A Threefold with $p_g = 0$ and $P_2 = 2$

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Abstract - We construct a nonsingular threefold $X$ with $q_1 = q_2 = p_g = 0$ and $P_2 = 2$ whose $m$-canonical transformation $\varphi_{[mK_X]}$ has the following properties

i) $\varphi_{[mK_X]}$ has the generic fiber of dimension $\geq 1$, for $2 \leq m \leq 5$;

ii) it is generically a transformation $2 : 1$, for $6 \leq m \leq 8$ and $m = 10$;

iii) it is birational for $m = 9$ and $m \geq 11$.

So, we have a gap for $m = 10$ in the birationality of $\varphi_{[mK_X]}$.

Introduction.

In the classification of nonsingular varieties $X$ of general type, the $m$-canonical tranformation $\varphi_{[mK_X]}$, where $K_X$ is a canonical divisor on $X$, plays an important part. The main problem concerning $\varphi_{[mK_X]}$ regards its birationality. The property of $\varphi_{[mK_X]}$ to have the generic fiber given by a finite set of points is important too.

In the case where $X$ is a threefold, Meng Chen has given several limitations for the birationality of $\varphi_{[mK_X]}$. In the particular case where $X$ has the geometric genus $p_g \geq 2$, Chen ([Che$_2$], [Che$_3$] ) proved that:

- if $p_g \geq 4$, then $\varphi_{[mK_X]}$ is birational for $m \geq 5$;
- if $p_g = 3$, then $\varphi_{[mK_X]}$ is birational for $m \geq 6$;
- if $p_g = 2$, then $\varphi_{[mK_X]}$ is birational for $m \geq 8$.

Such limitations are optimal, as demonstrated by examples costructed by Chen himself [Che$_2$] if $p_g \geq 4$, by S. Chiaruttini - R. Gattazzo ([CG]) if $p_g = 3$, by S. Chiaruttini ([Ch$\hat{i}$]) and by C. Hacon, considering an example of M. Reid [Re], if $p_g = 2$ (see [Che$_3$]).

In the case of $p_g = 1$ and $p_g = 0$, we have only partial results and the

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problem of finding an optimal limitation for the birationality of $\varphi_{[mK_X]}$ remains ([Che1]). If $p_g = 1$ and the bigenus of $X$ is $P_2 = 2$, then a Chen-Zuo’s limitation ([CZ]) states that $\varphi_{[mK_X]}$ is birational for $m \geq 11$. We constructed ([S1]) a threefold $X$ with $q_1 = q_2 = 0$ (where $q_1$ and $q_2$ are the first and second irregularities of $X$) $p_g = 1$ and $P_2 = 2$ such that $\varphi_{[mK_X]}$ is birational if and only if $m \geq 11$, (cf. also $X_{22}$ in [Re], p. 359, and [F]); so the above limitation is optimal.

As for threefolds with $p_g = 0$, we tried to find examples of $X$ with $q_1 = q_2 = 0$, $P_2 = 2$ and with the birationality of $\varphi_{[mK_X]}$ for $m$ large. The results obtained were worse than expected as regards the birationality of $\varphi_{[mK_X]}$, while an interesting result emerged for the gaps in the birationality of $\varphi_{[mK_X]}$. Having obtained the birationality of $\varphi_{[mK_X]}$ if and only if $m \geq 11$ in the case of $p_g = 1$ and $P_2 = 2$, the expected result in the new case of $p_g = 0$ and $P_2 = 2$ is birationality if and only if $m > 11$. Instead, all our constructions of threefolds $X$ with $q_1 = q_2 = p_g = 0$ and $P_2 = 2$ have the 9-canonical transformation $\varphi_{[9K_X]}$, which is birational, but some of them also have $\varphi_{[10K_X]}$, which is not birational, and $\varphi_{[mK_X]}$, which is birational if and only if $m = 9$ and $m \geq 11$.

So, the threefolds with this property have a gap in the birationality of $\varphi_{[mK_X]}$ for $m = 10$. This came as a surprise because the only cases of gaps in the birationality of $\varphi_{[mK_X]}$ that we found were in threefolds with $q_1 = q_2 = p_g = P_2 = P_3 = 0$ or $q_1 = q_2 = p_g = P_2 = 0$. Such examples with gaps are in [S3], where an example is constructed with the same properties as the example $X_{46}$ in Reid’s list ([Re]), and in [Ro2].

In the present paper, we construct a threefold $X$ with the properties described – i.e. $\varphi_{[mK_X]}$ is birational if and only if $m = 9$ and $m \geq 11$, $q_1 = q_2 = 0$ and $p_g = 0$, $P_2 = 2$ – and with further plurigenera $P_3 = 2$, $P_4 = P_5 = 4$, $P_6 = P_7 = 8$, $P_8 = 13$, $P_9 = 15$, $P_{10} = 19$, $P_{11} = 22$.

We note that $X$ is birationally distinct from the threefolds appearing in the lists of [Re], pp. 358-359 and [F], pp. 151-154, 169-170, because $X$ has different plurigenera from those of the threefolds in said lists.

The example $X$ is constructed as a desingularization of a degree six hypersurface $V \subset \mathbb{P}^4$ endowed with a singularity at each of the five vertices $A_0, A_1, A_2, A_3$ and $A_4$ of the fundamental pentahedron. The construction is similar to those in [S4]. Precisely, we put a triple point with an infinitely-near double surface at $A_0$ on $V$, we put a triple point with an infinitely-near triple curve at $A_1, A_2, A_3$, and an ordinary 4-ple point at $A_4$. Other unimposed singularities appear on $V$, but they do not affect the birational invariants of $X$.

The ground field $k$ is an algebraically closed field of characteristic zero, which we can assume to be the field of complex numbers.
1. Imposing singularities on a degree six hypersurface $V$ in $\mathbb{P}^4$.

Let $(x_0, x_1, x_2, x_3, x_4)$ be homogeneous coordinates in $\mathbb{P}^4$ and let us indicate as $f_6(X_0, X_1, X_2, X_3, X_4)$ a form (homogeneous polynomial) of degree 6, in the variables $X_0, X_1, X_2, X_3, X_4$, defining a hypersurface $V \subset \mathbb{P}^4$ of degree six. We impose a triple point on $V$ at each of the four vertices $A_0 = (1,0,0,0,0), \ A_1 = (0,1,0,0,0), \ A_2 = (0,0,1,0,0), \ A_3 = (0,0,0,1,0)$ and an ordinary 4-ple (quadruple) point at $A_4 = (0,0,0,0,1)$ of the fundamental pentahedron $X_0X_1X_2X_3X_4 = 0$.

The equation for $V$, with the imposed singularities, is of the following type

$$V : f_6(X_0, X_1, X_2, X_3, X_4)$$

$$= X_0^3(a_{33000} X_1^3 + \cdots) + X_0^3(a_{23100} X_0^2 X_2 + \cdots) + X_2^3(\cdots) + X_3^3(\cdots) + X_4^3(\cdots)$$

$$+ a_{22000} X_0^2 X_1^2 X_2^2 + a_{22110} X_0^2 X_1^2 X_2 X_3 + \cdots + a_{00222} X_2^2 X_3^2 X_4^2 = 0,$$

where $a_{ijklm} \in k$ denotes the coefficient of the monomial $X_0^i X_1^j X_2^k X_3^l X_4^m$.

Moreover, we impose a double surface $S_0$ infinitely near $A_0$ in the first neighbourhood. We impose the same double surface $S_0$, which is locally isomorphic to a plane as in $[S_2]$. In addition, we impose a triple curve $C_i$ infinitely near $A_i, \ i = 1, 2, 3$ in the first neighbourhood. $C_i$ is locally isomorphic to a straight line as in $[S_1]$.

As an example, we provide a few details on the realization of the singularity at $A_0$ on $V$. This will also enable a better understanding in the sequel of the computation of the $m$-canonical adjoints to $V$ and of the $m$-genus $P_m$ of a desingularization $X$ of $V$, $\sigma : X \longrightarrow V$ (cf. section 5). Let us consider the affine open set $U_0 \ni A_0$ in $\mathbb{P}^4$ given by $X_0 \neq 0$ of affine coordinates $\left( x = \frac{X_1}{X_0}, y = \frac{X_2}{X_0}, z = \frac{X_3}{X_0}, t = \frac{X_4}{X_0} \right)$. The affine equation of $V \cap U_0$ is given by $f_6(1, x, y, z, t) = 0$.

The affine coordinates of $A_0$ are $(0,0,0,0)$, so the blow-up of $\mathbb{P}^4$ at the point $A_0$ is locally given by the formulas:

$$B_{x_1} : \begin{cases} x = x_1 \\ y = x_1 y_1 \\ z = x_1 z_1 \\ t = x_1 t_1 \end{cases} \quad B_{y_1} : \begin{cases} x = x_2 y_2 \\ y = y_2 \\ z = y_2 z_2 \\ t = y_2 t_2 \end{cases} \quad B_{z_1} : \begin{cases} x = x_3 z_3 \\ y = y_3 z_3 \\ z = z_3 \\ t = z_3 t_3 \end{cases} \quad B_{t_1} : \begin{cases} x = x_4 t_4 \\ y = y_4 t_4 \\ z = z_4 t_4 \\ t = t_4 \end{cases}$$

and we consider $B_{t_4}$. The strict (or proper) transform $V'$ of $V$ with respect to
the local blow-up $B_{t_4}$ has an affine equation given by

$$V' : \frac{1}{t_4^3} f_6(1, x_4 t_4, y_4 t_4, z_4 t_4, t_4) = a_{31200} x_4 y_4^2 + \cdots + a_{00222} y_4^2 z_4^2 t_4 = 0.$$ 

On this threefold $V'$ we impose the plane $S_0 \cap U_0$ given affinely by

$$\begin{align*}
x_4 &= 0 \\
t_4 &= 0
\end{align*}$$

as a singular plane of multiplicity two (i.e. as a double plane). The conditions on the coefficients $a_{ijkl}$, such that $V$ has the double plane $S_0 \cap U_0$ infinitely near $A_0$, are given by

$$\begin{align*}
a_{31200} &= 0 & a_{30210} &= 0 & a_{30012} &= 0 & a_{20220} &= 0 \\
a_{31110} &= 0 & a_{30201} &= 0 & a_{30003} &= 0 & a_{20211} &= 0 \\
a_{31101} &= 0 & a_{30120} &= 0 & a_{20310} &= 0 & a_{20202} &= 0 \\
a_{30120} &= 0 & a_{30111} &= 0 & a_{20301} &= 0 & a_{20121} &= 0 \\
a_{31011} &= 0 & a_{30102} &= 0 & a_{20130} &= 0 & a_{20112} &= 0 \\
a_{31002} &= 0 & a_{30003} &= 0 & a_{20031} &= 0 & a_{20022} &= 0. \\
a_{30300} &= 0 & a_{30021} &= 0
\end{align*}$$

In much the same way as above and precisely as in $[S_1]$, we impose a triple curve $C_i$ infinitely near $A_i$ and in the first neighbourhood, which is locally isomorphic to a straight line, for $i = 1, 2, 3$. Further information on the above singularities can be found in $[S_4]$.

We give the final equation for our hypersurface $V$ after imposing all the above-mentioned singularities. We have chosen several coefficients as equal to zero because they are inessential for the computation of the birational invariants of a desingularization $\sigma : X \to V$ of $V$. The shortest equation with the essential coefficients is

$$V : f_6(X_0, X_1, X_2, X_3, X_4)$$

$$\begin{align*}
&= a_{33000} X_0^3 X_1^3 + a_{32100} X_0^3 X_1^2 X_2 + a_{32001} X_0^3 X_2^2 X_3 + a_{23010} X_0^2 X_1^2 X_3^2 + a_{13020} X_0 X_1 X_2 X_3^3 \\
&+ a_{10302} X_0 X_2^3 X_4 + a_{03030} X_1^3 X_3 + a_{02031} X_1^2 X_3^3 X_4 + a_{01032} X_1 X_3^2 X_4^2 + a_{22000} X_0^2 X_1^2 X_2^2 \\
&+ a_{22020} X_0 X_1^2 X_3^2 + a_{22002} X_0^2 X_1 X_2 X_4 + a_{21201} X_0^2 X_2 X_3 X_4 + a_{21120} X_0^2 X_1 X_2^2 X_4 + \\
&+ a_{21011} X_0^2 X_1 X_2 X_3^2 + a_{21021} X_0^2 X_1 X_3^2 X_4 + a_{21012} X_0^2 X_2 X_3 X_4^2 + a_{12012} X_0 X_1 X_3^2 X_4^2 \\
&+ a_{02022} X_1^2 X_3^2 X_4 + a_{00222} X_3^2 X_4^2 = 0.
\end{align*}$$

From here on, $V$ denotes this last hypersurface defined by the above form $f_6(X_0, X_1, X_2, X_3, X_4)$ for a generic choice of the parameters $a_{ijkl}$. As a reminder of this generic choice, we sometimes call $V$: the generic $V$.  

2. **Imposed and unimposed singularities of \( V \): the actual singularities.**

We consider the hypersurface \( V \) given at the end of section 1.

New unimposed singularities appear on the (generic) \( V \) close to the singularities imposed on \( V \); they are actual or infinitely-near singularities. We call a singularity on \( V \) actual to distinguish it from those which are infinitely near. We call a singularity of \( V \) unimposed if it does not appear in the list of singularities in section 1.

There are six unimposed actual double (straight) lines on \( V \) given by \( A_0A_2, A_0A_3, A_0A_4, A_1A_2, A_1A_4, A_2A_3 \) and the unimposed double plane cubic

\[
\begin{cases}
X_1 = 0 \\
X_2 = 0 \\
ar_{01032}X_3^2X_4 + ar_{21021}X_0^2X_3 + ar_{21012}X_0^2X_4 = 0.
\end{cases}
\]

The generic \( V \) has no other actual singularities. It follows that the generic \( V \) is reduced, irreducible and normal.

The cubic lies on the plane \( \begin{cases} X_1 = 0 \\
X_2 = 0 \end{cases} \), which is simple on \( V \). The picture of the six double lines is as follows, where the double lines are drawn in bold type.

![Diagram of singularities](image)

3. **The infinitely-near singularities of \( V \).**

In section 2, we described the actual singularities on \( V \); in the present section, we briefly describe the infinitely-near singularities. Here again, new infinitely-near singularities appear on the generic \( V \) alongside the infinitely-near singularities imposed on \( V \). They are only double singular curves and isolated double points, so none of the unimposed singularities (be they actual or otherwise) affect the birational invariants of a desingularization \( \sigma : X \rightarrow V \) of \( V \), such as the irregularities and the plurigenera.
of $X$. This means that, in calculating these invariants, we can assume that there are only the imposed singularities on $V$.

We compute said birational invariants of $X$ using the theory of adjoints and pluricanonical adjoints developed in [S$_1$]. We can apply this theory because the singularities on the hypersurface $V$ satisfy the hypotheses of [S$_1$], i.e. it must be possible to resolve the singularities on $V$ with local blow-ups along linear affine subspaces; moreover, the degree six hypersurfaces in $\mathbb{P}^4$ must have singularities of codimension $\geq 2$ (i.e. the hypersurfaces must be normal).

Such hypotheses on the singularities are satisfied by either actual or infinitely-near singularities of $V$. In particular, $V$ is normal (section 2). To be precise, all the singularities of $V$ are resolved with local blow-ups either along straight lines, that are double on $V$ and on strict transforms of $V$, or along planes containing double curves and points. These planes are simple on $V$ and on strict transforms of $V$, e.g. the simple plane \[
\begin{aligned}
X_1 &= 0 \\
X_2 &= 0
\end{aligned}
\] containing the cubic curve on $V$ in section 2.

Having said as much, we only give details on the imposed infinitely-near singularities of $V$ that are needed in the sequel.

From section 1, we already have the information that we need about the triple point $A_0$ and the double surface $S_0$ infinitely near $A_0$.

Next, we consider the triple point $A_1$ on $V$ and the blow-up at $A_1$. Let us consider the affine open set $U_1 \ni A_1$ in $\mathbb{P}^4$ given by $X_1 \neq 0$ of affine coordinates \[
\left( x = \frac{X_0}{X_1}, \, y = \frac{X_2}{X_1}, \, z = \frac{X_3}{X_1}, \, t = \frac{X_4}{X_1} \right).\]

The affine equations of $V \cap U_1$ are given by $f_6(x, 1, y, z, t) = 0$. The affine coordinates of $A_1$ are $(0, 0, 0, 0, 0)$.

We can assume that the blow-up at $A_1$ is the first to be performed, so we can use the local blows-up $B_{x_1}, B_{y_2}, B_{z_2}, B_{t_4}$ in section 1.

The strict transform of $V \cap U_1$, with respect to $B_{t_4}$, is given by

\[ V_{t_4} : \frac{1}{t_4^3} f_6(x_4 t_4, 1, y_4 t_4, z_4 t_4, t_4) \]
\[ = a_{33000} x_4^3 + \cdots + a_{03030} z_4^3 + \cdots + a_{12012} x_4 z_4 t_4 + \cdots = 0. \]

We are interested in the triple curve infinitely near $A_1$. So, we focus locally on the triple line on $V_{t_4}$ belonging to the exceptional divisor $t_4 = 0$ of the local blow-up $B_{t_4}$. This triple line is given by \[
\begin{aligned}
x_4 &= 0 \\
z_4 &= 0 \\
t_4 &= 0
\end{aligned}
\]

Let us go on to consider the triple point $A_2$ on $V$, the blow-up at $A_2$ and the affine open set $U_2 \ni A_2$ in $\mathbb{P}^4$ given by $X_2 \neq 0$ of affine coordinates
\[
(x = \frac{X_0}{X_2}, y = \frac{X_1}{X_2}, z = \frac{X_3}{X_2}, t = \frac{X_4}{X_2}).
\]

The affine equations of \( V \cap U_2 \) are given by \( f_0(x, y, 1, z, t) = 0 \). The affine coordinates of \( A_2 \) are \((0, 0, 0, 0)\).

Here again, we can assume that the blow-up at \( A_2 \) is the first to be performed, so we can use the local blow-ups \( B_{x_1}, B_{y_2}, B_{z_2}, B_{t_4} \) in section 1.

The strict transform of \( V \cap U_2 \), with respect to \( B_{y_2} \), is given by

\[
V'_{y_2} : \frac{1}{y_2^2} f_0(x_2 y_2, y_2, 1, y_2 z_2, y_2 t_2)
= a_{10302} x_2 t_2^2 + \cdots + a_{22200} x_2^2 y_2 + \cdots + a_{00222} y_2 z_2^2 t_2^2 = 0.
\]

We are interested in the triple curve infinitely near \( A_2 \), so we focus locally on the triple line on \( V'_{y_2} \) belonging to the exceptional divisor \( y_2 = 0 \) of the local blow-up \( B_{y_2} \). This triple line is given by

\[
\begin{cases}
  x_2 = 0 \\
  y_2 = 0 \\
  t_2 = 0
\end{cases}
\]

Finally, let us consider the triple point \( A_3 \) on \( V \), the blow-up at \( A_3 \) and the affine open set \( U_3 \ni A_3 \) in \( \mathbb{P}^4 \) given by \( X_3 \neq 0 \) of affine coordinates

\[
(x = \frac{X_0}{X_3}, y = \frac{X_1}{X_3}, z = \frac{X_2}{X_3}, t = \frac{X_4}{X_3}).
\]

The affine equations of \( V \cap U_3 \) are given by \( f_0(x, y, z, 1, t) = 0 \). The affine coordinates of \( A_3 \) are \((0, 0, 0, 0)\).

We can again assume that the blow-up at \( A_3 \) is the first to be performed, so we can use the local blow-ups \( B_{x_1}, B_{y_2}, B_{z_2}, B_{t_4} \) in section 1.

The strict transform of \( V \cap U_3 \), with respect to \( B_{x_1} \), is given by

\[
V'_{x_1} : \frac{1}{x_1^2} f_0(x_1, x_1 y_1, x_1 z_1, 1, x_1 t_1)
= a_{03030} y_1^3 + \cdots + a_{22020} x_1 y_1^2 + \cdots + a_{21021} x_1 y_1 t_1 + \cdots = 0.
\]

We are interested in the triple curve infinitely near \( A_3 \), so we focus locally on the triple line on \( V'_{x_1} \) belonging to the exceptional divisor \( x_1 = 0 \) of the local blow-up \( B_{x_1} \). This triple line is given by

\[
\begin{cases}
  x_1 = 0 \\
  y_1 = 0 \\
  t_1 = 0
\end{cases}
\]

To end this section, we add one more item of information, drawing the picture of the tree of local blow-ups resolving the singularity at \( A_0 \) and those infinitely near.
where “ns” means “nonsingular”.

4. The m-canonical adjoints to $V \subset P^4$.

Let

$$P_r \xrightarrow{\pi_r} \cdots \xrightarrow{\pi_3} P_2 \xrightarrow{\pi_2} P_1 \xrightarrow{\pi_1} P_0 = P^4$$

be a sequence of blow-ups solving the singularities of $V$.

If we call $V_i \subset P_i$ the strict transform of $V_{i-1}$ with respect to $\pi_i$, then the above sequence gives us

$$X = V_r \xrightarrow{\pi'_r} \cdots \xrightarrow{\pi'_3} V_2 \xrightarrow{\pi'_2} V_1 \xrightarrow{\pi'_1} V_0 = V,$$

where $\pi'_j = \pi_j|_{V_i} : V_i \to V_{i-1}$ and $\sigma_{i,j} : X \to V$, $\sigma = \pi_r \circ \cdots \circ \pi_1$, is a desingularization of $V \subset P^4$. 
Let us assume that \( \pi_i \) is a blow-up along a subvariety \( Y_{i-1} \) of \( \mathbb{P}_{i-1} \), of dimension \( j_{i-1} \), which can be either a singular or a nonsingular subvariety of \( V_{i-1} \subset \mathbb{P}_{i-1} \) (i.e. \( Y_{i-1} \) is a locus of singular or simple points of \( V_{i-1} \)). Let \( m_{i-1} \) be the multiplicity of the variety \( Y_{i-1} \) on \( V_{i-1} \).

Let us set \( n_{i-1} = -3 + j_{i-1} + m_{i-1} \), for \( i = 1, ..., r \) and \( \deg(V) = d \).

A hypersurface \( \Phi_{m(d-5)} \) of degree \( m(d-5) \), \( m \geq 1 \), in \( \mathbb{P}^4 \) is an \textit{m-canononical adjoint} to \( V \) (with respect to the sequence of blow-ups \( \pi_1, ..., \pi_r \)) if the restriction to \( X \) of the divisor

\[
D_m = \pi_r^* \{ \pi_{r-1}^* \cdots \pi_1^* (\Phi_{m(d-5)}) - mn_0 E_1 \cdots \} - mn_{r-2} E_{r-1} - mn_{r-1} E_r
\]

is effective, i.e. \( D_{m|_X} \geq 0 \), where \( E_i = \pi^{-1}(Y_{i-1}) \) is the exceptional divisor of \( \pi_i \) and \( \pi_i^* : \text{Div}(\mathbb{P}_{i-1}) \to \text{Div}(\mathbb{P}_i) \) is the homomorphism of the Cartier (or locally principal) divisor groups (cf. [S1], sections 1,2).

An \textit{m-canononical adjoint} \( \Phi_{m(d-5)} \) is an \textit{global m-canononical adjoint} to \( V \) (with respect to \( \pi_1, ..., \pi_r \)) if the divisor \( D_m \) is effective on \( \mathbb{P}_r \), i.e. \( D_m \geq 0 \) (loc. cit.).

Note that, if \( \Phi_{m(d-5)} \) is an \( m \)-canonical adjoint to \( V \), then \( D_{m|_X} \equiv mK \), where ‘\( \equiv \)’ denotes linear equivalence and \( K \) denotes a canonical divisor on \( X \).

In our above example, an order can be established in the sequence of blow-ups, e.g. let us assume that \( \pi_1 \) is the blow-up at the triple point \( A_0 \), \( \pi_2 \) is the blow-up along the double surface \( S_0 \) infinitely near \( A_0 \), \( \pi_3 \) is the blow-up at the triple point \( A_1 \), \( \pi_4 \) is the blow-up along the triple curve \( C_1 \) infinitely near \( A_1 \), \( \pi_5 \) is the blow-up at the triple point \( A_2 \), \( \pi_6 \) is the blow-up along the triple curve \( C_2 \) infinitely near \( A_2 \), \( \pi_7 \) is the blow-up at the triple point \( A_3 \), \( \pi_8 \) is the blow-up along the triple curve \( C_3 \) infinitely near \( A_3 \) and the blow-up \( \pi_9 \) is the one at the 4-ple point \( A_4 \).

The example \( V \) has degree \( d = 6 \) and \( D_m \), relative to our \( X \), is given by:

\[
(*) \quad D_m = \pi_r^* \cdots \pi_3^* \{ \pi_2^* (\Phi_m) - mE_2 \} - mE_4 - mE_6 - mE_8 - mE_9,
\]

where \( E_i \) is the exceptional divisor of the blow-up \( \pi_i \) and, to be more specific, \( E_1 \) is the exceptional divisor of the blow-up \( \pi_1 \) at the triple point \( A_0 \), \( E_2 \) is the exceptional divisor of the blow-up \( \pi_2 \) along \( C_1 \), ... and \( E_9 \) is the exceptional divisor of the blow-up \( \pi_9 \) at the 4-ple point \( A_4 \).

No other exceptional divisors are subtracted in \( D_m \) because, as we said before, the unimposed singularities are either actual or infinitely-near double singular curves or isolated double points on our (generic) \( V \). Put more precisely, the exceptional divisors of the blow-ups along the double curves appear with coefficient \( n_i = 0 \) in the above expression of \( D_m \) and the exceptional divisors of the blow-ups along simple planes appear again with
coefficient \( n_j = 0 \). Since we have resolved all the unimproved singularities with blow-ups either along double curves or along simple planes, only the exceptional divisors \( E_2, E_4, E_6, E_8 \) and \( E_9 \) appear in \( D_m \). Note, moreover, that the exceptional divisor of a blow-up at a triple point also appears with coefficient \( n_h = 0 \) in \( D_m \).

5. The plurigenera of a desingularization \( X \) of \( V \).

Let us consider the equation of \( V: f_6(X_0, X_1, X_2, X_3, X_4) = 0 \) at the end of section 1 and arrange the form \( f_6 \) according to the powers of \( X_4 \).

\[ (** \quad f_6 = \psi_4(X_0, X_1, X_2, X_3)X_4^2 + \psi_5(X_0, X_1, X_2, X_3)X_4 + \psi_6(X_0, X_1, X_2, X_3), \]

where \( \psi_i(X_0, X_1, X_2, X_3) \) is a form of degree \( i \) in \( X_0, X_1, X_2, X_3 \) and precisely

\[ \psi_4(X_0, X_1, X_2, X_3) = a_{1032}X_0X_2^3 + a_{01032}X_1X_3^3 + a_{22002}X_0^2X_1^2 + \cdots + a_{00222}X_2^2X_3^2. \]

Next, let us consider the hypersurface \( \Phi_m \), appearing in (***) section 4 and assume that its equation is \( F_m(X_0, X_1, X_2, X_3, X_4) = 0, \) of degree \( m \). Arranging the form \( F_m \) according to the powers of \( X_4 \), we can write

\[ (***) \quad F_m(X_0, X_1, X_2, X_3, X_4) = \psi_s(X_0, X_1, X_2, X_3)X_4^{m-s} + \psi_{s+1}(X_0, X_1, X_2, X_3)X_4^{m-s-1} + \cdots + \psi_m(X_0, X_1, X_2, X_3), \]

where \( \psi_j(X_0, X_1, X_2, X_3) \) is a form of degree \( j \) in \( X_0, X_1, X_2, X_3 \) and \( s \) is an integer satisfying \( 0 \leq s \leq m \).

Under the sole hypothesis that \( V \) has a 4-ple point at \( A_4 \) the following lemma holds.

**Lemma 1.** *With the above notations, if \( \Phi_m \) is an m-canonical adjoint (be it global or not), then, modulo \( V: f_6 = 0 \), we can assume that \( s \geq m - 1 \) in (***) ; i.e. if \( \Phi_m: F_m = 0 \) is an m-canonical adjoint, then we can assume that

\[ F_m = \psi_{m-1}(X_0, X_1, X_2, X_3)X_4 + \psi_m(X_0, X_1, X_2, X_3). \]

Moreover, we have the equality

\[ \psi_{m-1}(X_0, X_1, X_2, X_3) = A_{m-5}(X_0, X_1, X_2, X_3)\psi_4(X_0, X_1, X_2, X_3), \]

where \( A_{m-5}(X_0, X_1, X_2, X_3) \) is a form of degree \( m - 5 \) in \( X_0, X_1, X_2, X_3 \) and \( \psi_4(X_0, X_1, X_2, X_3) \) is defined above in (**).
The idea for the proof of the above lemma came from M. C. Ronconi [CR], [Ro1]. A detailed proof can be found in [S4] (Lemma 1, section 5).

**Remark 1.** In Lemma 1, we have $F_m = A_{m-5} \varphi_4 X_4 + \psi_m$. We see that, if $A_{m-5} = 0$, then $F_m$ defines a global $m$-canonical adjoint $\Phi_m$ to $V$, whereas if $A_{m-5} \neq 0$, then $\Phi_m$ is a “non-global” $m$-canonical adjoint to $V$. The non-global $m$-canonical adjoints to $V$ are important for establishing the birationality of the $m$-canonical transformation $\varphi_{[mK_X]}$ (see next section).

The following lemma is proved in [S4], Lemma 2, section 12, where the singularities at three fundamental points on a degree six hypersurface $V'$ differ from those on $V$ in the present case. More precisely, $V'$ has three triple points with an infinitely-near double plane, whereas $V$ has three triple points with an infinitely-near triple curve. But the proof remains the same in both cases.

**Lemma 2.** The $m$-canonical adjoint to $V$ given by

$$\Phi_m : A_{m-5}(X_0, X_1, X_2, X_3)\varphi_4(X_0, X_1, X_2, X_3)X_4 + \psi_m(X_0, X_1, X_2, X_3, X_4) = 0,$$

has the following property

$$D_{m|X} \geq 0 \iff D_m + E_9 \geq 0,$$

where $D_m = \pi^*_r \cdots \pi^*_3(\pi^*_2(\pi^*_1(\Phi_m)) - mE_2) - mE_4 - mE_6 - mE_8 - mE_9$, is defined in (*), section 4.

**Remark 2.** Roughly speaking, the result in Lemmas 1 and 2, that permits us an easy computation of the $m$-genus $P_m$ ($\forall m$) of a desingularization $\sigma : X \longrightarrow V$ of $V$, is the following. Our degree six hypersurface $V$ has a 4-ple point, so from Lemma 1 we can assume that the $m$-canonical adjoint $\Phi_m$ is defined by a form of the type $F_m = A_{m-5} \varphi_4 X_4 + \psi_m$, where the variable $X_4$ appears to the power 1. In order to compute the linear conditions given by the other singularities to the hypersurfaces $\Phi_m$ so that they are $m$-canonical adjoints to $V$, i.e. to obtain $D_{m|X} \geq 0$, we find that we do not need to restrict $D_m$ to $X$ and, after imposing $D_{m|X} \geq 0$, we only need to have $D_m + E_9 \geq 0$. This follows from the fact that $F_m$ contains the variable $X_4$ to the power 2, whereas the form $f_6$ defining $V$ contains the variable $X_4$ to the power 2, and also from the particular singularities obtained in our examples. We note that $E_9$ has to be added to $D_m$, otherwise $D_m$ may not be effective (when $A_{m-5} \neq 0$,}
see Remark 1). So it is very easy to compute the conditions on \( F_m \) such that \( D_m + E_9 \) is effective and, since \( P_m = \) number of linearly independent forms contained in \( F_m \) (cf. \([S_1]\)), the computation of \( P_m, \forall m \), is very easy too.

Now, we are ready to compute the plurigenera of a desingularization \( \sigma : X \longrightarrow V \) of \( V \). Let us write

\[
A_{m-5}(X_0, X_1, X_2, X_3)X_4 = \left( \sum_{i+j+k+h=m-5} a_{ijkh} X_i^j Y^k X_3^h \right) X_4,
\]

\[
\psi_m(X_0, X_1, X_2, X_3) = \sum_{i'+j'+h' = m} b_{ijkh} X_0^{i'} X_1^{j'} X_2^{k'} X_3^{h'},
\]

where \( a_{ijkh}, b_{ijkh} \in k \).

**• First let us consider the two blows-up \( \pi_1 \) and \( \pi_2 \).** We know that the blow-up \( \pi_1 \) of \( \mathbb{P}^4 \) at \( A_0 \) is given by \( B_{x_1}, B_{y_2}, B_{z_2}, B_{t_1} \) (cf. section 1). Let us consider the affine open set \( U_0 = \{ X_0 \neq 0 \} \) as in section 1.

The total transform of \( \Phi_m \cap U_0 \) with respect to \( B_{t_4} \) is given by

\[
B_{t_4}^*(\Phi_m \cap U_0) : A_{m-5}(1, x_4 t_4, y_4 t_4, z_4 t_4) \varphi_4(1, x_4 t_4, y_4 t_4, z_4 t_4) t_4 + \psi_m(1, x_4 t_4, y_4 t_4, z_4 t_4, t_4) = 0.
\]

The double surface \( S_0 \) infinitely near \( A_0 \) in affine coordinates \((x_4, y_4, z_4, t_4)\) is given by \( \{ x_4 = 0 \} \) (cf. section 1).

The blow-up \( \pi_2 \) along \( S_0 \) is locally given by the formulas:

\[
B_{x_41} : \begin{cases} x_4 = x_{41} \\ y_4 = y_{41} \\ z_4 = z_{41} \\ t_4 = x_{41} t_{41} \end{cases} ; \quad B_{t_{42}} : \begin{cases} x_4 = x_{42} t_{42} \\ y_4 = y_{42} \\ z_4 = z_{42} \\ t_4 = t_{42} \end{cases}.
\]

The total transform of \( B_{t_4}^*(\Phi_m \cap U_0) \) with respect to \( B_{x_41} \) is given by

\[
B_{x_41}^* [B_{t_4}^*(\Phi_m \cap U_0)] : \quad A_{m-5}(1, x_{41}^2 t_{41}, x_{41} y_{41} t_{41}, x_{41} z_{41} t_{41}) \varphi_4(1, x_{41}^2 t_{41}, x_{41} y_{41} t_{41}, x_{41} z_{41} t_{41}) x_{41} t_{41} + \psi_m(1, x_{41}^2 t_{41}, x_{41} y_{41} t_{41}, x_{41} z_{41} t_{41}, x_{41} t_{41}) = 0.
\]
With the above notations, this total transform is given by

\[ B_{x_{41}}^* [B_{t_4}^*(\Phi_m \cap U_0)] : \]

\[
\sum_{i+j+k+h=m-5} a_{ijkh} x_{41}^{2j+k} y_{41}^k z_{41}^h \varphi_4(1, x_{41}^2 t_{41}, x_{41} y_{41} t_{41}, x_{41} z_{41} t_{41}) x_{41} t_{41} \\
+ \sum_{i'+j'+k'+l'=m} b_{ijkh} x_{41}^{2j'+k'+l'} y_{41}^k z_{41}^h = 0.
\]

The following claims hold true; they are corollaries to Lemma 1 and 2 and consequences of the desingularization of \( V \).

**Claim 1.** The composition of the two local blows-up \( B_{x_{41}} \circ B_{t_4} \) coincides, up to isomorphisms, with the desingularization \( \sigma_{x} \) on the affine open set \( V_{x_{41}} \), because \( V_{x_{41}} \) is nonsingular (see the tree of blow-ups at the end of section 3). In fact, \( V_{x_{41}} \) is isomorphic to an open set on \( X \) and the two above morphisms can be identified on \( V_{x_{41}} \).

**Claim 2.** Since \( \Phi_m \) is an \( m \)-canonical adjoint to \( V \), by definition we have \( D_{m|_x} \geq 0 \); so, from Lemma 2, we can say that: \( D_m + E_9 \geq 0 \).

**Claim 3.** From Claims 1 and 2, we deduce (up to isomorphisms) that

\[ B_{x_{41}}^* [B_{t_4}^*(\Phi_m \cap U_0)] - mE_2 + E_9 \geq 0. \]

This last inequality is equivalent to the following equality of polynomials

\[
\sum_{i+j+k+h=m-5} a_{ijkh} x_{41}^{2j+k} y_{41}^k z_{41}^h \varphi_4(1, x_{41}^2 t_{41}, x_{41} y_{41} t_{41}, x_{41} z_{41} t_{41}) x_{41} t_{41} \\
+ \sum_{i'+j'+k'+l'=m} b_{ijkh} x_{41}^{2j'+k'+l'} y_{41}^k z_{41}^h = x_{41}^m(...)
\]

**Claim 4.** Since \( \varphi_4(1, x_{41}^2 t_{41}, x_{41} y_{41} t_{41}, x_{41} z_{41} t_{41}) = x_{41}^3(...) \), the latter equality of polynomials is equivalent to the inequalities

\[
\begin{align*}
2j + k + h + 3 + 1 & \geq m, \\
2j' + k' + h' & \geq m,
\end{align*}
\]

i.e.

\[
\begin{align*}
j & \geq i + 1, \\
j' & \geq i'.
\end{align*}
\]

**Next, let us consider the two blow-ups** \( \pi_3 \) and \( \pi_4 \). As in section 3, we can assume that the first blow-up that we perform is \( \pi_3 \) at \( A_1 \), so we can use the local blows-up \( B_{x_1}, B_{y_2}, B_{z_3}, B_{t_4} \) in section 1.

As in the above case of \( \pi_1 \) and \( \pi_2 \), here too for \( \pi_3 \) and \( \pi_4 \), we find that the
total transform of \(\Phi_m \cap U_1\) with respect to \(B_{t_4}\) is given by
\[
B^*_{t_4}(\Phi_m \cap U_1) : A_{m-5}(x_4 t_4, 1, y_4 t_4, z_4 t_4) \varphi_4(x_4 t_4, 1, y_4 t_4, z_4 t_4) t_4 \\
+ \psi_m(x_4 t_4, 1, y_4 t_4, z_4 t_4, t_4) = 0.
\]

The triple curve \(C_1\) infinitely near \(A_1\) in affine coordinates \((x_4, y_4, z_4, t_4)\) is given by (section 3)
\[
\begin{align*}
x_4 &= 0 \\
z_4 &= 0 \\
t_4 &= 0
\end{align*}
\]

The blow-up \(\pi_4\) along \(C_1\) is locally given by the formulas:
\[
B_{x_{41}} : \begin{cases} 
  x_4 = x_{41} \\
y_4 = y_{41} \\
z_4 = x_{41} z_{41} \\
t_4 = x_{41} t_{41}
\end{cases}; \quad B_{z_{41}} : \begin{cases} 
  x_4 = x_{42} z_{42} \\
y_4 = y_{42} \\
z_4 = z_{42} \\
t_4 = z_{42} t_{42}
\end{cases}; \quad B_{t_{41}} : \begin{cases} 
  x_4 = x_{43} t_{43} \\
y_4 = y_{43} \\
z_4 = z_{43} t_{43} \\
t_4 = t_{43}
\end{cases}.
\]

The total transform of \(B^*_{t_4}(\Phi_m \cap U_1)\) with respect to \(B_{x_{41}}\) is given by
\[
B^*_{x_{41}}[B^*_{t_4}(\Phi_m \cap U_1)] :
\begin{align*}
&\left(\sum_{i+j+k+h=m-5} a_{ijkh} x_{41}^{2i+k+2h} y_{41}^{k} z_{41}^{h} \right) \varphi_4(x_{41}^2 t_{41}, 1, x_{41} y_{41} t_{41}, x_{41}^2 z_{41} t_{41}) x_{41} t_{41} \\
&+ \sum_{i'+j'+k'+l'=m} b_{ijkh} x_{41}^{2i'+k'+2h'} y_{41}^{k'} z_{41}^{h'} = 0.
\end{align*}
\]

From the analogous four claims written above and from the equality
\[
\varphi_4(x_{41}^2 t_{41}, 1, x_{41} y_{41} t_{41}, x_{41}^2 z_{41} t_{41}) = x_{41}^4(...),
\]
we obtain the inequalities
\[
\begin{align*}
2i + k + 2h + 4 + 1 &\geq m \\
2i' + k' + 2h' &\geq m
\end{align*}
\]
e.g. \(i + h \geq j\)
\(i' + h' \geq j'\).

* * * Let us move on now to consider the two blows-up \(\pi_5\) and \(\pi_6\). Once again, we can assume that the blow-up \(\pi_3\) at \(A_2\) is performed first, so we can again use the local blows-up \(B_{x_1}, B_{y_2}, B_{z_2}, B_{t_4}\) in section 1.

As in the above cases, here for \(\pi_5\) and \(\pi_6\) we obtain that the total transform of \(\Phi_m \cap U_2\), with respect to \(B_{y_2}\), is given by
\[
B^*_{y_2}(\Phi_m \cap U_2) : A_{m-5}(x_2 y_2, 1, y_2 z_2) \varphi_4(x_2 y_2, 1, y_2 z_2) y_2 t_2 \\
+ \psi_m(x_2 y_2, 1, y_2 z_2, y_2 t_2) = 0.
\]
The triple curve $C_2$ infinitely near $A_2$ in affine coordinates $(x_2, y_2, z_2, t_2)$ is given by (section 3) \[
\begin{align*}
x_2 &= 0 \\
y_2 &= 0 \\
t_2 &= 0
\end{align*}
\]
The blow-up $\pi_6$ along $C_2$ is locally given by the formulas:

$$
\begin{align*}
B_{x_21} : & \begin{cases}
x_2 = x_{21} \\
y_2 = x_{21} y_{21} \\
z_2 = z_{21} \\
t_2 = x_{21} t_{21}
\end{cases} & B_{y_2} : & \begin{cases}
x_2 = x_{22} y_{22} \\
y_2 = y_{22} \\
z_2 = z_{22} \\
t_2 = y_{22} t_{22}
\end{cases} & B_{t_2} : & \begin{cases}
x_2 = x_{23} t_{23} \\
y_2 = y_{23} t_{23} \\
z_2 = z_{23} \\
t_2 = t_{23}
\end{cases}
\end{align*}
$$

The total transform of $B^*_y (\Phi_m \cap U_2)$ with respect to $B_{x_21}$ is given by

$$
\left( \sum_{i+j+k+h=m-5} a_{ijkl} x_{21}^{2i+j+h} y_{21}^i z_{21}^j \right) \varphi_4 (x_{21}^2 y_{21}, x_{21} y_{21}, 1, x_{21} z_{21}) + \sum_{i'+j'+h'+l'=m} b_{ijkl} x_{21}^{2i'+j'+h'} y_{21}^i z_{21}^j = 0.
$$

From the same four claims written above, and from the equality

$$
\varphi_4 (x_{21}^2 y_{21}, x_{21} y_{21}, 1, x_{21} z_{21}) = x_{21}^2 (\ldots),
$$

we obtain the inequalities

$$
\begin{align*}
& \begin{cases}
2i + j + h + 2 + 2 \geq m \\
i \geq k + 1
\end{cases}, \quad \text{i.e.} \quad \begin{cases}
i \geq k + 1 \\
i' \geq k'
\end{cases}
\end{align*}
$$

- **Finally, considering the two blows-up $\pi_7$ and $\pi_8$, as in the case of $\pi_5$ and $\pi_6$, we obtain the inequalities**

$$
\begin{align*}
& \begin{cases}
i + 2j + k + 2 + 2 \geq m \\
i' + 2j' + k' \geq m
\end{cases}, \quad \text{i.e.} \quad \begin{cases}
j \geq h + 1 \\
j' \geq h'
\end{cases}
\end{align*}
$$

Joining the above inequalities, we obtain

$$
\begin{align*}
\text{(**)} & \begin{cases}
i + h \geq j \geq i + 1 \geq k + 2, \\
i' + h' \geq j' \geq i' \geq k',
\end{cases} \\
j \geq h + 1 \quad j' \geq h'
\end{align*}
$$

From the inequalities in the first line of (**), we deduce $j \geq 2$, $i \geq 1$, $h \geq 1$. Bearing in mind that $i + j + k + h = m - 5$,

i) there are no values of $i, j, k, h$ satisfying (***) and corresponding to $m$, for $m \leq 8$;

ii) the values $[i = 1, j = 2, k = 0, h = 1]$ correspond to $m = 9;$
iii) there are no values of $i,j,k,h$ satisfying $(**)$ and corresponding to $m = 10$;
   iv) the two sets of values $[i = 2, j = 3, k = 0, h = 1]$ and $[i = 1, j = 3, k = 0, h = 2]$ satisfy $(**)$ and correspond to $m = 11$, and so on; there are values of $i, j, k, h$ satisfying $(**)$ that correspond to any value of $m \geq 12$.

As for the inequalities in the second line of $(**)$, and given that $i' + j' + k' + h' = m$,

1) there are no values of $i', j', k', h'$ satisfying $(**)$ and corresponding to $m = 1$;
2) the two sets of values $[i' = j' = 1, k' = h' = 0]$ and $[j' = h' = 1, i' = k' = 0]$ satisfy $(**)$ and correspond to $m = 2$;
3) the two sets of values $[i' = j' = k' = 1, h' = 0]$ and $[i' = j' = h' = 1, k' = 0]$ satisfy $(**)$ and correspond to $m = 3$;
4) there are 4 sets of values satisfying $(**)$ and corresponding to $m = 4$,
   there are also 4 sets of values satisfying $(**)$ and corresponding to $m = 5$, 8 sets satisfying $(**)$ and corresponding to $m = 6$ and 8 sets satisfying $(**)$ and corresponding to $m = 7$.
5) The following sets $[i' = j' = 3, k' = h' = 0]$, $[i' = j' = h' = 2, k' = 0]$, $[i' = j' = 2, k' = h' = 1]$, $[i' = 2, j' = 3, k' = 0, h' = 1]$ are 4 of the 8 sets of values satisfying $(**)$ that correspond to $m = 6$.
   The following sets $[i' = j' = 3, k' = 1, h' = 0]$, $[i' = h' = 2, j' = 3, k' = 0]$, $[i' = 1, j' = 3, k' = 1, h' = 2]$, $[i' = 1, j' = h' = 3, k' = 0]$ are 4 of the 8 sets of values satisfying $(**)$ that correspond to $m = 7$.

Consequences. Let us just recall that we have written the equation of an $m$-canonical adjoint $\Phi_m$ as follows:

$$
\Phi_m : A_{m-5}(X_0, X_1, X_2, X_3)\varphi_4(X_0, X_1, X_2, X_3)X_4 + \psi_m(X_0, X_1, X_3, X_4) = 0,
$$

where

$$
A_{m-5}(X_0, X_1, X_2, X_3)X_4 = \left( \sum_{i+j+k+h=m-5} a_{ijkl} X_0^{i} X_1^{j} X_2^{k} X_3^{h} \right) X_4
$$

and

$$
\psi_m(X_0, X_1, X_3) = \sum_{i'+j'+k'+h'=m} b_{ijkl} X_0^{i'} X_1^{j'} X_2^{k'} X_3^{h'}.
$$

From i),...,vi), we deduce that the form $A_{m-5}$ is zero if and only if $m \leq 8$ and $m = 10$. 

A threefold with $p_g = 0$ and $P_2 = 2$

Since the $m$-genus $P_m$ of a desingularization $X$ of $V$ is the number of the linearly independent forms defining $m$-canonical adjoints to $V$ (cf. [S1]), from 1), ..., 4), we deduce the following results regarding the plurigenera of a desingularization $X$ of $V$.

From 1), we can establish that there are no 1-canonical adjoints (also called canonical adjoints) to $V$; this implies that the geometric genus of $X$ is $p_g = 0$.

From 2), we find that $\Phi_2 : \psi_2(X_0, X_1, X_3, X_4) = X_1(\lambda_1 X_0 + \lambda_2 X_3) = 0$, where $\lambda_i \in k$; this implies that the bigenus of $X$ is $P_2 = 2$.

From 3), we learn that $\Phi_3 : \psi_3(X_0, X_1, X_3, X_4) = X_0 X_1(\mu_1 X_2 + \mu_2 X_3) = 0$, $\mu_i \in k$; this implies that the trigenus of $X$ is $P_3 = 2$.

From 4), we obtain that $P_4 = P_5 = 4$, $P_6 = 8$ and $P_7 = 8$.

In addition, $X$ has the plurigenera $P_8 = 13$, $P_9 = 15$, $P_{10} = 19$, $P_{11} = 22$.

6. The $m$-canonical transformation $\varphi_{|mK_X|}$, $m \geq 2$.

Let us use $\alpha_m : V \dashrightarrow \mathbb{P}^{P_m-1}$ to indicate the rational transformation associated with the linear system of $m$-canonical adjoints $\Phi_m$ to $V$. The following triangle

\[
\begin{array}{c}
X \\
\sigma |_X \\
V \end{array} \quad \dashrightarrow \quad \begin{array}{c}
\mathbb{P}^{P_m-1} \\
\alpha_m \\
\end{array}
\]

is commutative.

Let us consider the linear system of $m$-canonical adjoints $\Phi_m$. From i) and 1), ..., 4) and the Consequences, we can see that if $2 \leq m \leq 5$, then $\Phi_m$ is given by $\psi_m(X_0, X_1, X_3, X_4) = 0$; moreover, the rational transformation $\alpha_m$ has the generic fiber of dimension $\geq 1$. From the commutativity of the above triangle, $\varphi_{|mK_X|}$ also has the generic fiber of dimension $\geq 1$.

From i) and 5) and the Consequences, we know that $\Phi_m$, for $m = 6, 7$, is again given by $\psi_m(X_0, X_1, X_3, X_4) = 0$, and that the rational transformation $\alpha_m$, as well as $\varphi_{|mK_X|}$, is generically $2 : 1$. As a consequence of this and of the
fact that $P_2 \neq 0$, $\varphi_{[mK_X]}$ is either generically $2 : 1$ or birational (to its image) for $m \geq 8$. It is not difficult to prove that $\varphi_{[5K_X]}$ and $\varphi_{[7K_X]}$ are generically $2 : 1$, since all we have to do is consider the rational transformation defined by the 4 sets of values given in 5) (in both cases $m = 6, 7$).

Next, we note that a necessary condition for the birationality of $\varphi_{[mK_X]}$ is that $A_{m-5} \neq 0$ in the equation $A_{m-5} \varphi_4 X_4 + \psi_m = 0$ of $\Phi_m$; in other words, $\Phi_m$ must be a non-global canonical adjoint to $V$ (cf. Remark 1, section 5).

To be more precise, let us consider $\Phi_m : A_{m-5} \varphi_4 X_4 + \psi_m = 0$ and assume that the rational transformation $\varphi'_m : V \dashrightarrow \mathbb{P}^{P_m-1}$ defined by the linear system $\psi_m = 0$ of global m-canonical adjoints to $V$ (see Remark 1, section 5) is generically $2 : 1$, then $\varphi_{[mK_X]}$ is birational if and only if $A_{m-5} \neq 0$. This is immediately proved by the presence of the addendum $A_{m-5} X_4$, which contains $X_4$ to the power 1; indeed, this addendum separates the two distinct points on $V : \varphi_4 X_4^2 + \varphi_5 X_4 + \varphi_6 = 0$ that are mapped to one point.

As a corollary of this latter fact, in the light of i),...iv) and the Corollaries, $\varphi_{[mK_X]}$ is birational if and only if $m = 9$ and $m \geq 11$. So, for $m = 10$, there is a gap in the birationality of $\varphi_{[mK_X]}$.

This concludes our examination of $\varphi_{[mK_X]}$, for $m \geq 2$.

7. Computing the irregularities of $X$.

This brings us to the demonstration that $q_i = \dim_k H^i(X, \mathcal{O}_X) = 0$, for $i = 1, 2$. We know that $q_1 = \dim_k H^1(X, \mathcal{O}_X) = q(S_r) = \dim_k H^1(S_r, \mathcal{O}_{S_r})$, where $S_r \subset X$ is the strict transform of a generic hyperplane section $S$ of $V$ (cf. [S_1], section 4, for instance). $S$ has several isolated (actual or infinitely-near) double points and no other singularities. This follows from the fact that, outside the points $A_0, A_1, A_2, A_3$ and $A_4$, the hypersurface $V$ only has actual or infinitely-near double curves and isolated double points. So, $q_1 = 0$.

To prove that $q_2 = 0$, we use the formula (36) in section 4 of [S_1], which states that:

$$q_2 = p_g(X) + p_g(S_r) - \dim_k(W_2),$$

where $W_2$ is the vector space of the degree 2 forms defining global adjoints $\Phi_2$ to $V$, i.e. defining hyperquadrics $\Phi_2$ such that

$$\pi_r^* \cdots \pi_2^*[\pi_1^*(\Phi_2)] - E_2 - E_4 - E_6 - E_8 - E_9 \geq 0,$$

(cf. the expression of $D_m$ in (*), section 4). So the above hyperquadrics $\Phi_2$
are those passing through the points $A_0, A_1, A_2, A_3$ and $A_4$. Thus, $\dim_k(W_2) = 15 - 5 = 10$. It follows from $p_g(S_r) = 10$ and $p_g(X) = 0$ (cf. Consequences at the end of section 5) that $q_2 = 0$.

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