A Property of Generalized McLain Groups

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Abstract - In this short note we show that if \( S \) is a connected unbounded poset and \( R \) a ring with no zero divisors, then a generalized McLain group \( G(R, S) \) is a product of two proper normal subgroups.

1. Introduction.

McLain groups were defined in [3] for the first time. These groups are characteristically simple and locally nilpotent with some further interesting properties.

Let \( S \) be an unbounded partially ordered set (poset, in short) and \( R \) be a ring with \( 1 \neq 0 \). Define the generalized McLain group \( G(R, S) \) as in [2]. Now every element of \( G(R, S) \) can be uniquely expressed as

\[
1 + \sum_{i=1}^{n} a_i e_{x_i, y_i}
\]

where \( a_i \in R, x_i, y_i \in S, x_i < y_i \) for \( i = 1, \ldots, r \) and \( n \in \mathbb{N} \).

In [5], some properties of \( G(\mathbb{F}_p, S) \) are considered for some orderings where \( \mathbb{F}_p \) is the field of \( p \) elements. The generalized McLain groups are considered in a general context in [2] and the automorphism groups of these groups are considered in [1], [2] and [4].

[2, Theorem 7.1] gives a necessary and sufficient condition to be \( G(R, S) \) indecomposable. In this short note we ask the following question:

Does \( G(R, S) \) have proper normal subgroups \( K \) and \( N \) such that \( G(R, S) = KN \)?

Two elements \( x, \beta \in S \) are called connected, if there are elements \( x_0, \ldots, x_n \in S \) such that \( x_0 = x, x_n = \beta \) and for each \( 0 \leq i < n \), either


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\[ \alpha_i \leq \alpha_{i+1} \text{ or } \alpha_{i+1} \leq \alpha_i. \text{ } S \text{ is called connected if every pair of elements in } S \text{ is connected.} \]

We shall prove the following:

**Theorem.** Let \( S \) be a connected unbounded poset and \( R \) a ring with no zero divisors. Then \( M := G(R, S) \) has proper normal subgroups \( K \) and \( N \) such that \( M = KN \), \( C_M(K) \neq 1 \) and \( C_M(N) = 1 \). Furthermore if \( 1 + ce_{\alpha} \in M \), then \( 1 + ce_{\alpha} \in K \) or \( 1 + ce_{\alpha} \in N \).

2. **Proof of the Theorem.**

**Lemma 2.1.** Let \( S \) be a connected unbounded poset and \( R \) a ring with no zero divisors. Then every finite family of non-trivial normal subgroups of \( M \) intersects non-trivially.

**Proof.** Obviously it is sufficient to prove the lemma for two proper non-trivial normal subgroups of \( M \). Let \( N \) and \( K \) be such subgroups of \( M \). Assume \( N \cap K = 1 \) and follow the proof of [2, Theorem 7.1] to reach a contradiction. \( \square \)

**Proof of the Theorem.** Put \( M := G(R, S) \) and let \( w = 1 + ae_{\alpha} \) with \( 0 \neq a \in R, \ K := C_M(\langle w^M \rangle) \) and \( N := \langle (1 + ce_{\beta})^M : 1 + ce_{\beta} \notin K, c \in R \rangle \). Then we will prove that \( M = KN \). Since \( Z(M) = 1 \), we have \( K \neq M \). Clearly

\[
\langle w^M \rangle = \left\langle (1 + ae_{\alpha})^{1+\sum_{i=1}^r a_i e_{i,\beta} + \sum_{j=1}^s b_j e_{\alpha,j}} : a, a_i, b_j \in R, 1 \leq i \leq r, 1 \leq j \leq s \right\rangle \\
= \left\langle 1 + ae_{\alpha} - \sum_{i=1}^r a a_i e_{i,\beta} + \sum_{j=1}^s b_j e_{\alpha,j} + \sum_{1 \leq i \leq s, 1 \leq j \leq r} a a_i b_j e_{i,j} : a, a_i, b_j \in R, \right. \\
\left. 1 \leq i \leq r, 1 \leq j \leq s \right\rangle.
\]

Let \( \alpha < \sigma < \tau < \beta \), then we have \( 1 + de_{\alpha} \in K \) with \( 0 \neq a \in R \) by [2, Lemma 2.2]. Since a generator \( 1 + ce_{\beta} \) of \( N \) is not contained in \( K \), it must be of the form \( 1 + ce_{\beta} \) or \( 1 + ce_{\beta} (\mu > \beta) \) or \( 1 + ce_{\gamma} \) or \( 1 + ce_{\lambda} (\lambda < \alpha) \) and its conjugates must be of the form:

\[
(1 + ce_{\beta})^{1+\sum_{i=1}^r a_i e_{i,\beta} + \sum_{j=1}^s b_j e_{\alpha,j}} = 1 + ce_{\beta} - \sum_{i=1}^r a a_i e_{i,\beta} + \sum_{j=1}^s b_j e_{\alpha,j} + \sum_{1 \leq i \leq s, 1 \leq j \leq r} a a_i b_j e_{i,j}.
\]
or

\[(1 + ce^{\mu\delta})^{1+\sum_{i=1}^r a_i e_i + \sum_{j=1}^s b_j e_{\mu_j}} = 1 + ce^{\mu\delta} - \sum_{i=1}^r a_i e_i + \sum_{j=1}^s b_j e_{\mu_j} + \sum \frac{ca_i b_j e_{\delta i}}{1+i}$

or

\[(1 + ce^{\gamma^2})^{1+\sum_{i=1}^r a_i e_i + \sum_{j=1}^s b_j e_{\mu_j}} = 1 + ce^{\gamma^2} - \sum_{i=1}^r a_i e_i + \sum_{j=1}^s b_j e_{\mu_j} + \sum \frac{ca_i b_j e_{\gamma i}}{1+i}$

or

\[(1 + ce^{\gamma^2})^{1+\sum_{i=1}^r a_i e_i + \sum_{j=1}^s b_j e_{\mu_j}} = 1 + ce^{\gamma^2} - \sum_{i=1}^r a_i e_i + \sum_{j=1}^s b_j e_{\mu_j} + \sum \frac{ca_i b_j e_{\gamma i}}{1+i}$

For all terms \(e_{\theta e}\) that appear in each case, \(\theta \notin [x, \beta] \) or \(e \notin [x, \beta] \). Hence we have that there is no product of these elements which equals \(1 + e_{\sigma i} \), i.e., \(1 + e_{\sigma i} \notin N \). Hence \(N \neq M \) and obviously \(M = KN \).

Clearly we have \(C_M(K) \neq 1 \). Assume \(C_M(N) \neq 1 \), then \(C_M(K) \cap \cap C_M(N) \neq 1 \) by Lemma 2.1. But since \(Z(M) = 1 \), this is a contradiction. The final part of the theorem follows by the construction of \(K \) and \(N \). Now the proof is complete.

**Corollary 2.2.** Let \(S \) be a connected unbounded poset and \(R \) a ring with no zero divisors. Put \(M := G(R, S) \) and let \(N \) be the subgroup defined in the theorem. Then

\[
C_M(\langle x^M \rangle) / C_M(\langle x^M \rangle) \cap N
\]

is perfect for every generator \(x \) of \(M \) of the form \(1 + ae_{\delta^2} \) (\(a \in R \)).

**Corollary 2.3.** Let \(S \) be a connected unbounded poset and \(R \) a ring with no zero divisors. Put \(M := G(R, S) \). Then \(M \) has a decomposable non-trivial epimorphic image.

The author is grateful to the referee for careful reading and many valuable suggestions.
REFERENCES


Manoscritto pervenuto in redazione il 13 novembre 2007.