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Right Sided Ideals and Multilinear Polynomials with Derivations on Prime Rings

BASUDEB DHARA (*) - RAJENDRA K. SHARMA (**)

ABSTRACT - Let R be an associative prime ring of char $R \neq 2$ with center $Z(R)$ and extended centroid $C, f(x_1, \ldots, x_n)$ a nonzero multilinear polynomial over C in n noncommuting variables, d a nonzero derivation of R and ρ a nonzero right ideal of R. We prove that: (i) if $[d^2(f(x_1, \ldots, x_n)), f(x_1, \ldots, x_n)] = 0$ for all $x_1, \ldots, x_n \in \rho$ then $\rho C = eRC$ for some idempotent element e in the socle of RC and $f(x_1, \ldots, x_n)$ is central-valued in $eRCe$ unless d is an inner derivation induced by $b \in Q$ such that $b^2 = 0$ and $b\rho = 0$; (ii) if $[d^2(f(x_1, \ldots, x_n)), f(x_1, \ldots, x_n)] \in Z(R)$ for all $x_1, \ldots, x_n \in \rho$ then $\rho C = eRC$ for some idempotent element e in the socle of RC and either $f(x_1, \ldots, x_n)$ is central in $eRCe$ or $eRCe$ satisfies the standard identity $S_4(x_1, x_2, x_3, x_4)$ unless d is an inner derivation induced by $b \in Q$ such that $b^2 = 0$ and $b\rho = 0$.

Throughout this paper, R always denotes a prime ring with extended centroid C and Q its two-sided Martindale ring of quotient. By d we mean a nonzero derivation of R. For $x, y \in R$, the commutator of x, y is denoted by $[x, y]$ and defined by $[x, y] = xy - yx$. We denote $[x, y]_2 = [[x, y], y] = [x, y]y - y[x, y]$.

A well known result proved by Posner [17] states that R must be commutative if $[d(x), x] \in Z(R)$ for all $x \in R$. In [10] Lanski generalized the Posner's result to a Lie ideal. More precisely Lanski proved that if L is a noncommutative Lie ideal of R such that $[d(x), x] \in Z(R)$ for all $x \in L$,

(*) Indirizzo dell'A.: Department of Mathematics, Belda College, Belda, Paschim Medinipur-721424, India.
E-mail: basu_dhara@yahoo.com
(**) Indirizzo dell'A.: Department of Mathematics, Indian Institute of Technology, Delhi, Hauz Khas, New Delhi-110016, India.
E-mail: rksharma@maths.iitd.ac.in
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then char $R = 2$ and R satisfies $S_4(x_1, x_2, x_3, x_4)$, the standard identity. Note that a noncommutative Lie ideal of R contains all the commutators $[x_1, x_2]$ for x_1, x_2 in some nonzero ideal of R (see [10, Lemma 2 (i), (ii)]). So, it is natural to consider the situation when $[d(x), x] \in Z(R)$ for all commutators $x = [x_1, x_2]$ or more general case $x = f(x_1, \ldots, x_n)$ where $f(x_1, \ldots, x_n)$ is a multilinear polynomial. In [11] Lee and Lee proved that if $[d(f(x_1, \ldots, x_n)), f(x_1, \ldots, x_n)] \in Z(R)$ for all $x_1, \ldots, x_n$ in some nonzero ideal of R, then $f(x_1, \ldots, x_n)$ is central-valued on R, except when char $R = 2$ and R satisfies $S_4(x_1, x_2, x_3, x_4)$. Recently, De Filippis and Di Vincenzo (see [7]) consider the situation $\delta([d(f(x_1, \ldots, x_n)), f(x_1, \ldots, x_n)]) = 0$ for all $x_1, \ldots, x_n \in R$, where d and δ are two derivations of R. The statement of De Filippis and Di Vincenzo's theorem is the following:

THEOREM A ([7, Theorem 1]). *Let K be a noncommutative ring with unity, R a prime K-algebra of characteristic different from 2, d and δ nonzero derivations of R and $f(x_1, \ldots, x_n)$ a multilinear polynomial over K. If $\delta([d(f(x_1, \ldots, x_n)), f(x_1, \ldots, x_n)]) = 0$ for all $x_1, \ldots, x_n \in R$, then $f(x_1, \ldots, x_n)$ is central-valued on R.*

In case δ and d are two same derivations, the differential identity becomes $[d^2(f(x_1, \ldots, x_n)), f(x_1, \ldots, x_n)] = 0$ for all $x_1, \ldots, x_n \in R$. So, it is natural to ask, what happen in cases $[d^2(f(x_1, \ldots, x_n)), f(x_1, \ldots, x_n)] \in Z(R)$ for all $x_1, \ldots, x_n \in R$ and $[d^2(f(x_1, \ldots, x_n)), f(x_1, \ldots, x_n)] \in Z(R)$ for all $x_1, \ldots, x_n \in \rho$, where ρ is a non-zero right ideal of R. In the present paper our object is to study these cases.

For the sake of completeness we recall some basic notations, definitions and some easy consequences of the result of Kharchenko [8] about the differential identities on a prime ring R. First, we denote by $Der(Q)$ the set of all derivations on Q. By a derivation word Δ of R we mean $\Delta = d_1 d_2 d_3 \ldots d_m$ for some derivations d_i of R. For $x \in R$, we denote by x^{Δ} the image of x under Δ, that is $x^{\Delta} = (\cdots(x^{d_1})^{d_2}\cdots)^{d_m}$. By a differential polynomial, we mean a generalized polynomial, with coefficients in Q, of the form $\Phi(x_i^{\Delta_j})$ involving noncommutative indeterminates x_i on which the derivations words Δ_j act as unary operations. $\Phi(x_i^{\Delta_j}) = 0$ is said to be a differential identity on a subset T of Q if it vanishes for any assignment of values from T to its indeterminates x_i.

Now let D_{int} be the C-subspace of $Der(Q)$ consisting of all inner derivations on Q. By Kharchenko's theorem [8, Theorem 2], we have the following result:

Let R be a prime ring of characteristic different from 2. If two nonzero derivations d and δ are C-linearly independent modulo D_{int} and $\Phi(x_i^{\Delta_j})$ is a differential identity on R, where Δ_j are derivations words of the following form $\delta, d, \delta^2, \delta d, d^2$, then $\Phi(y_{ji})$ is a generalized polynomial identity on R, where y_{ji} are distinct indeterminates.

As a particular case, we have:

If d is a nonzero derivation on R and $\Phi(x_1, \dots, x_n, x_1^d, \dots, x_n^d, x_1^{d^2}, \dots, x_n^{d^2})$ is a differential identity on R, then one of the following holds:

(i) either $d \in D_{int}$

or

(ii) R satisfies the generalized polynomial identity $\Phi(x_1, \dots, x_n, y_1, \dots, y_n, z_1, \dots, z_n)$

Denote by $Q *_C C\{X_1, \dots, X_n\}$ the free product of the C-algebra Q and $C\{X_1, \dots, X_n\}$, the free C-algebra in noncommuting indeterminates $X_1, \dots, X_n$.

Since $f(x_1, \dots, x_n)$ is a multilinear polynomial, we can write

$$f(x_1, \dots, x_n) = x_1 x_2 \dots x_n + \sum_{I \neq \sigma \in S_n} \alpha_\sigma x_{\sigma(1)} \dots x_{\sigma(n)}$$

where S_n is the permutation group over n elements and any $\alpha_\sigma \in C$. We denote by $f^d(x_1, \dots, x_n)$ the polynomial obtained from $f(x_1, \dots, x_n)$ by replacing each coefficient α_σ with $d(\alpha_\sigma.1)$. In this way we have

$$d(f(x_1, \dots, x_n)) = f^d(x_1, \dots, x_n) + \sum_i f(x_1, \dots, d(x_i), \dots, x_n)$$

and

$$\begin{aligned}
d^2(f(x_1, \dots, x_n)) &= d(f^d(x_1, \dots, x_n)) + d\left(\sum_i f(x_1, \dots, d(x_i), \dots, x_n)\right) \\
&= f^{d^2}(x_1, \dots, x_n) + \sum_i f^d(x_1, \dots, d(x_i), \dots, x_n) \\
&+ \sum_i f^d(x_1, \dots, d(x_i), \dots, x_n) + \sum_{i \neq j} f(x_1, \dots, d(x_i), \dots, d(x_j), \dots, x_n) \\
&+ \sum_i f(x_1, \dots, d^2(x_i), \dots, x_n) \\
&= f^{d^2}(x_1, \dots, x_n) + 2\sum_i f^d(x_1, \dots, d(x_i), \dots, x_n) \\
&+ 2\sum_{i<j} f(x_1, \dots, d(x_i), \dots, d(x_j), \dots, x_n) + \sum_i f(x_1, \dots, d^2(x_i), \dots, x_n).
\end{aligned}$$

1. The case for $\rho = R$.

LEMMA 1.1. *Let $R = M_k(F)$ be the ring of all $k \times k$ matrices over a field F of characteristic $\neq 2$, $b \in R$ and $f(x_1, \ldots, x_n)$ is a multilinear polynomial over F. If $k \geq 2$ and $[[b,[b,f(x_1,\ldots,x_n)]], f(x_1,\ldots,x_n)] = 0$ for all $x_1, \ldots, x_n \in R$ or if $k \geq 3$ and $[[b,[b,f(x_1,\ldots,x_n)]], f(x_1,\ldots,x_n)] \in Z(R)$ for all $x_1, \ldots, x_n \in R$, then either $b \in F \cdot I_k$ or $f(x_1, \ldots, x_n)$ is central-valued on R.*

PROOF. Let $b = (b_{ij})_{k\times k}$. Let e_{ij} be the usual matrix unit with 1 in (i,j) entry and zero else where. Now we proceed to show that $b \in Z(R)$ if $\in f(x_1, \ldots, x_n)$ is non central valued on R.

For simplicity, we write $f(x_1, \ldots, x_n) = f(x)$, where $x = (x_1, \ldots, x_n)$ $R^n = R \times \cdots \times R$ (n times). Then by assumption,

$$[[b,[b,f(x)]], f(x)] = [b^2 f(x) - 2bf(x)b + f(x)b^2, f(x)] \in Z(R)$$

for all $x \in R^n$. Since $f(x_1, \ldots, x_n)$ is assumed to be noncentral on R, by [15, Lemma 2, Proof of Lemma 3] there exists a sequence of matrices $r = (r_1, \ldots, r_n)$ in R such that $f(r) = f(r_1, \ldots, r_n) = \alpha e_{ij} \neq 0$ where $0 \neq \alpha \in F$ and $i \neq j$. Thus

$$[b^2\alpha e_{ij} - 2b\alpha e_{ij} b + \alpha e_{ij} b^2, \alpha e_{ij}] \in Z(R).$$

Since the rank of $[b^2\alpha e_{ij} - 2b\alpha e_{ij}b + \alpha e_{ij}b^2, \alpha e_{ij}]$ is ≤ 2, $[b^2\alpha e_{ij} - 2b\alpha e_{ij}b + {}+ \alpha e_{ij}b^2, \alpha e_{ij}] = 0$. Left multiplying by e_{ij}, we get $0 = e_{ij}(-2b\alpha e_{ij}b\alpha e_{ij}) = {}= -2\alpha^2 b_{ji}^2 e_{ij}$. Since char $F \neq 2$, $b_{ji} = 0$. For $s \neq t$, let σ be a permutation in the symmetric group S_m such that $\sigma(i) = s$ and $\sigma(j) = t$. Let ψ be the automorphism of R defined by $x^\psi = \left(\sum_{p,q} \xi_{pq} e_{pq}\right)^\psi = \sum_{p,q} \xi_{pq} e_{\sigma(p),\sigma(q)}$. Then $f(r^\psi) = f(r_1^\psi, \ldots, r_n^\psi) = f(r)^\psi = \alpha e_{st} \neq 0$ and we have as above $b_{ts} = 0$ for $s \neq t$. Thus b is a diagonal matrix. For any F-automorphism θ of R, b^θ enjoys the same property as b does, namely, $[[b^\theta,[b^\theta, f(x)]], f(x)] \in Z(R)$ for all $x \in R^n$. Hence, b^θ must be diagonal. Write $b = \sum_{i=1}^{k} a_{ii} e_{ii}$; then for each $j \neq 1$, we have

$$(1 + e_{1j})b(1 - e_{1j}) = \sum_{i=1}^{k} a_{ii}e_{ii} + (b_{jj} - b_{11})e_{1j}$$

diagonal. Therefore, $b_{jj} = b_{11}$ and so b is a scalar matrix.

LEMMA 1.2. *Let R be a prime ring of characteristic different from 2 and $f(x_1,\ldots,x_n)$ a multilinear polynomial over C. If for any $i=1,\ldots,n$,*

$$[f(x_1,\ldots,z_i,\ldots,x_n), f(x_1,\ldots,x_n)] = 0$$

for all $x_1,\ldots,x_n,z_i \in R$, then the polynomial $f(x_1,\ldots,x_n)$ is central-valued on R.

PROOF. Let a be a noncentral element of R. Then replacing z_i with $[a,x_i]$ we have that for any $i=1,\ldots,n$

$$[f(x_1,\ldots,[a,x_i],\ldots,x_n), f(x_1,\ldots,x_n)] = 0$$

and so

$$\left[\sum_{i=0}^{n} f(x_1,\ldots,[a,x_i],\ldots,x_n), f(x_1,\ldots,x_n)\right] = 0$$

which implies, $[a,f(x_1,\ldots,x_n)]_2 = 0$ for all $x_1,\ldots,x_n \in R$. By [11, Theorem], $f(x_1,\ldots,x_n)$ is central-valued on R.

THEOREM 1.3. *Let R be a prime ring of characteristic different from 2, d a nonzero derivation of R, $f(x_1,\ldots,x_n)$ a multilinear polynomial over C. If*

$$[d^2(f(x_1,\ldots,x_n)), f(x_1,\ldots,x_n)] \in Z(R) \quad \text{for all } x_1,\ldots,x_n \in R,$$

then either $f(x_1,\ldots,x_n)$ is central-valued on R or R satisfies the standard identity $S_4(x_1,x_2,x_3,x_4)$.

PROOF. Let I be any nonzero two-sided ideal of R. If for every $r_1,\ldots,r_n \in I$, $[d^2(f(r_1,\ldots,r_n)), f(r_1,\ldots,r_n)] = 0$, then by [14], this generalized differential identity is also satisfied by Q and hence by R as well. By Theorem A, $f(r_1,\ldots,r_n)$ is then central-valued on R and we are done. Now we assume that for some $r_1,\ldots,r_n \in I$, $0 \neq [d^2(f(r_1,\ldots,r_n)), f(r_1,\ldots,r_n)] \in I \cap Z(R)$. Thus $I \cap Z(R) \neq 0$. Let K be a nonzero two-sided ideal of R_Z, the ring of central quotients of R. Since $K \cap R$ is a nonzero two-sided ideal of R, $(K \cap R) \cap Z(R) \neq 0$. Therefore, K contains an invertible element in R_Z and so R_Z is a simple ring with identity 1.

By assumption, R satisfies the differential identity

$$\begin{aligned} &g(x_1,\ldots,x_n,d(x_1),\ldots,d(x_n),d^2(x_1),\ldots,d^2(x_n)) \\ &= [[f^{d^2}(x_1,\ldots,x_n) + 2\sum_i f^d(x_1,\ldots,d(x_i),\ldots,x_n) \\ &\quad + 2\sum_{i<j} f(x_1,\ldots,d(x_i),\ldots,d(x_j),\ldots,x_n) \\ &\quad + \sum_i f(x_1,\ldots,d^2(x_i),\ldots,x_n), f(x_1,\ldots,x_n)], x_{n+1}]. \end{aligned}$$

If d is not Q-inner, then by Kharchenko's theorem [8],

$$\begin{aligned} &\Bigg[\Bigg[f^{d^2}(x_1,\ldots,x_n) + 2\sum_i f^d(x_1,\ldots,y_i,\ldots,x_n) \\ &\quad + 2\sum_{i<j} f(x_1,\ldots,y_i,\ldots,y_j,\ldots,x_n) \\ &\quad + \sum_i f(x_1,\ldots,z_i,\ldots,x_n), f(x_1,\ldots,x_n)\Bigg], x_{n+1}\Bigg] = 0 \end{aligned} \tag{1}$$

for all $x_i, y_i, z_i, x_{n+1} \in R$ for $i = 1, 2, \ldots, n$. In particular, for any i, assuming $y_1 = \cdots = y_{i-1} = y_{i+1} = \cdots = y_n = 0, z_1 = \cdots = z_n = 0$, we have

$$[[f^{d^2}(x_1,\ldots,x_n) + 2f^d(x_1,\ldots,y_i,\ldots,x_n), f(x_1,\ldots,x_n)], x_{n+1}] = 0$$

and so

$$\Bigg[\Bigg[f^{d^2}(x_1,\ldots,x_n) + 2\sum_i f^d(x_1,\ldots,y_i,\ldots,x_n), f(x_1,\ldots,x_n)\Bigg], x_{n+1}\Bigg] = 0$$

for all $x_i, y_i, x_{n+1} \in R$, $i = 1, 2, \ldots, n$. Thus from (1), we obtain

$$\begin{aligned} &\Bigg[\Bigg[2\sum_{i<j} f(x_1,\ldots,y_i,\ldots,y_j,\ldots,x_n) \\ &\quad + \sum_i f(x_1,\ldots,z_i,\ldots,x_n), f(x_1,\ldots,x_n)\Bigg], x_{n+1}\Bigg] = 0 \end{aligned} \tag{2}$$

for all $x_i, y_i, z_i, x_{n+1} \in R$ for $i = 1, 2, \ldots, n$.

By localizing R at $Z(R)$, we obtain that (2) is also an identity of R_Z. Since R and R_Z satisfy the same polynomial identities, in order to prove that R satisfies S_4, we may assume that R is a simple ring with 1. Thus R satisfies the identity (2). Now putting $y_i = [b, x_i] = \delta(x_i)$ and $z_i = [b, [b, x_i]] =$

$= \delta^2(x_i)$, $i = 1, 2, \ldots, n$ for some $b \notin Z(R)$, where δ is an inner derivation induced by some $b \in R$, we obtain that R satisfies

$$[[\delta^2(f(x_1, \ldots, x_n)), f(x_1, \ldots, x_n)], x_{n+1}] = 0.$$

Thus by Martindale's theorem [16], R is a primitive ring with a minimal right ideal, whose commuting ring D is a division ring which is finite dimensional over $Z(R)$. However, since R is simple with 1, R must be Artinian. Hence $R = D_{k'}$, the ring of $k' \times k'$ matrices over D, for some $k' \geq 1$. Again, by [9, Lemma 2], it follows that there exists a field F such that $R \subseteq M_k(F)$, the ring of all $k \times k$ matrices over the field F, and $M_k(F)$ satisfies

$$[[\delta^2(f(x_1, \ldots, x_n)), f(x_1, \ldots, x_n)], x_{n+1}] = 0.$$

If $k \geq 3$, then by Lemma 1.1, we have $b \in Z(R)$, a contradiction. Thus $k = 2$, that is, R satisfies $S_4(x_1, x_2, x_3, x_4)$.

Similarly, the same conclusion can be drawn in case d is an Q-inner derivation induced by some $b \in Q$.

2. The case for one-sided ideal.

We begin with the following lemmas

LEMMA 2.1. *Let ρ be a nonzero right ideal of R and d a derivation of R. Then the following conditions are equivalent:*

(i) *d is an inner derivation induced by some $b \in Q$ such that $b\rho = 0$;*
(ii) *$d(\rho)\rho = 0$.*

For its proof, we refer to [2, Lemma].

LEMMA 2.2. *Let R be a prime ring, ρ a nonzero right ideal of R, $f(x_1, \ldots, x_t)$ a multilinear polynomial over C, $a \in R$ and n a fixed positive integer. If $f(x_1, \ldots, x_t)^n a = 0$ for all $x_1, \ldots, x_t \in \rho$, then either $a = 0$ or $f(\rho)\rho = 0$.*

For its proof, we refer to [3, Lemma 2 (II)].

LEMMA 2.3. *Let R be a prime ring. If $[d^2(f(x_1, \ldots, x_n)), f(x_1, \ldots, x_n)] \in Z(R)$ for all $x_1, \ldots, x_n \in \rho$, then R satisfies nontrivial generalized polynomial identity unless d is an inner derivation induced by $b \in Q$ such that $b^2 = 0$ and $b\rho = 0$.*

PROOF. Suppose on the contrary that R does not satisfy any nontrivial generalized polynomial identity (GPI). Thus we may assume that R is noncommutative, otherwise R satisfies trivially a nontrivial GPI. Now we consider the following two cases:

CASE I. Suppose that d is a Q-inner derivation induced by an element $b \in Q$ such that $b^2 \neq 0$. Then for any $x_0 \in \rho$

$$[[b,[b,f(x_0X_1,\dots,x_0X_n)]],f(x_0X_1,\dots,x_0X_n)] \in Z(R)$$

that is

$$(3)\quad \begin{aligned}&[[b^2f(x_0X_1,\dots,x_0X_n)-2bf(x_0X_1,\dots,x_0X_n)b\\&+f(x_0X_1,\dots,x_0X_n)b^2,f(x_0X_1,\dots,x_0X_n)],x_0X_{n+1}]\end{aligned}$$

is a GPI for R, so it is the zero element in $Q *_C C\{X_1,\dots,X_{n+1}\}$. Denote $l_R(\rho)$ the left annihilator of ρ in R. Suppose first that $\{1,b,b^2\}$ are linearly C-independent modulo $l_R(\rho)$, that is $(\alpha b^2+\beta b+\gamma)\rho=0$ if and only if $\alpha=\beta=\gamma=0$. Since R is not a GPI-ring, a fortiori it can not be a PI-ring. Thus, by [13, Lemma 3] there exists $x_0 \in \rho$ such that $\{b^2x_0, bx_0, x_0\}$ are linearly C-independent. Then we have that

$$\begin{aligned}&[[b^2f(x_0X_1,\dots,x_0X_n)-2bf(x_0X_1,\dots,x_0X_n)b\\&+f(x_0X_1,\dots,x_0X_n)b^2,f(x_0X_1,\dots,x_0X_n)],x_0X_{n+1}]=0\end{aligned}$$

is a nontrivial GPI for R, a contradiction.

Therefore, $\{1,b,b^2\}$ are linearly C-dependent modulo $l_R(\rho)$, that is there exist $\alpha,\beta,\gamma \in C$, not all zero, such that $(\alpha b^2+\beta b+\gamma)\rho=0$. Suppose that $\alpha=0$. Then $\beta \neq 0$, otherwise $\gamma=0$. Thus by $(\beta b+\gamma)\rho=0$, we have that $(b+\beta^{-1}\gamma)\rho=0$. Since b and $b+\beta^{-1}\gamma$ induce the same inner derivation, we may replace b by $b+\beta^{-1}\gamma$ in the basic hypothesis. Therefore, in any case we may suppose $b\rho=0$ and then from (3), R satisfies $x_0X_{n+1}f^2(x_0X_1,\dots,x_0X_n)b^2=0$. Since R does not satisfy any nontrivial GPI, $b^2=0$, a contradiction.

Next suppose that $\alpha \neq 0$. In this case there exist $\lambda,\mu \in C$ such that $b^2x_0=\lambda bx_0+\mu x_0$ for all $x_0 \in \rho$. If bx_0 and x_0 are linearly C-dependent for all $x_0 \in \rho$, then again we obtain $b\rho=0$ and so $b^2=0$. Therefore choose $x_0 \in \rho$ such that bx_0 and x_0 are linearly C-independent. Then replacing b^2x_0 with $\lambda bx_0+\mu x_0$, we obtain from (3) that R satisfies

$$\Big[\big\{(\lambda b+\mu)f^2(x_0X_1,\dots,x_0X_n)-2bf(x_0X_1,\dots,x_0X_n)bf(x_0X_1,\dots,x_0X_n)$$
$$+f(x_0X_1,\dots,x_0X_n)(\lambda b+\mu)f(x_0X_1,\dots,x_0X_n)\big\}$$
$$-\big\{f(x_0X_1,\dots,x_0X_n)(\lambda b+\mu)f(x_0X_1,\dots,x_0X_n)$$
$$-2f(x_0X_1,\dots,x_0X_n)bf(x_0X_1,\dots,x_0X_n)b+f^2(x_0X_1,\dots,x_0X_n)b^2\big\},x_0X_{n+1}\Big].$$

This is a nontrivial GPI for R, because the term

$$\big(\lambda bf^2(x_0X_1,\dots,x_0X_n)-2bf(x_0X_1,\dots,x_0X_n)bf(x_0X_1,\dots,x_0X_n)\big)x_0X_{n+1}$$

appears nontrivially, a contradiction.

CASE II. Suppose that d is an inner derivation induced by an element $b\in Q$ such that $b^2=0$. Thus we have that $[-2bf(X_1,\dots,X_n)b, f(X_1,\dots,X_n)]\in Z(R)$ is satisfied by ρ. In case there exists $x_0\in\rho$ such that $\{bx_0,x_0\}$ are linearly C-independent, we have that $[[-2bf(x_0X_1,\dots,x_0X_n)b, f(x_0X_1,\dots,x_0X_n)],x_0X_{n+1}]$ is a non trivial GPI for R, a contradiction. Hence $\{bx_0,x_0\}$ are linearly C-dependent for all $x_0\in\rho$, that is there exists $\alpha\in C$ such that $(b-\alpha)\rho=0$. Thus we have that $[\alpha f^2(X_1,\dots,X_n)(\alpha-b),X_{n+1}]$ is satisfied by ρ, in particular R satisfies:

$$[\alpha f^2(x_0X_1,\dots,x_0X_n)(\alpha-b),f(x_0X_1,\dots,x_0X_n)]=\alpha f^3(X_1,\dots,X_n)(\alpha-b)$$

for any $x_0\in\rho$. Since R is not GPI, it follows that either $b=\alpha\in C$, which is a contradiction, or $\alpha=0$ which means $b\rho=0$, as required.

CASE III. Suppose that d is an inner derivation induced by an element $b\in Q$ such that $b\rho=0$. Thus we have that $[-f^2(X_1,\dots,X_n)b^2,X_{n+1}]$ is satisfied by ρ, in particular R satisfies:

$$[-f^2(x_0X_1,\dots,x_0X_n)b^2,f(x_0X_1,\dots,x_0X_n)]=f^3(x_0X_1,\dots,x_0X_n)b^2$$

for any $x_0\in\rho$. Again since R is not GPI we conclude that $b^2=0$.

CASE IV. Next suppose that d is not Q-inner derivation. By our assumption we have that R satisfies

$$0=\Big[\big[f^{d^2}(xX_1,\dots,xX_n)+2\sum_i f^d(xX_1,\dots,d(x)X_i+xd(X_i),\dots,xX_n)$$
$$+2\sum_{i<j} f(xX_1,\dots,d(x)X_i+xd(X_i),\dots,d(x)X_j+xd(X_j),\dots,xX_n)$$
$$+\sum_i f(xX_1,\dots,d^2(x)X_i+2d(x)d(X_i)+xd^2(X_i),\dots,xX_n),f(xX_1,\dots,xX_n)\big],X_{n+1}\Big].$$

By Kharchenko's theorem [8],

$$\Big[\big[f^{d^2}(xX_1,\dots,xX_n)+2\sum_i f^d(xX_1,\dots,d(x)X_i+xr_i,\dots,xX_n)$$
$$+2\sum_{i<j} f(xX_1,\dots,d(x)X_i+xr_i,\dots,d(x)X_j+xr_j,\dots,xX_n)$$
$$+\sum_i f(xX_1,\dots,d^2(x)X_i+2d(x)r_i+xs_i,\dots,xX_n), f(xX_1,\dots,xX_n)\big],X_{n+1}\Big]=0$$

for all $X_1,\dots,X_n,r_1,\dots,r_n,s_1,\dots,s_n\in R$. In particular, for $r_1 = = r_2 = \cdots = r_n = 0$, we have

$$\Big[\big[f^{d^2}(xX_1,\dots,xX_n)+2\sum_i f^d(xX_1,\dots,d(x)X_i,\dots,xX_n)$$
$$2\sum_{i<j} f(xX_1,\dots,d(x)X_i,\dots,d(x)X_j,\dots,xX_n)+\sum_i f(xX_1,\dots,d^2(x)X_i,\dots,xX_n)+$$
$$+\sum_i f(xX_1,\dots,xs_i,\dots,xX_n), f(xX_1,\dots,xX_n)\big],X_{n+1}\Big]=0.$$

Hence R satisfies the blended component

$$[[f(xs_1,\dots,xX_n), f(xX_1,\dots,xX_n)],X_{n+1}]=0$$

which is a nontrivial GPI for R, a contradiction.

THEOREM 2.4. *Let R be an associative prime ring of char $R\neq 2$ with center $Z(R)$ and extended centroid C, $f(x_1,\dots,x_n)$ a nonzero multilinear polynomial over C in n noncommuting variables, d a nonzero derivation of R and ρ a nonzero right ideal of R. If $[d^2(f(x_1,\dots,x_n)), f(x_1,\dots,x_n)]=0$ for all $x_1,\dots,x_n\in\rho$ then $\rho C=eRC$ for some idempotent e in the socle of RC and $f(x_1,\dots,x_n)$ is central-valued on $eRCe$ unless d is an inner derivation induced by $b\in Q$ such that $b^2=0$ and $b\rho=0$.*

PROOF. Suppose d is not a Q-inner derivation induced by an element $b\in Q$ such that $b^2=0$ and $b\rho=0$.

Now assume first that $f(\rho)\rho=0$, that is $f(x_1,\dots,x_n)x_{n+1}=0$ for all $x_1,x_2,\dots,x_{n+1}\in\rho$. Then by [12, Proposition], $\rho C=eRC$ for some idempotent $e\in soc(RC)$. Since $f(\rho)\rho=0$, we have $f(\rho R)\rho R=0$ and hence $f(\rho Q)\rho Q=0$ by [4, Theorem 2]. In particular, $f(\rho C)\rho C=0$, or equivalently, $f(eRC)e=0$. Then $f(eRCe)=0$, that is, $f(x_1,\dots,x_n)$ is a PI for $eRCe$ and, a fortiori, central valued on $eRCe$.

Next assume that $f(\rho)\rho\neq 0$, that is $f(x_1,\dots,x_n)x_{n+1}$ is not an identity for ρ and then we derive a contradiction. By Lemma 2.3, R is a GPI-ring and so is also Q (see [1] and [4]). By [16], Q is a primitive ring with $H = soc(Q) \neq 0$. Moreover, we may assume $f(\rho H)\rho H \neq 0$, otherwise by [1] and [4], $f(\rho Q)\rho Q = 0$, which is a contradiction. Choose $a_0, a_1, \ldots, a_n \in \rho H$ such that $f(a_1, \ldots, a_n)a_0 \neq 0$. Let $a \in \rho H$. Since H is a regular ring, there exists $e^2 = e \in H$ such that

$$eH = aH + a_0H + a_1H + \cdots + a_nH.$$

Then $e \in \rho H$ and $a = ea, a_i = ea_i$ for $i = 0, 1, \ldots, n$. Thus we have $f(eHe) = f(eH)e \neq 0$. By our assumption and by [14, Theorem 2], we also assume that

$$[d^2(f(x_1, \ldots, x_n)), f(x_1, \ldots, x_n)]$$

is an identity for ρQ. In particular $[d^2(f(x_1, \ldots, x_n)), f(x_1, \ldots, x_n)]$ is an identity for ρH and so for eH. It follows that, for all $r_1, \ldots, r_n \in H$,

$$0 = [d^2(f(er_1, \ldots, er_n)), f(er_1, \ldots, er_n)].$$

We may write $f(x_1, \ldots, x_n) = t(x_1, \ldots, x_{n-1})x_n + h(x_1, \ldots, x_n)$, where x_n never appears as last variable in any monomials of h. Let $r \in H$. Then replacing r_n with $r(1-e)$, we have

$$0 = [d^2(t(er_1, \ldots, er_{n-1})er(1-e)), t(er_1, \ldots, er_{n-1})er(1-e)]. \tag{4}$$

Now, we know the fact that $d(x(1-e))e = -x(1-e)d(e)$ and $(1-e)d(ex) = = (1-e)d(e)ex$ and so

$$\begin{aligned}(1-e)d^2(ex(1-e))e &= (1-e)d\{d(e)ex(1-e) + ed(ex(1-e))\}e \\ &= (1-e)d(e)d(ex(1-e))e + (1-e)d(e)d(ex(1-e))e \\ &= -2(1-e)d(e)ex(1-e)d(e).\end{aligned}$$

Thus using this facts, we have from (4),

$$\begin{aligned}0 &= (1-e)[d^2(t(er_1, \ldots, er_{n-1})er(1-e)), t(er_1, \ldots, er_{n-1})er(1-e)] \\ &= (1-e)d^2(t(er_1, \ldots, er_{n-1})er(1-e))t(er_1, \ldots, er_{n-1})er(1-e) \\ &= -2(1-e)d(e)t(er_1, \ldots, er_{n-1})er(1-e)d(e)t(er_1, \ldots, er_{n-1})er(1-e) \\ &= -2((1-e)d(e)t(er_1, \ldots, er_{n-1})er)^2(1-e).\end{aligned}$$

This implies

$$0 = -2\{(1-e)d(e)t(er_1, \ldots, er_{n-1})er\}^3$$

that is

$$0 = -2\{(1-e)d(e)t(er_1, \ldots, er_{n-1})eH\}^3.$$

By [6], $(1-e)d(e)t(er_1,\ldots,er_{n-1})eH=0$ which implies

$$(1-e)d(e)t(er_1e,\ldots,er_{n-1}e)=0.$$

Since eHe is a simple Artinian ring and $t(eHe)\neq 0$ is invariant under the action of all inner automorphisms of eHe, by [5, Lemma 2],$(1-e)d(e)=0$ and so $d(e)=ed(e)\in eH$. Thus $d(eH)\subseteq d(e)H+ed(H)\subseteq eH\subseteq \rho H$ and $d(a)=d(ea)\in d(eH)\subseteq \rho H$. Therefore, $d(\rho H)\subseteq \rho H$. Denote the left annihilator of ρH in H by $l_H(\rho H)$. Then $\overline{\rho H}=\dfrac{\rho H}{\rho H\cap l_H(\rho H)}$, a prime C-algebra with the derivation $\overline{d}$ such that $\overline{d}(\overline{x})=\overline{d(x)}$, for all $x\in\rho H$. By assumption, we have that

$$[\overline{d}^2(f(\overline{x_1},\ldots,\overline{x_n})),f(\overline{x_1},\ldots,\overline{x_n})]=0$$

for all $\overline{x_1},\ldots,\overline{x_n}\in\overline{\rho H}$. By Theorem A, either $\overline{d}=0$ or $f(\overline{x_1},\ldots,\overline{x_n})$ is central-valued on $\overline{\rho H}$.

If $\overline{d}=0$, then $d(\rho H)\rho H=0$ and so $d(\rho)\rho=0$. By Lemma 2.1, d is an inner derivation induced by an element $b\in Q$ such that $b\rho=0$. Then for all $x_1,\ldots,x_n\in\rho$, we have by assumption that

$$0=[[b,[b,f(x_1,\ldots,x_n)]],f(x_1,\ldots,x_n)]=-f^2(x_1,\ldots,x_n)b^2.$$

By [3, Lemma 4], either $b^2=0$ or $f(\rho)\rho=0$. In both cases we have contradiction.

If $f(\overline{x_1},\ldots,\overline{x_n})$ is central-valued on $\overline{\rho H}$, then ρH, as well as ρ, satisfies $[f(x_1,\ldots,x_n),x_{n+1}]x_{n+2}=0$. Then $\rho C=eRC$ for some idempotent element $e\in soc(RC)$ by [12, Proposition] and $f(x_1,\ldots,x_n)$ is central-valued on $eRCe$ and we are done.

THEOREM 2.5. *Let R be an associative prime ring of char $R\neq 2$ with center $Z(R)$ and extended centroid C, $f(x_1,\ldots,x_n)$ a nonzero multilinear polynomial over C in n noncommuting variables, d a nonzero derivation of R and ρ a nonzero right ideal of R. If $[d^2(f(x_1,\ldots,x_n)),f(x_1,\ldots,x_n)]\in Z(R)$ for all $x_1,\ldots,x_n\in\rho$ then $\rho C=eRC$ for some idempotent e in the socle of RC and either $f(x_1,\ldots,x_n)$ is central-valued on $eRCe$ or $eRCe$ satisfies $S_4(x_1,x_2,x_3,x_4)$ unless d is an inner derivation induced by $b\in Q$ such that $b^2=0$ and $b\rho=0$.*

PROOF. Suppose d is not a Q-inner derivation induced by an element $b\in Q$ such that $b^2=0$ and $b\rho=0$.

If $[f(\rho),\rho]\rho=0$, that is $[f(x_1,\ldots,x_n),x_{n+1}]x_{n+2}=0$ for all

$x_1, x_2, \ldots, x_{n+2} \in \rho$, then by [12, Proposition], $\rho C = eRC$ for some idempotent $e \in soc(RC)$ and $f(x_1, \ldots, x_n)$ is central-valued on $eRCe$.

So, assume that $[f(\rho), \rho]\rho \neq 0$, that is $[f(x_1, \ldots, x_n), x_{n+1}]x_{n+2}$ is not an identity for ρ and then we derive that $eRCe$ satisfies S_4. By Lemma 2.3, R is a GPI-ring and so is also Q (see [1] and [4]). By [16], Q is a primitive ring with $H = soc(Q) \neq 0$. Moreover, we may assume $[f(\rho H), \rho H]\rho H \neq 0$, otherwise by [1] and [4], $[f(\rho Q), \rho Q]\rho Q = 0$, which is a contradiction. Choose $a_1, \ldots, a_{n+2}, b_1, \ldots, b_5 \in \rho H$ such that $[f(a_1, \ldots, a_n), a_{n+1}]a_{n+2} \neq 0$ and $S_4(b_1, b_2, b_3, b_4)b_5 \neq 0$. Let $a \in \rho H$. Since H is a regular ring, there exists $e^2 = e \in H$ such that

$$eH = aH + a_1H + \cdots + a_{n+2}H + b_1H + \cdots + b_5H.$$

Then $e \in \rho H$ and $a = ea, a_i = ea_i$ for $i = 1, \ldots, n+2$, $b_i = eb_i$ for $i = 1, \ldots, 5$. Thus we have $f(eHe) = f(eH)e \neq 0$. Moreover, by [14, Theorem 2], we may also assume that

$$[[d^2(f(x_1, \ldots, x_n)), f(x_1, \ldots, x_n)], x_{n+1}]$$

is an identity for ρQ. In particular, $[[d^2(f(x_1, \ldots, x_n)), f(x_1, \ldots, x_n)], x_{n+1}]$ is an identity for ρH and so for eH. It follows that, for all $r_1, \ldots, r_{n+1} \in H$,

$$0 = [[d^2(f(er_1, \ldots, er_n)), f(er_1, \ldots, er_n)], er_{n+1}].$$

We may write $f(x_1, \ldots, x_n) = t(x_1, \ldots, x_{n-1})x_n + h(x_1, \ldots, x_n)$, where x_n never appears as last variable in any monomials of h. Let $r \in H$. Then replacing r_n with $r(1-e)$ and r_{n+1} with $r_{n+1}(1-e)$, we have

$$(5) \quad 0 = [[d^2(t(er_1, \ldots, er_{n-1})er(1-e)), t(er_1, \ldots, er_{n-1})er(1-e)], er_{n+1}(1-e)].$$

Now, we know the fact that $d(x(1-e))e = -x(1-e)d(e)$, $(1-e)d(ex) = (1-e)d(e)ex$ and $(1-e)d^2(ex(1-e))e = -2(1-e)d(e)ex(1-e)d(e)$. Thus using these facts, we have from (5),

$$\begin{aligned} 0 &= [[d^2(t(er_1, \ldots, er_{n-1})er(1-e)), t(er_1, \ldots, er_{n-1})er(1-e)], er_{n+1}(1-e)] \\ &= [d^2(t(er_1, \ldots, er_{n-1})er(1-e)), t(er_1, \ldots, er_{n-1})er(1-e)]er_{n+1}(1-e) \\ &\quad - er_{n+1}(1-e)[d^2(t(er_1, \ldots, er_{n-1})er(1-e)), t(er_1, \ldots, er_{n-1})er(1-e)] \\ &= -t(er_1, \ldots, er_{n-1})er(1-e)d^2(t(er_1, \ldots, er_{n-1})er(1-e))er_{n+1}(1-e) \\ &\quad - er_{n+1}(1-e)d^2(t(er_1, \ldots, er_{n-1})er(1-e))t(er_1, \ldots, er_{n-1})er(1-e) \\ &= t(er_1, \ldots, er_{n-1})er(1-e)d(e)t(er_1, \ldots, er_{n-1})er(1-e)d(e)r_{n+1}(1-e) \\ &\quad + er_{n+1}(1-e)d(e)t(er_1, \ldots, er_{n-1})er(1-e)d(e)t(er_1, \ldots, er_{n-1})er(1-e). \end{aligned}$$

Replacing r_{n+1} with $t(er_1, \dots, er_{n-1})er$ in the above relation, we get

$$2t(er_1, \dots, er_{n-1})er((1-e)d(e)t(er_1, \dots, er_{n-1})er)^2(1-e) = 0.$$

This implies

$$2((1-e)d(e)t(er_1, \dots, er_{n-1})er)^4 = 0$$

that is

$$2\{(1-e)d(e)t(er_1, \dots, er_{n-1})eH\}^4 = 0.$$

By [6], $(1-e)d(e)t(er_1, \dots, er_{n-1})eH = 0$ which implies

$$(1-e)d(e)t(er_1e, \dots, er_{n-1}e) = 0.$$

Since eHe is a simple Artinian ring and $t(eHe) \neq 0$ is invariant under the action of all inner automorphisms of eHe, by [5, Lemma 2],$(1-e)d(e) = 0$ and so $d(e) = ed(e) \in eH$. Thus $d(eH) \subseteq d(e)H + ed(H) \subseteq eH \subseteq \rho H$ and $d(a) = d(ea) \in d(eH) \subseteq \rho H$. Therefore, $d(\rho H) \subseteq \rho H$. Denote the left annihilator of ρH in H by $l_H(\rho H)$. Then $\overline{\rho H} = \dfrac{\rho H}{\rho H \cap l_H(\rho H)}$, a prime C-algebra with the derivation $\overline{d}$ such that $\overline{d}(\overline{x}) = \overline{d(x)}$, for all $x \in \rho H$. By assumption, we have that

$$[[\overline{d}^2 f(\overline{x_1}, \dots, \overline{x_n}), f(\overline{x_1}, \dots, \overline{x_n})], \overline{x_{n+1}}]] = 0$$

for all $\overline{x_1}, \dots, \overline{x_n} \in \overline{\rho H}$. By Theorem 1.3, either $\overline{d} = 0$ or $f(\overline{x_1}, \dots, \overline{x_n})$ is central-valued on $\overline{\rho H}$ or $\overline{\rho H}$ satisfies the standard identity $S_4(\overline{x_1}, \dots, \overline{x_4})$.

If $\overline{d} = 0$, then as in the proof of Theorem 2.4, we have $d(\rho)\rho = 0$ and hence by Lemma 2.1, d is an inner derivation induced by an element $b \in Q$ such that $b\rho = 0$. Thus for all $r_1, \dots, r_n \in \rho H$,

$$[d^2(f(r_1, \dots, r_n)), f(r_1, \dots, r_n)] = -f(r_1, \dots, r_n)^2 b^2 \in C.$$

Commuting both sides with $f(r_1, \dots, r_n)$, we obtain $f(r_1, \dots, r_n)^3 b^2 = 0$. In this case by Lemma 2.2, since $b^2 \neq 0$, $f(\rho H)\rho H = 0$. If $f(\rho H)\rho H = 0$, then $[f(\rho H), \rho H]\rho H = 0$, a contradiction.

If $f(\overline{x_1}, \dots, \overline{x_n})$ is central-valued on $\overline{\rho H}$, then we obtain that

$$[f(x_1, \dots, x_n), x_{n+1}]x_{n+2}$$

is an identity for ρ, a contradiction.

Finally, if $S_4(\overline{x_1}, \dots, \overline{x_4})$ is an identity for $\overline{\rho H}$, $S_4(x_1, \dots, x_4)x_5$ is an identity for ρH and so for ρC and this contradicts the choices of the elements $b_1, \dots, b_5 \in \rho H$. Therefore, we conclude that in any case ρC satisfies a polynomial identity, hence by [12, Proposition], there exists an idempotent $e \in Soc(RC)$ such that $\rho C = eRC$, as desired.

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