Finitely Presented Modules over Right Non-Singular Rings.

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Abstract - This paper characterizes the right non-singular rings $R$ for which $M/Z(M)$ is projective whenever $M$ is a cyclically (finitely) presented module. Several related results investigate right semi-hereditary rings.

1. Introduction.

The straightforward attempt to extend the notion of torsion-freeness from integral domains to non-commutative rings encounters immediate difficulties. To overcome these, one can concentrate on either the computational or the homological properties of torsion-free modules. Goodearl and others took the first approach when they introduced the notion of a non-singular module [8]. A right $R$-module $M$ is non-singular if $Z(M) = 0$ where $Z(M) = \{ x \in M \mid xI = 0 \text{ for some essential right ideal } I \text{ of } R \}$ denotes the singular submodule of $M$. On the other hand, $M$ is singular if $Z(M) = M$. Moreover, a submodule $U$ of an $R$-module $M$ is $S$-closed if $M/U$ is non-singular. Finally, $R$ is a right non-singular ring if $R_R$ is non-singular.

The right non-singular rings are precisely the rings which have a regular, right self-injective maximal right ring of quotients, which will be denoted by $Q^r$ (see [8] and [11] for details). Following [11, Chapter XI], $Q^r$ is a perfect left localization of $R$ if $Q^r$ is flat as a right $R$-module and the multiplication map $Q^r \otimes_R Q^r \rightarrow Q^r$ is an isomorphism. In particular, $Q^r$ is a perfect left localization of $R$ if and only if every finitely generated non-singular right $R$-module can be embedded into a projective module ([8] and [11]). We call a right non-singular ring with this property right strongly non-singular.

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Hattori took the second approach by defining $M$ to be torsion-free if $\text{Tor}_1^R(M, R/Rr) = 0$ for all $r \in R$ [9]. The classes of torsion-free and non-singular right $R$-modules coincide if and only if $R$ is a right Utumi p.p.-ring without an infinite set of orthogonal idempotents [3, Theorem 3.7]. Here, $R$ is a right p.p.-ring if all principal right ideals of $R$ are projective. Moreover, a right non-singular ring $R$ is a right Utumi-ring if every $S$-closed right ideal of $R$ is a right annihilator.

Closely related to the notion of torsion-freeness are those of purity and relative divisibility. A sequence of right $R$-modules is pure-exact (RD-exact) if every finitely presented (cyclically presented) module is projective with respect to it. Investigating RD- and pure-projective modules leads to the investigation of the condition that $M/Z(M)$ is projective. The dual question when $Z(M)$ is injective has been addressed in [8, Page 48, Example 24]. Section 2 discusses the question for which rings $M/Z(M)$ is projective for all RD-projective modules $M$. Theorem 2.1 shows that, provided $R$ has no infinite set of orthogonal idempotents, these are precisely the right Utumi p.p.-rings discussed in [3]. The structure of pure-projective right $R$-modules was described in [4] in case that $R$ is a right strongly non-singular, right semi-hereditary ring $R$ without an infinite set of orthogonal idempotents. Theorem 2.3 shows that these conditions on $R$ are not only sufficient, but also necessary for the structure-theorem (part b) of Theorem 2.3 to hold.

Prüfer domains can be characterized as the domains with the property that, whenever a torsion-free module $M$ contains a projective submodule $U$ with $M/U$ finitely generated, then $M$ is projective and $M/U$ is finitely presented [7, Chapter VI]. We show that the right non-singular rings having the corresponding property for non-singular modules are precisely the right strongly non-singular, right semi-hereditary rings of finite right Goldie dimension (Theorem 3.1).

Section 4 investigates pure-projective modules over right hereditary rings. As part of our discussion, we obtain a characterization of the right Noetherian right hereditary rings with the restricted right minimum condition which are right strongly non-singular. The last results of this paper demonstrate that right invertible submodules of $Q_R$ which were introduced in [11, Chapters II.4 and IX.5] may fail to share many of the important properties of invertible modules over integral domains. For instance, the lattice of finitely generated right ideals over a Prüfer domain is distributive, i.e. $I \cap (J + K) = (I \cap J) + (I \cap K)$ for all finitely generated ideals $I, J,$ and $K$ of $R$ [7, Theorem III.1.1]. Example 4.7 shows that there exists a right strongly non-singular, hereditary, right and left Noetherian ring whose finitely generated right ideals do not have this property.
2. RD-Projective Modules.

Let $U$ be a submodule of a non-singular module $M$. The $S$-closure of $U$ in $M$ is the submodule $V$ of $M$ which contains $U$ such that $V/U = Z(M/U)$.

**Theorem 2.1.** The following are equivalent for a right non-singular ring $R$ without an infinite set of orthogonal idempotents:

a) $R$ is a right Utumi p.p.-ring.

b) $M/Z(M)$ is projective for every RD-projective module $M$.

**Proof.** a) $\Rightarrow$ b): Since every RD-projective module is a direct summand of a direct sum of cyclically presented modules, it suffices to verify b) in case that $M \cong R/aR$ for some $a \in R$. Let $J$ be the right ideal of $R$ which contains $aR$ such that $J/aR = Z(R/aR)$. Then, $R/J \cong M/Z(M)$ is a non-singular cyclic module which is projective by [3, Corollary 3.4].

b) $\Rightarrow$ a): Let $I$ be the $S$-closure of $rR$ for some $r \in R$. Since $R/rR$ is RD-projective, and $R/I \cong (R/rR)/Z(R/rR)$, we obtain that $R/I$ is projective. Thus, $I$ is generated by an idempotent.

By [3, Lemma 3.5], it suffices to show that every $S$-closed right ideal $J$ of $R$ is generated by an idempotent. For this, select $0 \neq r_0 \in J$. Since $J$ is $S$-closed in $R$, it contains the $S$-closure $I_0$ of $r_0R$. By what has been shown so far, $I_0 = e_0R$ for some idempotent $e_0$ of $R$. Hence, $J = e_0R \oplus [J \cap (1 - e_0)R]$. If $J \cap (1 - e_0)R \neq 0$, select a non-zero $r_1 = (1 - e_0)r_1 \in J$; and observe that $J \cap (1 - e_0)R$ is $S$-closed in $R$. Hence, it contains the $S$-closure $I_1$ of $r_1R$ in $R$. By the previous paragraph, $I_1 = fR$ for some idempotent $f$ of $R$. Write $f = (1 - e_0)s$ for some $s \in R$, and set $e_1 = f(1 - e_0)$. Since $e_0f = 0$, we have $e_1e_0 = e_0e_1 = 0$ and $e_1^2 = f^2 - f^2e_0 - fe_0f + (fe_0)^2 = f - fe_0 = e_1$. Thus, $e_0$ and $e_1$ are non-zero orthogonal idempotents with $e_1R \subseteq fR$. On the other hand, $f = f(1 - e_0)s = e_1s$ yields $fR = e_1R$. Consequently, $R = e_0R \oplus e_1R \oplus [J \cap (1 - e_0 - e_1)R]$. Continuing inductively, we can construct non-zero orthogonal idempotents $e_0, \ldots, e_n, e_{n+1} \in J$ as long as $J \cap (1 - e_0 - \ldots - e_n)R \neq 0$. Since $R$ does not contain an infinite family of orthogonal idempotent, this process has to stop, say $J \cap (1 - e_0 - \ldots - e_n)R = 0$. Then, $e_0 + \ldots + e_m$ is an idempotent with $J = (e_0 + \ldots + e_m)R$.

We now investigate which conditions $R$ has to satisfy to ensure the validity of the structure theorem for pure-projectives in [4]. We want
to remind the reader that a right $R$-module is essentially finitely generated if contains an essential, finitely generated submodule.

**Lemma 2.2.** The following are equivalent for a right non-singular ring $R$:

a) $R$ is right semi-hereditary and has finite right Goldie-dimension.

b) A finitely generated right $R$-module $M$ is finitely presented if and only if $p.d.M \leq 1$.

**Proof.** $a) \Rightarrow b)$: Since $R$ is right semi-hereditary, every finitely presented module has projective dimension at most 1. Conversely, whenever $M \cong R^n / U$ for some projective module $U$, then $U$ is essentially finitely generated since $R$ has finite right Goldie dimension. By Sandomierski's Theorem [5, Proposition 8.24], essentially finitely generated projective modules are finitely generated.

$b) \Rightarrow a)$: Clearly, $R$ has to be right semi-hereditary. If $R$ has infinite right Goldie-dimension, then it contains a family $\{I_n\}_{n<\omega}$ of non-zero, finitely generated right ideals whose sum is direct. Since $R$ is right semi-hereditary, each $I_n$ is projective, and the same holds for $\oplus_{n<\omega} I_n$. By b), $R/ \oplus_{n<\omega} I_n$ is finitely presented, a contradiction. \qed

**Theorem 2.3.** The following conditions are equivalent for a right non-singular ring $R$ without an infinite set of orthogonal idempotents:

a) $R$ is a (right Utumi), right semi-hereditary ring such that $Q^e$ is a perfect left localization of $R$.

b) A right $R$-module $M$ is pure-projective if and only if

i) $Z(M)$ is a direct summand of a direct sum of finitely generated modules of projective dimension at most 1.

ii) $M/Z(M)$ is projective.

**Proof.** A right strongly non-singular, right semi-hereditary ring without an infinite set of orthogonal idempotents is a right Utumi ring [3, Theorems 3.7 and 4.2].

$a) \Rightarrow b)$: By [3], $R$ has finite right Goldie-dimension. Because of Lemma 2.2, $Z(M)$ is pure-projective if and only if condition i) in b) holds. It remains to show that $M/Z(M)$ is projective whenever $M$ is finitely presented. However, this follows from [11] since $M/Z(M)$ is a finitely generated non-singular module, and $Q^e$ is a perfect left localization of $R$. 


b) ⇒ a): To see that \( R \) is right semi-hereditary, consider a finitely generated right ideal \( I \) if \( R \). The \( S \)-closure \( J \) of \( I \) satisfies \( J/I = Z(R/I) \). Hence, \( R/J \) is projective by b), and \( J/I \) has projective dimension at most 1. Since \( J \) is projective, this is only possible if \( I \) is projective.

To show that \( R \) has finite right Goldie-dimension, consider a right ideal of \( R \) of the form \( a_0 R \oplus \ldots \oplus a_n R \oplus \ldots \) where each \( a_n \neq 0 \). For \( m < \omega \), let \( J_m \) be the \( S \)-closure of \( a_0 R \oplus \ldots \oplus a_m R \). Since \( R/I_m \) is projective by b), \( I_m \) is generated by an idempotent \( e_m \) of \( R \). Write \( I_{m+1} = I_m \oplus [I_{m+1} \cap (1 - e_m)R] \). Observe that \( [I_{m+1} \cap (1 - e_m)R] \) is generated by an idempotent \( f \) of \( R \) as a direct summand of \( R \). Setting \( d_m = f(1 - e_m) \) yields an idempotent \( d_m \) of \( R \) such that \( e_m d_m = d_m e_m = 0 \), and \( I_{m+1} = I_m \oplus d_m R \) as in the proof of Theorem 2.1. Inductively, one obtains an infinite family of orthogonal idempotents \( \{d_m | m < \omega \} \) of \( R \), which is not possible. Thus, \( R \) has finite right Goldie-dimension; and every \( S \)-closed right ideal \( J \) of \( R \) is the \( S \)-closure of a finitely generated right ideal. By b), \( R/J \) is projective; and \( R \) is a right Utumi-ring since \( J = eR \) for some idempotent \( e \) of \( R \).

To establish that \( Q^r \) is a perfect left localization of \( R \), it suffices to show that every finitely generated non-singular right \( R \)-module \( M \) is projective. Write \( M \cong R^n/U \) and observe that \( U \) is essentially finitely generated. Select a finitely generated essential submodule \( V \) of \( U \). Then, \( U/V = Z(R^n/U) \), and \( M \) is projective by b). □

In the following, the injective hull of a module \( M \) is denoted by \( E(M) \).

**Corollary 2.4.** Let \( R \) be a right semi-hereditary ring of finite right Goldie-dimension such that \( Q^r \) is a perfect left localization of \( R \). A right \( R \)-module \( M \) is pure-projective if and only if \( M/Z(M) \) is projective and \( Z(M) \) is isomorphic to a direct summand of a direct sum of finitely generated submodules of \( (Q^r/R)^n \).

**Proof.** We first show that a finitely generated singular module \( M \) has projective dimension at most 1 if and only if it can be embedded into a finite direct sum of copies of \( Q^r/R \). If \( p.d. M \leq 1 \), then there exist a finitely generated free module \( F \) and an essential projective submodule \( P \) of \( F \) with \( M \cong F/P \). Since \( R \) has finite right Goldie-dimension, \( P \) is essentially finitely generated, and hence itself finitely generated by Sandomierski’s Theorem [5, Proposition 8.24]. We can find a finitely generated projective module \( U \) such that \( P \oplus U \) is a finitely generated free module, say \( P \oplus U \cong R^n \). Since \( M \) is singular, \( P \oplus U \) is an essential submodule of \( F \oplus U \). Therefore, \( F \oplus U \subseteq E(P \oplus U) = (Q^r)^n \), and \( M \subseteq (Q^r/R)^n \).
On the other hand, if $M$ is a finitely generated submodule of $(Q'/R)^n$, then there is a finitely generated submodule $U$ of $(Q')^n$ containing $R^n$ such that $M = U/R^n$. Since $R$ is right semi-hereditary, and since $Q'$ is a perfect left localization of $R$, $U$ is projective and $p.d.M \leq 1$. The corollary now follows directly from Theorem 2.3.

\[ \square \]

3. Essential Extensions of Projective Modules.

In the commutative setting, Prüfer domains are characterized by conditions b) and c.ii) [7].

**Theorem 3.1.** The following are equivalent for a right non-singular ring $R$:

a) $R$ is a right semi-hereditary ring of finite right Goldie-dimension for which $Q'$ is a perfect left localization of $R$.

b) Whenever a non-singular module $M$ contains a projective submodule $U$ such that $M/U$ is finitely generated, then $M$ is projective and $M/U$ is finitely presented.

c) i) $R$ is a right p.p.-ring.

 ii) If a finitely generated non-singular right $R$-module $M$ contains an essential projective submodule $U$, then $M$ is projective, and $M/U$ is finitely presented.

**Proof.** a) $\Rightarrow$ b): Let $W$ be the $S$-closure of $U$ in $M$. Since $M/W$ is finitely generated as an image of $M/U$ and non-singular, it is projective by a). Hence, $M = W \oplus P$ for some finitely generated projective module $P$. We may thus assume that $M/U$ is singular.

Since $R$ is right semi-hereditary, $U = \bigoplus_i U_i$ where each $U_i$ is finitely generated [1]. Because $M/U$ is singular and $M$ is non-singular, $U$ is essential in $M$. Thus, $E(M) = E(U) = \bigoplus_i E(U_i)$ in view of the fact that direct sums of non-singular injectives are injective if $R$ has finite right Goldie-dimension [11, Proposition XIII.3.3]. Choose a finitely generated submodule $V$ of $M$ such that $M = U + V$. There is a finite subset $J$ of $I$ such that $V \subseteq \bigoplus_J E(U_i)$. Then, $W_1 = V + \bigoplus_J U_i$ is a finitely generated submodule of $\bigoplus_J E(U_i)$ such that $V \cap (\bigoplus_J U_i) = 0$. Consequently, $M = W_1 \oplus \bigoplus_J U_i$. But $W_1$ is projective by a) showing that $M$ is projective and that $M/U = W_1/U$ is finitely presented.

b) $\Rightarrow$ c): Observe that every finitely generated non-singular module is projective by choosing $U = 0$ in b).
c) $\Rightarrow$ a): Assume that $R$ contains a right ideal $U$ of the form $U = \bigoplus_{n<\omega} a_n R$ where each $a_n \neq 0$. By part i) of c), $U$ is projective. Choose a right ideal $V$ of $R$ which is maximal with respect to the property $U \cap V = 0$. Since $R$ is right non-singular, $V$ is an $S$-closed submodule of $R$ and $[R/V]/[U \oplus V/V] \cong R/(U \oplus V)$ is singular. Therefore, the projective module $U \oplus V/V$ is essential in the non-singular module $R/V$. By c), $R/V$ is projective; and $R/(U \oplus V)$ is finitely presented. Hence, $U$ is finitely generated which is not possible.

To see that $R$ is a right strongly non-singular, right semi-hereditary ring, it suffices to show that a finitely generated non-singular right $R$-module $M$ is projective. By [11, Proposition XII.7.2], $M \subseteq (Q^n)^n$ for some $n < \omega$. Since $R$ has finite right Goldie dimension and $R^n$ is essential in $(Q^n)^n$, $M$ has finite Goldie-dimension. Therefore, $M$ contains uniform submodules $U_1, \ldots, U_m$ such that $U_1 \oplus \ldots \oplus U_m$ is essential in $M$. Furthermore, we may assume that each $U_i$ is cyclic, say $U_i = b_i R$. Since $M$ is non-singular, $\text{ann}_R(b_i) = \{ r \in R \mid b_i r = 0 \}$ is not essential in $R$. Select $c_i \in R$ with $c_i R \cap \text{ann}_R(b_i) = 0$. Then, $U_i$ contains a submodule $V_i \cong c_i R$. Since $R$ is a right p.p.-ring, $V_i$ is projective. Hence, $M$ contains the essential projective submodule $V_1 \oplus \ldots \oplus V_m$. By c), $M$ is projective. \[\Box\]

A submodule $U$ of a module $M$ is tight if both $U$ and $M/U$ have projective dimension at most 1. A module is coherent if all its finitely generated submodules are finitely presented.

**Corollary 3.2.** Let $R$ be a right semi-hereditary ring of finite right Goldie-dimension such that $Q^n$ is a perfect localization of $R$.

a) A right $R$-module $M$ of projective dimension at most 1 is coherent. Moreover, all its finitely generated submodules are tight.

b) If $M$ is singular and a direct sum of countably generated modules, then $p.d. M \leq 1$ if and only if $M \subseteq (Q^n/R)^{(I)}$ for some index-set $I$.

**Proof.** a) Write $M \cong F/P$ where $F$ and its submodule $P$ are projective. If $U$ is a finitely generated submodule of $M$, then there is a submodule $W$ of $F$ containing $P$ with $W/P \cong U$. By Theorem 3.1, $W$ is projective and $W/P$ is finitely presented. Clearly, $U$ and $M/U$ have projective dimension at most 1.

b) Without loss of generality, we may assume that $M$ is countably generated. If $p.d. M \leq 1$, then $M = F/P$ where $F$ is projective and $P \cong R^{(I)}$ for some index-set $I$. Since $P$ is essential in $F$, we have $F \subseteq E(P) \cong (Q^n)^{(I)}$
by [11, Proposition XIII.3.3] because $R$ has finite right Goldie-dimension. Hence, $M \subseteq (Q^r/R)^{(f)}$.

Conversely, suppose that $M \subseteq (Q^r/R)^{(\omega)}$, and select a submodule $U$ of $(Q^r)^{(\omega)}$ containing $R^{(\omega)}$ such that $M = U/R^{(\omega)}$. Choose $\{u_n \mid n < \omega\} \subseteq U$ such that $U = \sum_{n < \omega} u_n R + R^{(\omega)}$ and $u_0 = 0$. Set $V_\ell = R^{(\omega)} + \sum_{n=1}^{\ell} u_n R$. By Theorem 3.1, each $V_\ell$ is projective. Let $W_\ell = V_\ell/R^{(\omega)} \subseteq M$. Then, $W_0 = 0$ and $M = \cup_{n=1}^{\omega} W_n$. Observe that $W_{\ell+1}/W_\ell \cong V_{\ell+1}/V_\ell$ has projective dimension at most 1. By Auslander’s Theorem, $p.d.M \leq 1$. 

$\square$


The first result describes the right strongly non-singular, right Noetherian, right hereditary rings.

PROPOSITION 4.1. The following conditions are equivalent for a right non-singular ring $R$ of finite right Goldie dimension:

a) $R$ is a right strongly non-singular, right hereditary ring without an infinite set of orthogonal idempotents.

b) $R$ is a right strongly non-singular, right Noetherian and right hereditary.

c) i) $R$ has finite right Goldie dimension.

ii) $M$ is pure projective if and only if $M/Z(M)$ is projective, and $Z(M)$ is a direct summand of a direct sum of finitely generated modules.

PROOF. a) $\Rightarrow$ b): By [3, Theorems 3.7 and 4.2], $R$ has finite right Goldie dimension. However, essentially finitely generated projective modules are finitely generated [5, Proposition 8.24]. b) $\Rightarrow$ c) is obvious in view of Theorem 2.3.

c) $\Rightarrow$ a): Let $I$ be a right ideal of $R$, and $J$ its $S$-closure in $R$. Since $R$ has finite right Goldie dimension, $I$ contains a finitely generated right ideal $K$ as an essential submodule. Thus, $J$ is the $S$-closure of $K$, and $J/K$ is the singular submodule of the finitely presented module $R/K$. By c), $R/J$ is projective, and $R = J \oplus J_1$. Then, $R/I \cong J/I \oplus J_1$. In particular, $J/I$ is a finitely generated singular module, which is pure-projective by c). Hence, $J/I$ is a direct summand of a direct sum of finitely presented modules. Clearly, this sum can be chosen to be finite. Therefore, $J/I$ is finitely presented, and $I$ is finitely generated since $J$ is a direct summand of $R$. Once we have shown that every finitely generated non-singular right $R$-
module $M$ is projective, we will have established that $R$ is a right hereditary ring with the property that $Q^r$ is a perfect left localization of $R$.

There exists a finitely generated free module $F$ and a submodule $U$ of $F$ such that $M \cong F/U$. Since $R$ has finite right Goldie-dimension, $U$ contains a finitely generated essential submodule $V$. Because, $F/U$ is non-singular, $U/V$ is the singular submodule of the finitely presented module $F/V$. By c), $F/U \cong (F/V)/(U/V)$ is projective. □

**Corollary 4.2.** The following are equivalent for a right non-singular ring $R$ with finite right Goldie-dimension:

a) $R$ is a right Noetherian, right hereditary ring which satisfies the restricted right minimum condition such that $Q^r$ is a perfect left localization of $R$.

b) $M$ is pure projective if and only if $M/Z(M)$ is projective and $Z(M)$ is a direct summand of a direct sum of finitely generated Artinian modules.

**Proof.** a) $\Rightarrow$ b): Since $R$ has the restricted minimum condition, every finitely generated singular right module is Artinian.

b) $\Rightarrow$ a): Let $I$ be an essential right ideal of $R$. Since $R$ has finite right Goldie-dimension, $I$ contains a finitely generated essential right ideal $J$. By c), the finitely presented module $R/J$ is a direct summand of a (finite) direct sum of finitely generated Artinian modules. But this is only possible if $R/J$ is Artinian. But then, $R/I$ is Artinian too. Arguing as in the proof of Proposition 4.1, we obtain that $R$ is a right strongly non-singular, right hereditary.

By [8, Proposition 5.27], a right hereditary, right Noetherian ring which is the product of prime rings and rings Morita equivalent to lower triangular matrix rings over a division algebra is right strongly non-singular and has the restricted right minimum condition, i.e. $R/I$ is Artinian for every essential right ideal $I$ of $R$.

Let $U$ be a subset of $Q^r$, and set $(R: U)_r = \{q \in Q^r \mid Uq \subseteq R\}$ and $(R: U)_l = \{q \in Q^r \mid qU \subseteq R\}$.

**Theorem 4.3.** The following are equivalent for a ring $R$:

a) $R$ is a right Noetherian right hereditary ring satisfying the restricted right minimum condition such that $Q^r$ is a perfect left localization of $R$. 
b) $R$ is a left Noetherian left hereditary ring satisfying the restricted left minimum condition such that $Q^r$ is a perfect right localization of $R$.

c) $R$ is a right and left Noetherian, hereditary, right and left Utumi-ring.

d) $R = R_1 \times \ldots \times R_n$ where each $R_i$ is either a prime right and left Noetherian hereditary ring or Morita equivalent to a lower triangular matrix ring over a division algebra.

Proof. a) $\Rightarrow$ c): By [3, Theorem 4.2], $Q^r = Q^l$ is a semi-simple Artinian ring, and $R$ is right and left Utumi. Furthermore, $R$ is left semi-hereditary by [3, Theorem 5.2]. It remains to show that $R$ is left Noetherian.

Suppose that $R$ contains a left ideal $I$ which is not finitely generated. Without loss of generality, we may assume that $I$ is essential in $R$. Since $Q^r = Q^l$ is semi-simple Artinian, $R$ has finite left Goldie-dimension [11, Theorem XII.2.5], and $I$ contains a finitely generated essential left ideal $J_0$. Since $I$ is not finitely generated, we can find an ascending chain $J_0 \subseteq \ldots \subseteq J_n \subseteq \ldots$ of finitely generated essential left ideals inside $I$ with $J_n \neq J_{n+1}$ for all $n$.

Since $Q^r$ is an injective left $R$-module being the maximal left ring of quotients of $R$, every map $\phi : J_i \to Q^r$ is right multiplication by some $q \in Q^r$, which is uniquely determined by $\phi$ since $J_i$ is essential. Therefore, we can identify $\text{Hom}_R(J_i, R)$ and $J_i = (R : J_i)_r$. Moreover, $J_i^*$ is a finitely generated projective right $R$-module because $J_i$ is projective since $R$ is left semi-hereditary. Furthermore, $J_i^{**} = (R : J_i^*)_r$ satisfies $J_i^{**} = J_i$. To see this, observe that $J_i \subseteq J_i^{**}$ by definition. Conversely, $J_i$ has a finite projective basis since it is finitely generated and projective. There are $a_1, \ldots, a_k \in J_i$ and $q_1, \ldots, q_k \in J_i^*$ such that $y = yq_1a_1 + \ldots + yq_ka_k$ for all $y \in J_i$. Since $Q^r$ also is the maximal left ring of quotients of $R$, it is non-singular as a left $R$-module. Because $J_i$ is an essential left ideal, $1 = q_1a_1 + \ldots + q_ka_k$. If $z \in J_i^{**}$, then $zq_i \in R$, and $z = (zq_1)a_1 + \ldots + (zq_k)a_k \in J_i$.

We obtain a descending chain $J_0^* \supseteq \ldots \supseteq J_n^* \supseteq \ldots \supseteq R^* = R$ of finitely generated submodules of $Q^r_R$. Since $Q^r/R$ is singular, $J_0^*/R$ is a finitely generated singular right $R$-module. By the restricted right minimum condition for $R$, $J_0^*/R$ is Artinian. Consequently, there is $m$ with $J_m = J_{m+1}^*$, from which one obtains $J_m = J_{m+1}$.

c) $\Rightarrow$ d): By [3, Theorem 5.2], the classes of torsion-free, non-singular and flat right $R$-modules coincide. Because of [3, Theorem 5.5], $R$ has the desired form.
\(d \Rightarrow a\): By [3, Theorem 5.5], \(R\) is a right and left Noetherian hereditary ring for which the classes of torsion-free, flat and non-singular modules coincide. Thus, \(Q'\) is a perfect left localization of \(R\) by [3, Theorem 5.2]. Finally, \(R\) satisfies the restricted right minimum condition by [5].

Since condition \(c\) is right-left symmetric, the equivalence of \(a\) and \(b\) follows immediately. \(\square\)

A submodule \(U\) of the right \(R\)-module \(Q'\) is right invertible if there exist \(u_1, \ldots, u_n \in U\) and \(q_1, \ldots, q_n \in (R : U)_{\ell}\) with \(u_1q_1 + \ldots + u_nq_n = 1\) [11, Chapters ii.4 and IX.5]. Over an integral domain \(R\), a right invertible module \(U\) has the additional property that \(U(R : U)_{\ell} = R\) which may fail in the non-commutative setting:

**Example 4.4.** There exists a right and left Artinian, hereditary ring \(R\) such that \(Q'\) is a perfect right and left localization of \(R\), which contains an essential right ideal \(I\) with \((R : I)_{\ell} = Q'\) and \((R : I)I = I\). Moreover, there exist \(r_1, r_2, s_1, s_2 \in I\) and \(q_1, q_2, p_1, p_2 \in (R : I)_{\ell}\) satisfying \(1 = r_1p_1 + r_2p_2 = s_1q_1 + s_2q_2\) such that \(r_1p_1, r_2p_2 \in R\), but \(s_1q_1, s_2q_2 \notin R\). Thus, \(I(R : I)_{\ell} \not\subseteq R\).

**Proof.** Let \(F\) be a field of characteristic different from 2, and \(R\) be the lower triangular matrix ring over \(F\). Clearly, \(R\) is a right and left Artinian ring. According to [8, Theorem 4.7], \(R\) is a right hereditary ring, which is also left hereditary by [5, Corollary 8.18]. Finally, by [8, Proposition 2.28 and Theorem 2.30], the maximal right and left ring of quotients of \(R\) is \(Q = \text{Mat}_2(F)\). Inside \(R\), we consider the idempotents \(e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\) and \(e_2 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}\). Let \(I = \begin{pmatrix} F & 0 \\ F & 0 \end{pmatrix}\), a two-sided ideal of \(R\) which is essential as a right ideal because \(\begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix}\) for all \(a, b, c \in F\). Since \(I = Q'e_1\), we have \((R : I)_{\ell} = Q'\) and \((R : I)I = I\).

Finally, consider the elements \(s_1 = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}\) and \(s_2 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}\) of \(I\) and \(q_1 = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\) and \(q_2 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\) of \(Q\). It is easy to see that \(s_1q_1, s_2q_2 \notin R\) although \(s_1q_1 + s_2q_2 = 1\). On the other hand, setting \(r_1 = p_1 = e_1 \in I\), \(r_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in I\), and \(p_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\) yields \(r_1p_1, r_2p_2 \in R\) and \(r_1p_1 + r_2p_2 = 1\). \(\square\)
In view of the previous example, we define a submodule $U$ of $Q^r_R$ to be \textit{strongly invertible} if there is a submodule $M$ of $RQ^r$ such that $MU = UM = R$.

**Lemma 4.5.** Let $R$ be a right and left non-singular, right and left Utumi-ring with maximal right and left ring of quotients $Q$. A submodule $U$ of $Q^r$ is strongly invertible if and only if it satisfies the following conditions:

i) $U$ is also a submodule of $RQ$.

ii) $U_R$ is a finitely generated projective generator of $\mathcal{M}_R$.

iii) $RU$ is a finitely generated projective generator of $R\mathcal{M}$.

**Proof.** Suppose that $U$ is strongly invertible, and choose a submodule $X$ of $RQ^r$ with $XU = UX = R$. Then, $X \subseteq (R : U)_r \cap (R : U)_l$ and $(R : U)_l U = R = U(R : U)_r$. Moreover, $RU = (UX)U = U(XU) = UR = U$ yields that $U$ is a submodule of $RQ^r$ too. By symmetry, $X$ is also a submodule of $Q^r_R$. Because of this, $(R : U)_l$ and $(R : U)_r$ are submodules of both $Q^r_R$ and $RQ^r$ too. Therefore, $U(R : U)_l = UR = U(R : U)_r U(R : U)_r = U(R : U)_r = R$. By symmetry, $(R : U)_r U = R$.

By what has been shown in the last paragraph, we can write $1 = u_1q_1 + \ldots + u_nq_n$ with $u_1, \ldots, u_n \in U$ and $q_1, \ldots, q_n \in (R : U)_l$. Let $\phi_i : U \to R$ be left multiplication by $q_i$. As in [11, Proposition IX.5.2], the set $\{(u_1, \phi_1), \ldots, (u_n, \phi_n)\}$ is a projective basis for $U$. Select $v_1, \ldots, v_m \in U$ and $p_1, \ldots, p_m \in (R : U)_l$ with $1 = p_1v_1 + \ldots + p_mv_m$. Define $\psi : U^m \to R$ by $\psi(x_1, \ldots, x_m) = \sum_{i=1}^{m} p_i x_i$. Then, $\psi$ is onto, and $U^m = R \oplus W$, i.e. $U$ is a generator of $\mathcal{M}_R$. By symmetry, $RU$ is a finitely generated projective generator of $R\mathcal{M}$.

Conversely, assume that $U$ satisfies the three conditions. Observe that $(R : U)_l$ and $(R : U)_r$ are submodules of both $Q^r_R$ and $RQ^r$. Since it is a projective generator of $\mathcal{M}_R$, there is $\ell < \omega$ such that $U^\ell = R \oplus W$. Let $\pi : U^\ell \to R$ be a projection with kernel $W$, and $\delta_j : U \to U^\ell$ be the embedding into the $j^{th}$-coordinate. The map $\pi \delta_j : U \to R$ is left multiplication by some $q_j \in Q^r$ since $Q^r$ is a right self-injective ring. Clearly, since $\pi$ is onto, there are $u_1, \ldots, u_\ell \in U$ with $q_1u_1 + \ldots + q_\ell u_\ell = 1$. Since $q_1, \ldots, q_\ell \in (R : U)_l$, we have $(R : U)_l U = R$. By symmetry, $U(U(R : U)_r = R$. Now, $U(R : U)_l = U(R : U)_U(R : U)_r = R$ yields $(R : U)_l \subseteq (R : U)_r$. In the same way, $(R : U)_l = (R : U)_r$, and $U$ is strongly invertible. \qed
Proposition 4.6. Let $R$ be a right and left non-singular, right and left Utumi-ring. If $I$ is a two-sided ideal of $R$ such that $R^I$ and $I_R$ are finitely generated projective generators of $R\mathcal{M}$ and $\mathcal{M}_R$ respectively, then $R/I$ is projective with respect to all RD-exact sequences.

Proof. The proof of [7, Lemmas 1.7.2 and 1.7.4] can be adapted to show that $R/I$ is projective with respect to all RD-exact sequences of $R$-modules provided there are $r_1, \ldots, r_n \in R$ and $q_1, \ldots, q_n \in (R : I)_I$ such that $r_1q_1 + \ldots + r_nq_n = 1$ and $r_1q_1, \ldots, r_nq_n \in R$. However, this is guaranteed by Lemma 4.5.

Finally, the lattice of finitely generated right ideals over right strongly nonsingular, hereditary right and left Noetherian rings may not be distributive:

Example 4.7. There exists a right strongly non-singular, hereditary, right and left Noetherian ring $R$ for which the lattice of right ideals is not distributive.

Proof. Let $R$ be the ring considered in Example 4.4, whose notation will be used in the following. Consider the right ideals $J = e_1R = \begin{pmatrix} Q & 0 \\ 0 & 0 \end{pmatrix}$ and $K = e_2R = \begin{pmatrix} 0 & 0 \\ Q & Q \end{pmatrix}$. If $I = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} Q & 0 \\ Q & Q \end{pmatrix} = \{ \begin{pmatrix} a & 0 \\ a & 0 \end{pmatrix} \mid a \in F \}$, then $I \cap J = I \cap K = 0$, while $I \cap (J + K) = I$. □

References


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