3} = Z_{\alpha+2}$. Since $W_{\alpha'} \leq G_{\alpha+2}$, we conclude that $W_{\alpha'} \leq Q_{\alpha+2} \leq G_\beta$. Because $W_{\alpha'}$ is abelian, $W_{\alpha'}$ acts quadratically on V_β and thus $|[W_{\alpha'}, V_\beta]| \leq 2^3$ by Proposition 2.5(ii). Hence, for $x \in V_\beta\backslash Q_{\alpha'}$, $|[W_{\alpha'}, x]| \leq |W_{\alpha'}, V_\beta]| \leq 2^3$ and so $[W_{\alpha'} : C_{W_{\alpha'}}(x)] \leq 2^3$, contrary to (13.3.1).

Therefore $W_{\alpha'} \cap Q_\beta \leq Q_\alpha$. If $W_{\alpha'} \leq Q_{\alpha+2}$, then $[W_{\alpha'} : C_{W_{\alpha'}}(Z_\alpha)] \leq$

$\leq |W_{\alpha'}Q_\beta/Q_\beta| \leq 2^3$ again contradicting (13.3.1). Thus $W_{\alpha'} \not\leq Q_{\alpha+2}$. Since $W_{\alpha'} \leq G_{\alpha+2}$, this implies $[W_{\alpha'}, Z_{\alpha+2}] \neq 1$. Now $V_{\alpha+3} \cap V_{\alpha+5}$ is centralized by $W_{\alpha'}$ and has index 2 in $V_{\alpha+3}$ and therefore

$$V_{\alpha+3} = Z_{\alpha+2}(V_{\alpha+3} \cap V_{\alpha+5}).$$

Consequently $W_{\alpha'} \cap Q_{\alpha+2}$ centralizes $V_{\alpha+3}$ and, in particular, centralizes $V_\beta \cap V_{\alpha+3} = [V_\beta, Q_{\alpha+2}]$. Hence $|(W_{\alpha'} \cap Q_{\alpha+2})Q_\beta/Q_\beta| \leq 2$ and so, as $[W_{\alpha'} : W_{\alpha'} \cap Q_{\alpha+2}] \leq 2$, we obtain $[W_{\alpha'} : W_{\alpha'} \cap Q_\alpha] \leq 2^2$ once again contradicting (13.3.1). Thus we have proved (13.3.2).

(13.3.3) $$V_\beta \cap V_{\alpha+3} \neq V_{\alpha+3} \cap V_{\alpha+5}\,.$$

If $V_\beta \cap V_{\alpha+3} = V_{\alpha+3} \cap V_{\alpha+5}$, then $P = \langle G_{\alpha+2\alpha+3}, G_{\alpha+3\alpha+4}\rangle \neq G_{\alpha+2}$. Now $W_{\alpha'}$ centralizing $V_{\alpha+3} \cap V_{\alpha+5}$ and Proposition 2.5(viii) yield $W_{\alpha'} \leq O_2(P) \leq$ $\leq G_{\alpha+2\alpha+3}$, which is impossible by (13.3.2). So (13.3.3) holds.

(13.3.4) (i) $W_{\alpha'} \cap Q_{\alpha+3} \leq Q_{\alpha+2}$
(ii) $|(W_{\alpha'} \cap Q_{\alpha+3})Q_\beta/Q_\beta| = 2^2$
(iii) $W_{\alpha'} \cap Q_{\alpha+3} \cap Q_\beta \not\leq Q_\alpha$

Suppose that $W_{\alpha'} \cap Q_{\alpha+3} \not\leq Q_{\alpha+2}$. So, since $W_{\alpha'} \cap Q_{\alpha+3} \leq G_{\alpha+2}$, $[W_{\alpha'} \cap Q_{\alpha+3}, Z_{\alpha+2}] \neq 1$. Hence, as $Z_{\alpha+2} = Z_\beta Z_{\alpha+3}$, $[W_{\alpha'} \cap Q_{\alpha+3}, Z_\beta] \neq 1$ and so $Z_\beta \not\leq W_{\alpha'}$. This then yields that $W_{\alpha'} \cap Q_\beta \leq Q_\alpha$. Now, just as in the proof of (13.3.2), $V_{\alpha+3} = (V_{\alpha+3} \cap V_{\alpha+5})Z_{\alpha+2}$, whence $[W_{\alpha'} \cap Q_{\alpha+3} \cap Q_{\alpha+2}, V_{\alpha+3}] = 1$. Therefore $|(W_{\alpha'} \cap Q_{\alpha+3} \cap Q_{\alpha+2})Q_\beta/Q_\beta| \leq 2$ and so $[W_{\alpha'} : W_{\alpha'} \cap Q_\beta] \leq 2^3$ which, as $W_{\alpha'} \cap Q_\beta \leq Q_{\alpha'}$, contradicts (13.3.1). Thus $W_{\alpha'} \cap Q_{\alpha+3} \leq Q_{\alpha+2}$, and we have (i).

Because $W_{\alpha'} \cap Q_{\alpha+3}$ acts quadratically on V_β and, by (i), $(W_{\alpha'} \cap Q_{\alpha+3})Q_\beta/Q_\beta \leq (Q_{\alpha+3} \cap Q_{\alpha+2})Q_\beta/Q_\beta$ we see that $|(W_{\alpha'} \cap Q_{\alpha+3})Q_\beta/Q_\beta| \leq 2^2$. In view of (13.3.1) and $[W_{\alpha'} : W_{\alpha'} \cap Q_{\alpha+3}] = 2$ (by (13.3.2)) we must have $|(W_{\alpha'} \cap Q_{\alpha+3})Q_\beta/Q_\beta| = 2^2$ and $W_{\alpha'} \cap Q_{\alpha+3} \cap Q_\beta \not\leq Q_\alpha$, so giving (ii) and (iii).

(13.3.5) $V_\beta \cap V_{\alpha+3} \geq [W_{\alpha'} \cap Q_{\alpha+3}, V_\beta] \geq Z_{\alpha+2}$ with $|[W_{\alpha'} \cap Q_{\alpha+3}, V_\beta]| = 2^3$.

Since $W_{\alpha'} \cap Q_{\alpha+3} \leq Q_{\alpha+2} \leq G_{\beta\alpha+2}$, we clearly have $[W_{\alpha'} \cap Q_{\alpha+3}, V_\beta] \leq$ $\leq V_\beta \cap V_{\alpha+3}$. From (13.3.4)(iii) we observe that $Z_\beta \leq [W_{\alpha'} \cap Q_{\alpha+3}, V_\beta]$. By (13.3.1), (13.3.4)(ii) and Proposition 2.5(ii) we have that $(W_{\alpha'} \cap Q_{\alpha+3})Q_\beta/Q_\beta$ is $Z(G_{\beta\alpha+2}/Q_\beta)$ or is $G_{\beta\alpha+2}/Q_\beta$-conjugate to $\langle s_1, t\rangle$. In either case we obtain $|[W_{\alpha'} \cap Q_{\alpha+3}, V_\beta]| = 2^3$ and $[W_{\alpha'} \cap Q_{\alpha+3}, V_\beta] \geq Z_{\alpha+2}$, as required.

(13.3.6) $$[W_{\alpha'} \cap Q_{\alpha+3}, V_\beta] = Z_{\alpha+2}Z_{\alpha+4}\,.$$

From (13.3.5) we see that $\langle W_{\alpha'}, G_{\alpha+2\alpha+3}\rangle$ is a parabolic subgroup of $G_{\alpha+3}$ which normalizes $Z_{\alpha+2}/Z_{\alpha+3}$. Hence $\langle W_{\alpha'}, G_{\alpha+2\alpha+3}\rangle$ also normalizes $[V_{\alpha+3}, Q_{\alpha+2}]/Z_{\alpha+3} = V_\beta \cap V_{\alpha+3}/Z_{\alpha+3}$. Using (13.3.3) we deduce that

$$\begin{aligned}[V_{\alpha+3}, W_{\alpha'}] &= [(V_\beta \cap V_{\alpha+3})(V_{\alpha+3} \cap V_{\alpha+5}), W_{\alpha'}]\\ &= [V_\beta \cap V_{\alpha+3}, W_{\alpha'}] \leq V_\beta \cap V_{\alpha+3}\,.\end{aligned}$$

Since $W_{\alpha'}$ acts as a central transvection on $V_{\alpha+3}/Z_{\alpha+3}$ (by (13.3.2)), it follows that $Z_{\alpha+4} \leq V_\beta \cap V_{\alpha+3}$. If $Z_{\alpha+4} \not\leq [W_{\alpha'} \cap Q_{\alpha+3}, V_\beta]$, then, by (13.3.5), $V_\beta \cap V_{\alpha+3} = Z_{\alpha+4}[W_{\alpha'} \cap Q_{\alpha+3}, V_\beta]$. But then $W_{\alpha'} \cap Q_{\alpha+3}$ centralizes $V_\beta \cap V_{\alpha+3}$, contradicting (13.3.4)(ii). Thus $Z_{\alpha+4} \leq [W_{\alpha'} \cap Q_{\alpha+3}, V_\beta]$. If $Z_{\alpha+2} = Z_{\alpha+4}$, then $\langle W_{\alpha'}, G_{\alpha+2\alpha+3}\rangle = \langle G_{\alpha+3\alpha+4}, G_{\alpha+2\alpha+3}\rangle$ which in turn implies $V_{\alpha+3} \cap V_{\alpha+5} = V_{\alpha+3} \cap V_\beta$, against (13.3.3). So $Z_{\alpha+2} \neq Z_{\alpha+4}$ and now (13.3.5) gives (13.3.6).

We now show that $Z_{\alpha+4} \leq V_{\alpha'}$, from which we will derive our final contradiction. Since $[V_{\alpha+3}, W_{\alpha'} \cap Q_{\alpha+3}] = Z_{\alpha+3}$ (else $W_{\alpha'} \cap Q_{\alpha+3}$ centralizes $V_\beta \cap V_{\alpha+3}$, contrary to (13.3.4)(ii)), it is clear that $[V_{\alpha+3}, W_{\alpha'}] = Z_{\alpha+4}$. Let $\alpha' + 2$ be such that $d(\alpha', \alpha' + 2) = 2$. By the minimality of b, $V_{\alpha+3} \leq G_{\alpha'+2}$ and $V_{\alpha+3}$ centralizes $V_{\alpha'}$. Hence $[V_{\alpha+3}, V_{\alpha'+2}] \leq Z_{\alpha'+1} \leq V_{\alpha'}$. Consequently $[V_{\alpha+3}, W_{\alpha'}] \leq V_{\alpha'}$, and thus $Z_{\alpha+4} \leq V_{\alpha'}$. Combining this with (13.3.6) gives

$$\begin{aligned}[V_\beta, (W_{\alpha'} \cap Q_{\alpha+3})/V_{\alpha'}] &= Z_{\alpha+2}Z_{\alpha+4}V_{\alpha'}/V_{\alpha'}\\ &= Z_{\alpha+2}V_{\alpha'}/V_{\alpha'} = Z_\beta V_{\alpha'}/V_{\alpha'}\,.\end{aligned}$$

So for $x = V_\beta \backslash Q_{\alpha'}$, $|[x, W_{\alpha'} \cap Q_{\alpha+3}/V_{\alpha'}]| \leq 2$, whence $[W_{\alpha'}/V_{\alpha'} : C_{W_{\alpha'}/V_{\alpha'}}(x)] \leq$ $\leq 2^2$, contrary to our supposition. This completes the proof of Lemma 13.3.

For the next result we require the following notation. Let $\delta = O(S_6)$ and $\gamma \in \Delta(\delta)$. Then we put

$$\Lambda(\delta, \gamma) = \{\tau \in \Delta(\delta) \mid Z_\tau \not\leq [V_\delta, Q_\gamma]\}.$$

LEMMA 13.4. *For any $\gamma \in \Delta(\beta)$,*

$$W_\beta = \langle U_\tau \mid \tau \in \Lambda(\beta, \gamma)\rangle U_\gamma\,.$$

PROOF. Let $\gamma \in \Delta(\beta)$. By Lemma 2.10(iii) $|\{Z_\tau \mid \tau \in \Lambda(\beta, \gamma)\}| = 8$ and so we have $V_\beta = \langle Z_\tau \mid \tau \in \Lambda(\beta, \gamma)\rangle$. We now investigate the sections $W_\beta/[W_\beta, Q_\beta]V_\beta$ and $[W_\beta, Q_\beta]V_\beta/V_\beta$. From $[V_\beta : V_\beta \cap V_{\alpha-1}] = 2$ it follows that $U_\alpha/V_\beta \cap V_{\alpha-1} \cong 2$ or $2 \oplus 1$ and so $[W_\beta, Q_\beta, Q_\beta] \leq V_\beta$. So these two sections are modules for G_β/Q_β. By Lemma 13.3 we may find an involution

$x \in G_{\alpha\beta} \backslash Q_\beta$ such that

$$[W_\beta/[W_\beta, Q_\beta]V_\beta : C_{W_\beta/[W_\beta,Q_\beta]V_\beta}(x)] \leq 2^2 \text{ and}$$

$$[[W_\beta\ ,\ Q_\beta\ ]\ V_\beta\ /V_\beta\ :\ C_{[W_\beta\ ,\ Q_\beta\ ]V_\beta\ /V_\beta}\ (x)] \leq 2^2\ .$$

Furthermore, $W_\beta/[W_\beta, Q_\beta]V_\beta$ is generated by $U_\alpha[W_\beta, Q_\beta]/[W_\beta, Q_\beta]V_\beta$ which has order 2 and is centralized by $G_{\alpha\beta}$. Similarly, $[W_\beta, Q_\beta]V_\beta/V_\beta$ is generated by $[U_\alpha, Q_\beta]V_\beta/V_\beta$ which has order at most 2 and is centralized by $G_{\alpha\beta}$. Applying Proposition 2.15 to each section yields that $W_\beta/[W_\beta, Q_\beta]V_\beta$ is a quotient of $\begin{pmatrix} 4 \\ 1 \end{pmatrix} \oplus 1$, as is $[W_\beta, Q_\beta]V_\beta/V_\beta$. Proceeding as in Lemma 5.17 gives the lemma.

LEMMA 13.5. $[V_{\alpha+3}, W_{\alpha'}] \leq Z_{\alpha'}$.

PROOF. By Lemma 13.4 $W_{\alpha'} = \langle U_\tau \mid \tau \in \Lambda(\alpha', \alpha' - 1)\rangle U_{\alpha'-1}$. The minimality of b implies $[V_{\alpha+3}, U_{\alpha'-1}] = 1$. Now let τ be an arbitrary element of $\Lambda(\alpha', \alpha' - 1)$, and let $\delta \in \Delta(\tau)\backslash\{\alpha'\}$. Since $V_{\alpha+3} \leq Q_\tau \leq G_\delta$ and $[V_{\alpha+3}, V_{\alpha'}] = 1$, $V_{\alpha+3}$ acts as at most a central transvection on V_δ/Z_δ. Hence $[V_{\alpha+3}, V_\delta] \leq Z_\tau$. Also, as $[V_{\alpha+3}, V_\delta] \leq V_{\alpha+3}$, Z_α centralizes $[V_{\alpha+3}, V_\delta]$. Since $\tau \in \Lambda(\alpha', \alpha' - 1)$. $Z_\tau \not\leq [V_{\alpha'}, Q_{\alpha'-1}] = V_{\alpha'} \cap V_{\alpha'-2} = C_{V_{\alpha'}}(Z_\alpha)$. Hence $[V_{\alpha+3}, V_\delta] \leq C_{V_{\alpha'}}(Z_\alpha) \cap Z_\tau = Z_{\alpha'}$. So $[V_{\alpha+3}, U_\tau] \leq Z_{\alpha'}$, which completes the verification of the lemma.

LEMMA 13.6.

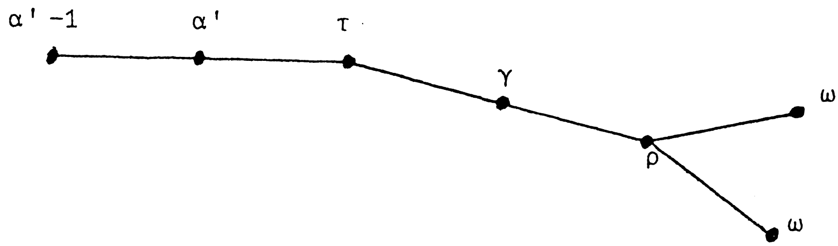
(i) *If* $[V_{\alpha+3}, W_{\alpha'}] = 1$, *then* $[V_\beta, W_{\alpha'}] = Z_{\alpha+2}$ *and* $\eta(G_{\alpha'}, W_{\alpha'}) = 2$.
(ii) *If* $\eta(G_\beta, W_\beta) = 2$, *then* $[U_\alpha, Q_\beta]V_\beta = [U_\gamma, Q_\beta]V_\beta$ *for all* $\gamma \in \Delta(\beta)$.

PROOF. (i) From $[V_{\alpha+3}, W_{\alpha'}] = 1$ it follows that $W_{\alpha'}$ acts upon V_β/Z_β as at most a central transvection of $G_{\beta\alpha+2}/Q_\beta$. Hence $[V_\beta, W_{\alpha'}] \leq Z_{\alpha+2}$ and so, as $\eta(G_{\alpha'}, W_{\alpha'}) \geq 2$, part (i) follows. The assumption $\eta(G_\beta, W_\beta) = 2$ implies that $\eta(G_\beta, [W_\beta, Q_\beta]V_\beta/V_\beta) = 0$, which gives (ii).

LEMMA 13.7. *Let* $\tau \in \Delta(\alpha', \alpha' - 1)$ *and* $\gamma \in \Delta(\tau)$. *If* $\eta(G_\beta, W_\beta) = 2$, *then* $[W_\gamma, V_{\alpha+5}] = 1$; *in particular* $W_\gamma \leq G_{\alpha+3}$.

PROOF. Using Lemma 13.4 gives $W_\gamma = \langle U_\rho \mid \rho \in \Lambda(\gamma, \tau)\rangle U_\tau$. Clearly $[U_\tau, V_{\alpha+5}] = 1$. Now let $\rho \in \Lambda(\gamma, \tau)$. Since $V_{\alpha+5}$ acts as at most a central transvection of $G_{\rho\omega}Q_\omega$ on V_ω/Z_ω for $\omega \in \Delta(\rho)$, we conclude that $[U_\rho, V_{\alpha+5}] \leq Z_\rho$.

Our present situation is as indicated.



Suppose that $[U_\rho, V_{\alpha+5}] \geq Z_\gamma$. Then, since $b \geq 7$, Z_α centralizes $Z_\gamma Z_{\alpha'} = Z_\tau$ and hence $Z_\tau \leq C_{V_{\alpha'}}(Z_\alpha) = [V_{\alpha'}, Q_{\alpha'-1}]$, contrary to $\tau \in \Delta(\alpha', \alpha'-1)$. Therefore $[U_\rho, V_{\alpha+5}] = Z_{\omega'}$ for some $\omega' \in \Delta(\rho)\backslash\{\gamma\}$ or $[U_\rho, V_{\alpha+5}] = 1$. Suppose that the former possibility holds. Then because $\rho \in \Lambda(\gamma, \tau)$ we have

$$V_\gamma = (V_\gamma \cap V_{\alpha'})Z_{\omega'} = (V_\gamma \cap V_{\alpha'})[U_\rho, V_{\alpha+5}].$$

Noting that $[U_\tau, Q_{\alpha'}]V_{\alpha'}$ has index 2 in U_τ and $V_\gamma \not\leq [U_\tau, Q_{\alpha'}]V_{\alpha'}$, we obtain

$$U_\tau = [U_\tau, Q_{\alpha'}]V_{\alpha'}V_\gamma = [U_\tau, Q_{\alpha'}]V_{\alpha'}[U_\rho, V_{\alpha+5}].$$

By hypothesis $\eta(G_\beta, W_\beta) = 2$ and thus $[U_\tau, Q_{\alpha'}]V_{\alpha'} \trianglelefteq G_{\alpha'}$ by Lemma 13.6(ii). Since $b \geq 7$, $[Z_\alpha, [U_\rho, V_{\alpha+5}]] = 1$, whence

$$[U_\tau, Z_\alpha] = [[U_\tau, Q_{\alpha'}]V_{\alpha'}, Z_\alpha] \leq [U_\tau, Q_{\alpha'}]V_{\alpha'} \leq U_\tau\,.$$

Consequently U_τ is normalized by $\langle Z_\alpha, G_{\alpha'\tau}\rangle = G_{\alpha'}$, using Lemma 2.10(v). This is impossible, and so $[U_\rho, V_{\alpha+5}] = 1$ must hold. Since ρ was an arbitrary vertex in $\Lambda(\gamma, \tau)$, $[W_\gamma, V_{\alpha+5}] = 1$.

Lemma 13.8. *Suppose that* $[W_{\alpha'}, V_{\alpha+3}] = 1$, *and let* $\tau \in \Lambda(\alpha', \alpha'-1)$ *and* $\gamma \in \Delta(\tau)\backslash\{\alpha'\}$. *Then* $[W_\gamma, V_{\alpha+3}] = 1$ *and, in particular,* $W_\gamma \leq G_\beta$.

Proof. From Lemma 13.6(i) we have $\eta(G_{\alpha'}, W_{\alpha'}) = 2$. Because $[V_{\alpha+3}, W_{\alpha'}] = 1$ by hypothesis, $[V_{\alpha+3}, V_\gamma] = 1$ and $V_{\alpha+3} \leq Q_\rho$ for all $\rho \in \Delta(\tau)$. Let $\rho \in \Lambda(\gamma, \tau)$. So $V_{\alpha+3}$ acts as at most a central transvection on each V_ω/Z_ω for $\omega \in \Delta(\rho)$. Hence $[V_{\alpha+3}, U_\rho] \leq Z_\rho$. If $[V_{\alpha+3}, U_\rho] \geq Z_\gamma$, then, as in Lemma 13.7, we obtain $\tau \notin \Lambda(\alpha', \alpha'-1)$ (note that $W_\gamma \leq G_{\alpha+3}$ by Lemma 13.7). Thus $[V_{\alpha+3}, U_\rho] \leq Z_{\omega'}$ for some $\omega' \in \Delta(\rho)\backslash\{\gamma\}$. If $[V_{\alpha+3}, U_\rho] = Z_{\omega'}$ holds, then arguing as in Lemma 3.7 we first obtain $U_\tau = [U_\tau, Q_{\alpha'}]V_{\alpha'}[U_\rho, V_{\alpha+3}]$ and thence $U_\tau \trianglelefteq \langle Z_\alpha, G_{\alpha'\tau}\rangle = G_{\alpha'}$. So we conclude that $[V_{\alpha+3}, U_\rho] = 1$ and consequently $[V_{\alpha+3}, W_\gamma] = 1$ by Lemma 13.4. Now Lemma 13.7 and $[V_{\alpha+3}, W_\gamma] = 1$ yield $W_\gamma \leq G_\beta$.

Lemma 13.9. $[W_{\alpha'}, V_{\alpha+3}] = Z_{\alpha'}$.

Proof. If the lemma is false, then $[W_{\alpha'}, V_{\alpha+3}] = 1$ by Lemma 13.5. So, for $\tau \in \Delta(\alpha', \alpha' - 1)$, Lemma 13.8 implies that $G_\tau^{[4]} \leq G_\beta$ with $G_\tau^{[4]}Q_\beta/Q_\beta$ at most a central transvection of $G_{\beta\alpha+2}/Q_\beta$ on V_β/Z_β. Using Lemma 13.6(i) we see that

$$[G_\tau^{[4]}, V_\beta] = [W_{\alpha'}, V_\beta] = Z_{\alpha+2} \leq W_{\alpha'} \leq G_\tau^{[4]},$$

whence $G_\tau^{[4]} \trianglelefteq \langle V_\beta, G_{\alpha'\tau}\rangle = G_\alpha$, a contradiction. Hence Lemma 13.9 holds.

In our next result our attention switches to W_β.

Lemma 13.10.

(i) $[W_\beta, V_{\alpha'-2}] = 1$.
(ii) $V_{\alpha'} \leq Q_\beta$.

Proof. (i) Suppose that $W_\beta \not\leq C_{G_{\alpha'-2}}(V_{\alpha'-2})$. Using Lemma 13.4 again gives

$$W_\beta = \langle U_\tau \mid \tau \in \Delta(\beta, \alpha+2)\rangle U_{\alpha+2} = \langle U_\tau \mid \tau \in \Delta(\beta), (\tau, \alpha') \in \mathscr{C}\rangle U_{\alpha+2}.$$

So there exists $\tau \in \Delta(\beta)$ with $(\tau, \alpha') \in \mathscr{C}$ and $\rho \in \Delta(\alpha' - 2)$ such that $[U_\tau, Z_\rho] \neq 1$. Hence $[V_{\tau-1}, Z_\rho] \neq 1$ for some $\tau - 1 \in \Delta(\tau)\backslash\{\beta\}$. If $Z_\rho \leq Q_{\tau-1}$, then $Z_{\tau-1} = [V_{\tau-1}, Z_\rho] \leq V_{\alpha'-2}$, contrary to $(\tau, \alpha') \in (\mathscr{C})$. Thus $(\rho, \tau - 1) \in \mathscr{C}$. But then applying Lemma 13.9 to $(\rho, \tau - 1)$ gives the contradiction

$$Z_{\tau-1} = [W_{\tau-1}, V_{\alpha'-4}] \leq V_{\alpha'-4} \leq Q_{\alpha'},$$

since $b > 5$. Therefore we have verified (i).

(ii) Suppose that $V_{\alpha'} \not\leq Q_\beta$ holds. Then there exists $\tau \in \Delta(\alpha')$ such that $(\tau, \beta) \in \mathscr{C}$. Lemma 13.9 applied to (τ, β) gives $[W_\beta, V_{\alpha'-2}] = Z_\beta$, contradicting part (i). Thus $V_{\alpha'} \leq Q_\beta$.

Proof of Theorem 13.1. Supposing the result false, we seek a contradiction. So the previous lemmas in this section are available to us. Let $\alpha - 1 \in \Delta(\alpha)\backslash\{\beta\}$.

(13.1.1) $$[W_{\alpha-1}, V_{\alpha'-4}] = 1$$

Suppose (13.1.1) is false. Then, by Lemma 13.4, there exists $\alpha - 2 \in \Lambda(\alpha - 1, \alpha)$ such that $[U_{\alpha-2}, V_{\alpha'-4}] \neq 1$. Since $[V_{\alpha'-4}, V_{\alpha-1}] = 1$, we may find $\alpha - 3 \in \Delta(\alpha - 2)\backslash\{\alpha - 1\}$ such that $[V_{\alpha-3}, V_{\alpha'-4}] \neq 1$. Moreover $V_{\alpha'-4}$ acts as at most a central transvection of $G_{\alpha-3\alpha-2}/Q_{\alpha-3}$ on $V_{\alpha-3}/Z_{\alpha-3}$, and so $[V_{\alpha-3}, V_{\alpha'-4}] \leq Z_{\alpha-2}$. Since $Z_{\alpha-1} \not\leq Q_{\alpha'}$ and $b > 5$, we deduce that $Z_{\alpha-1} \not\leq [V_{\alpha-3}, V_{\alpha'-4}]$ whence, as $\alpha - 2 \in \Lambda(\alpha - 1, \alpha)$,

$$V_{\alpha-1} = (V_\beta \cap V_{\alpha-1})[V_{\alpha-3}, V_{\alpha'-4}].$$

Combining $[V_{\alpha'}, [V_{\alpha-3}, V_{\alpha'-4}]] = 1$ and $[V_\beta, V_{\alpha'}] = Z_\beta$ (by Lemma 3.10(ii)) we obtain $[V_{\alpha-1}, V_{\alpha'}] \leq Z_\beta \leq V_{\alpha-1}$ and then $V_{\alpha-1} \trianglelefteq \langle V_{\alpha'}, G_{\alpha-1\alpha}\rangle = G_\alpha$, a contradiction. Thus we have (13.1.1).

(13.1.2) $$[W_{\alpha-1}, V_{\alpha'-2}] = 1$$

From Lemma 13.10(i) $[W_\beta, V_{\alpha'-2}] = 1$ and so $[U_\alpha, V_{\alpha'-2}] = 1$. Therefore $V_{\alpha'-2} \leq Q_{\alpha-2}$ for any $\alpha - 2 \in \Delta(\alpha - 1)$ and $V_{\alpha'-2}$ acts as at most a central transvection of $G_{\alpha-3\alpha-2}/Q_{\alpha-3}$ on $V_{\alpha-3}/Z_{\alpha-3}$ $(\alpha - 3 \in \Delta(\alpha - 2))$. Thus $[V_{\alpha'-2}, V_{\alpha-3}] \leq Z_{\alpha-2}$ and once again we deduce that either

(a) $[W_{\alpha-1}, V_{\alpha'-2}] = 1$ or

(b) there exists $\alpha - 2 \in \Lambda(\alpha - 1, \alpha)$ and $\alpha - 3 \in \Delta(\alpha - 2)\backslash\{\alpha - 1\}$ such that $[V_{\alpha'-2}, V_{\alpha-3}] \neq 1$. (Here we use (13.1.1) to get $[V_{\alpha'-2}, V_{\alpha-3}] \leq V_{\alpha'-2}$).

In case (b), just as in (13.1.1), we obtain

$$V_{\alpha-1} = (V_{\alpha-1} \cap V_\beta)[V_{\alpha'-2}, V_{\alpha-3}],$$

and then $V_{\alpha-1} \trianglelefteq \langle V_{\alpha'}, G_{\alpha-1\alpha}\rangle = G_\alpha$. So (a) must hold, as required.

Combining (13.1.1) and (13.1.2) gives $W_{\alpha-1} \leq G_{\alpha'}$ with $W_{\alpha-1}Q_{\alpha'}/Q_{\alpha'}$ acting as a central transvection on $V_{\alpha'}/Z_{\alpha'}$. Now, by Lemma 3.10(ii) and $\eta(G_\alpha, U_\alpha) = 2$, we have $|[U_\alpha, V_{\alpha'}]| \geq 2^2$. Hence

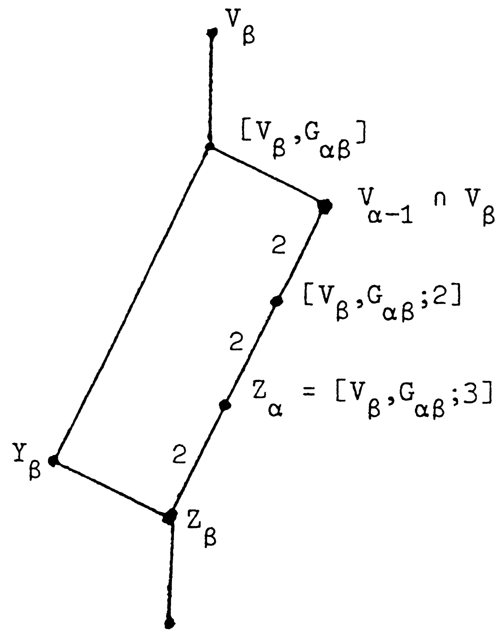
$$[W_{\alpha-1}, V_{\alpha'}] = Z_{\alpha'-1} = [U_\alpha, V_{\alpha'}] \leq U_\alpha \leq W_{\alpha-1}$$

and so $W_{\alpha-1} \trianglelefteq \langle V_{\alpha'}, G_{\alpha-1\alpha}\rangle = G_\alpha$. With this contradiction we have completed the proof of Theorem 13.1.

We now tackle Case 2, and will prove the following

THEOREM 13.11. *Suppose that* $(\alpha, \alpha') \in \mathscr{C}$, $V_\beta/Z_\beta \cong \begin{pmatrix} 4 \\ 1 \end{pmatrix}$ *and* $(V_{\alpha-1} \cap V_\beta)Y_\beta = [V_\beta, G_{\alpha\beta}]$ *(where* $Y_\beta = C_{V_\beta}(O^2(G_\beta))$*). Then* $b = 3$.

PROOF. Let $(\alpha, \alpha') \in \mathscr{C}$. Referring to Lemma 12.6(ii) we have



We shall show that the assumption $b \geq 5$ leads to a contradiction - the observations in (13.11.1) and (13.11.2) hold the key to doing this.

(13.11.1) Let $x \in G_{\alpha\beta}$. If x acts as the central transvection of $G_{\alpha\beta}/Q_\beta$ on V_β/Z_β, then

(i) $[V_\beta, x] \not\leq Z_\alpha$; and
(ii) $[V_\beta, x] \not\leq V_{\alpha-1} \cap V_\beta$.

Because $Z_\alpha = [V_\beta, G_{\alpha\beta}; 3]$ part (i) is a consequence of Proposition 2.5(i), (iii). Now if $[V_\beta, x] \leq V_{\alpha-1} \cap V_\beta$, then by part (i) $Y_\beta \leq [V_\beta, x]Z_\alpha \leq V_{\alpha-1} \cap V_\beta$, contrary to Lemma 12.6(ii). Hence (ii) holds as well.

(13.11.2) $$\eta(G_\alpha, U_\alpha) = 3.$$

Put $N = V_{\alpha-1} \cap V_\beta$ and $U_\alpha{}^{(1)} = \langle [V_\beta, G_{\alpha\beta}]^{G_\alpha} \rangle$. If $\eta(G_\alpha, U_\alpha{}^{(1)}/N) = 1$, then

$$U_\alpha{}^{(1)} = [V_\beta, G_{\alpha\beta}]N = [V_{\alpha-1}, G_{\alpha\beta}]N \leq V_{\alpha-1} \cap V_\beta$$

which is impossible since $[V_\beta, G_{\alpha\beta}] \not\leq V_{\alpha-1} \cap V_\beta$. Thus $\eta(G_\alpha, U_\alpha{}^{(1)}/N) = 1$ and so, using Lemma 1.2(v), we see that $\eta(G_\alpha, U_\alpha) = 3$.

(13.11.3) $V_\beta Q_{\alpha'}/Q_{\alpha'}$ is the central transvection of $G_{\alpha'-1\alpha'}/Q_{\alpha'}$ on $V_{\alpha'}/Z_{\alpha'}$.

Since V_β centralizes $V_{\alpha'-2} \cap V_{\alpha'}$, V_β acts as the central transvection of $G_{\alpha'-1\alpha'}/Q_{\alpha'}$ on $V_{\alpha'}/Y_{\alpha'}$, and now (13.11.3) follows by Proposition 2.5(iii).

(13.11.4) $$V_{\alpha'} \not\leq Q_\beta.$$

Suppose $V_{\alpha'} \leq Q_\beta$ holds, and choose $\rho \in \Delta(\alpha')$ such that $Z_\rho \not\leq Q_\alpha$. Now suppose that we have $\gamma \in \Delta(\alpha' - 2)$ for which $(\gamma, \alpha - 1) \in \mathscr{C}$. Because $b \geq 5$, Z_ρ centralizes $[V_{\alpha'-2}, V_{\alpha-1}]$ and hence, since Z_ρ is transitive on $\Delta(\alpha)\backslash\{\beta\}$, $[V_{\alpha'-2}, V_{\alpha-1}] \leq V_{\alpha-1} \cap V_\beta$. Applying (13.11.3) to $(\gamma, \alpha - 1)$ we obtain a contradiction. Therefore $V_{\alpha'-2} \leq Q_{\alpha-1}$. Furthermore, since $b \geq 5$, $Z_{\alpha-1} \not\leq V_{\alpha'-2}$ and so $[V_{\alpha-1}, V_{\alpha'-2}] = 1$. Thus $[U_\alpha, V_{\alpha'-2}] = 1$. Consequently $U_\alpha \leq G_{\alpha'}$ and $|U_\alpha Q_{\alpha'}/Q_{\alpha'}| = 2$. But then $[U_\alpha : C_{U_\alpha}(Z_\rho)] \leq 2^2$, against (13.11.2). This proves (13.11.4).

(13.11.5) $V_{\alpha'} \cap Q_\beta = [V_{\alpha'}, V_\beta](V_{\alpha'} \cap V_{\alpha'-2})$ and, in particular, $[V_\beta, V_{\alpha'} \cap Q_\beta] = 1$.

This is an immediate consequence of (13.11.1)(i), (13.11.3) and (13.11.4).

We recall, from Section 12, the definition of F_α and H_β.

$$F_\alpha = \langle Y_\lambda \mid \lambda \in \Delta(\alpha) \rangle$$

$$H_\beta = \langle F_\mu \mid \mu \in \Delta(\beta) \rangle$$

(13.11.6) $$[F_\alpha, V_{\alpha'-2}] = 1 \quad \text{and} \quad F_\alpha Q_{\alpha'} = V_\beta Q_{\alpha'}\,.$$

By the minimality of b $V_{\alpha'-2} \leq G_{\alpha-1}$ for $\alpha - 1 \in \Delta(\alpha)\backslash\{\beta\}$ and thus $[V_{\alpha'-2}, Y_{\alpha-1}] \leq V_{\alpha'-2} \cap Z_{\alpha-1} = 1$. So $[F_\alpha, V_{\alpha'-2}] = 1$ and (13.11.6) follows.

(13.11.7) There exists $\delta \in \Delta(\beta)$ such that $(\delta, \alpha') \in \mathscr{C}$ and $\langle G_{\alpha\beta}, V_{\alpha'} \rangle = G_\beta$.

By (13.11.4) we may find a $\delta \in \Delta(\beta)$ for which $\langle G_{\delta\beta}, V_{\alpha'} \rangle = G_\beta$. If $(\delta, \alpha') \notin \mathscr{C}$, then $Z_\delta \leq V_\beta \cap Q_{\alpha'} = [V_\beta, V_{\alpha'}](V_\beta \cap V_{\alpha+3})$ whence $[Z_\delta, V_{\alpha'}] = 1$. But then $Z_\delta \trianglelefteq G_\beta$, a contradiction. So $(\delta, \alpha') \in \mathscr{C}$ and we have (13.11.7).

Since the results in (13.11.2) - (13.11.6) hold for any critical pair we may suppose (α, α') is chosen so as $\langle G_{\alpha\beta}, V_{\alpha'} \rangle = G_\beta$.

(13.11.8) $$[(F_\alpha V_\beta) \cap Q_{\alpha'}, V_{\alpha'}] \neq 1.$$

From (13.11.6) we see that $F_\alpha V_\beta = V_\beta((F_\alpha V_\beta) \cap Q_{\alpha'})$. Thus, if $[(F_\alpha V_\beta) \cap Q_{\alpha'}, V_{\alpha'}] = 1$,

$$[F_\alpha V_\beta, V_{\alpha'}] = [V_\beta, V_{\alpha'}][(F_\alpha V_\beta) \cap Q_{\alpha'}, V_{\alpha'}] = [V_\beta, V_{\alpha'}] \leq V_\beta\,.$$

Then $F_\alpha V_\beta \trianglelefteq \langle G_{\alpha\beta}, V_{\alpha'}\rangle = G_\beta$ and so $H_\beta = F_\alpha V_\beta$ with $\eta(G_\beta, H_\beta/V_\beta) = 0$, contrary to Lemma 12.5(ii). Thus (13.11.8) holds.

In view of (13.11.8) we may choose $\lambda \in \Delta(\alpha')$ such that $(F_\alpha V_\beta) \cap Q_{\alpha'} \not\leq Q_\lambda$.

(13.11.9) (i) $(\lambda, \beta) \in \mathscr{C}$.
(ii) $[U_\tau, V_{\alpha+3}] = 1$.
(iii) $U_\tau Q_\beta = V_{\alpha'} Q_\beta$.
(iv) $[U_\tau, V_\beta \cap Q_{\alpha'}] = 1$.

Suppose that $(\lambda, \beta) \notin \mathscr{C}$. Then $Z_\lambda \leq V_{\alpha'} \cap Q_\beta$ and so $[Z_\lambda, V_\beta] = 1$ by (13.11.5). Hence $Z_\lambda \leq Q_\alpha \leq G_{\alpha-1}$ for $\alpha - 1 \in \Delta(\alpha)\backslash\{\beta\}$ and so $[Z_\lambda, Y_{\alpha-1}] \leq$ $\leq V_{\alpha'} \cap Z_{\alpha-1} = 1$. Thus we have $[F_\alpha V_\beta, Z_\lambda] = 1$, contrary to the choice of λ. Therefore $(\lambda, \beta) \in \mathscr{C}$.

The minimality of b implies that $[V_{\alpha'}, V_{\alpha+3}] = 1$ and thus, for $\lambda + 1 \in \Delta(\lambda)\backslash\{\alpha'\}$, either $V_{\alpha+3}Q_{\lambda+1}/Q_{\lambda+1}$ is the central transvection of $G_{\lambda\lambda+1}/Q_{\lambda+1}$ (on $V_{\lambda+1}/Z_{\lambda+1}$) or $V_{\alpha+3} \leq Q_{\lambda+1}$. Suppose the former holds. By (13.11.1)(ii) $[V_{\alpha+3}, V_{\lambda+1}] \not\leq V_{\lambda+1} \cap V_{\alpha'}$. Now, if $b \geq 7$, then $F_\alpha V_\beta$ centralizes $V_{\alpha+3}$, while when $b = 5$, $\alpha + 3 = \alpha' - 2$ and (13.11.6) gives us the same conclusion. Hence we have that $(F_\alpha V_\beta) \cap Q_{\alpha'}$ centralizes $[V_{\alpha+3}, V_{\lambda+1}]$ and then $[V_{\alpha+3}, V_{\lambda+1}] \leq V_{\lambda+1} \cap V_{\alpha'}$ by the choice of λ. So $V_{\alpha+3} \leq Q_{\lambda+1}$ and thus $[V_{\alpha+3}, V_{\lambda+1}] \leq V_{\alpha+3} \cap Z_{\lambda+1} \leq Q_\beta$. Now (i) yields $[V_{\alpha+3}, V_{\lambda+1}] = 1$ which completes the proof of (ii), and so of (iii). While part (iv) follows from $V_\beta \cap Q_{\alpha'} = [V_\beta, V_{\alpha'}](V_{\alpha+3} \cap V_\beta)$.

We now exhibit the desired contradiction. Noting that $Y_\beta \leq V_\beta \cap Q_{\alpha'}$ and $V_\beta = Z_\alpha(V_\beta \cap Q_{\alpha'})$, (13.11.8)(iv) implies that U_λ centralizes Y_β and $U_\lambda \cap Q_\alpha$ centralizes V_β. Also, since $Z_{\alpha-1} \not\leq Q_{\alpha'}$, $U_\lambda \cap Q_\alpha$ centralizes $Y_{\alpha-1}$ for $\alpha - 1 \in \Delta(\alpha)\backslash\{\beta\}$. So we have $[U_\lambda \cap Q_\alpha, F_\alpha V_\beta] = 1$. Then by (13.11.8)(iii) $[U_\alpha : C_{U_\alpha}((F_\alpha V_\beta) \cap Q_{\alpha'})] \leq 2^2$ and so $\eta(G_\lambda, U_\lambda) \leq 2$, contradicting (13.11.2) and hence completing the proof of the theorem.

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