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# A Group of Automorphisms of the Rooted Dyadic Tree and Associated Gelfand Pairs

DANIELE D'ANGELI (*) - ALFREDO DONNO (*)

ABSTRACT - In this paper we study the group $G$ of automorphisms of the rooted dyadic tree generated by translation by 1 and multiplication by an odd integer $q$, showing that $G$ admits a self-similar presentation and it is isomorphic to the Baumslag-Solitar group $BS_q$. Moreover, we show that the action of $G$ on each level of the tree gives rise to a Gelfand pair.

## 1. Introduction.

Let $T$ be the infinite binary rooted tree. Denote by $L_n$ the $n$-th level of $T$, which consists of $2^n$ vertices. The root of $T$ can be identified with the group $\mathbb{Z}$, each vertex, say at level $L_n$, can be regarded as a coset of $2^n\mathbb{Z}$ in $\mathbb{Z}$. Finally, the boundary $\partial T$ corresponds to the ring of dyadic integers $\mathbb{Z}_2$ (for more details see [F]).

Let $G$ be the group of automorphisms of $T$ generated by the translation by 1 and by the multiplication by an odd integer $q$ for each vertex in $T$. Denote by $a$ and $b$ such automorphisms, respectively. The action of $G$ on $T$ is *self-similar*: we will directly prove that these automorphisms admit the following form

$$a = (1, a)\varepsilon, \quad b = (b, ba^h),$$

with $q = 2h + 1$, according to the fact that every self-similar group can be embedded in a wreath product. By using the self-similarity, we deduce that $G$ is isomorphic to the Baumslag-Solitar group $BS_q = \langle s, t : t^{-1}st = s^q \rangle$, introduced in [BS]. We mention that self-similar groups are closely con-

(*) Indirizzo degli A.: Section de Mathématiques, Université de Genève, 2-4, Rue du Lièvre, Case Postale 64, 1211 Genève 4, Suisse.
E-mail: daniele.dangeli@unige.ch alfredo.donno@unige.ch

nected with automata. For example, Bartholdi and Šunik in [BŠ] obtained the same result in the more general context of a family of solvable groups generated by finite automata.

Let $G_n$ be the finite homomorphic image of $G$ acting faithfully on $L_n$. Also denote by $K_n \leq G_n$ the parabolic subgroup which stabilizes a fixed vertex in $L_n$. This way, $L_n$ will be identified with the corresponding homogeneous space $G_n/K_n$. We then prove that $(G_n, K_n)$ is a Gelfand pair for every $n \geq 1$. In particular, we show, by direct computations involving characters, that the decomposition of the corresponding permutation representation into irreducible $G-$representations is multiplicity-free. Actually the result can also be obtained from the general theory of representations of semidirect products developed in [CST]. Finally, a formula for the relative spherical functions is given.

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## 2. Preliminaries.

Let $T$ be the binary infinite rooted homogeneous tree, that is the tree in which each vertex has two children. For every $n \geq 1$, denote by $L_n$ the $n$-th level of $T$, formed by $2^n$ vertices. Identify each vertex of $L_n$ with a word $x_0 \ldots x_{n-1}$ of length $n$ in the alphabet $X = \{0, 1\}$. Let $X^*$ be the set of all words of finite length in the alphabet $X$. The root will be represented by the empty word $\emptyset$. We will denote by $T_x$ the subtree of $T$ rooted at the vertex $x$ and isomorphic to $T$.

The set $L_n$ can be endowed with an ultrametric distance $d$, defined in the following way: if $x = x_0 \ldots x_{n-1}$ and $y = y_0 \ldots y_{n-1}$, then

$$d(x, y) = n - \max\{i : x_k = y_k, \quad \forall k \leq i\}.$$

We observe that $d = d'/2$, where $d'$ denotes the usual geodesic distance on $T$.

In this way $(L_n, d)$ becomes an ultrametric space, in particular a metric space, on which the automorphisms group $Aut(T)$ acts isometrically. Note that the diameter of $(L_n, d)$ is exactly $n$.

Each automorphism $g \in Aut(T)$ can be represented by its *labelling*. The labelling of $g \in Aut(T)$ can be realized as follows: given a vertex $x = x_0 \ldots x_{n-1} \in T$, we associate with $x$ a permutation $g_x \in S_2$ ($S_2$ denotes the symmetric group on 2 elements) that gives the action of $g$ on the two children of $x$. Formally, the action of $g$ on the vertex labelled with the word $x = x_0 \dots x_{n-1}$ is

$$x^g = x_0^{g_\emptyset} x_1^{g_{x_0}} \dots x_{n-1}^{g_{x_0 \dots x_{n-2}}}.$$

It will be used also $g(x)$ to indicate the element $x^g$. Let $G \leq Aut(T)$. We define the *stabilizer* of $x \in T$ as $Stab_G(x) = \{g \in G : g(x) = x\}$ and the stabilizer of the $n$-th level as $Stab_G(n) = \bigcap_{x \in L_n} Stab_G(x)$. Observe that $Stab_G(n)$ is a normal subgroup of $G$ of finite index for all $n \geq 1$.

An automorphism $g \in Stab_G(1)$ can be identified with the elements $g_i, i = 0, 1$ that describe the action of $g$ on the respective subtrees $T_i$. So we get the following embedding

$$\varphi : Stab_G(1) \longrightarrow Aut(T) \times Aut(T)$$

that associates to $g$ the pair $(g_0, g_1)$.

The following definitions can be found in [BGŠ].

- $G$ is *spherically transitive* if the action of $G$ on $L_n$ is transitive for every $n$.

In this case the subgroups $Stab_G(x), x \in L_n$ are conjugate each other.

- $G$ is *fractal* if $Stab_G(x)|_{T_x} \cong G$, for each $x \in T$. This is equivalent to require that the map $\varphi : Stab_G(1) \longrightarrow G \times G$ is a subdirect embedding, that is, surjective on each factor.

Clearly, a fractal group is spherically transitive if and only if it is transitive on the first level of $T$.

- $G$ is *self-similar* if for every $g \in G$, $x \in X$, there exists $g_x \in G$, $x' \in X$ such that $g(xw) = x' g_x(w)$ for all $w \in X^*$.

A self-similar group $G$ can be embedded in the wreath product $G \wr S_2 = G^2 \rtimes S_2$, so that an element $g \in G$ is represented as $g = (g_0, g_1)\sigma$, where $g_i \in G$ describe the action on the subtree $T_i$, and $\sigma \in S_2$ corresponds to the action on $L_1$. With this notation, the product between two elements $g = (g_0, g_1)\sigma$ and $h = (h_0, h_1)\tau$ is

$$gh = (g_0, g_1)(h_0, h_1)^\sigma \sigma\tau = (g_0 h_{\sigma^{-1}(0)}, g_1 h_{\sigma^{-1}(1)})\sigma\tau.$$

Observe that in the product $gh$ the automorphism $g$ acts before the automorphism $h$, to have coherence with the wreath recursion. Consider again the binary tree but from another point of view. Observe that the infinite sequence $x_0x_1x_2\ldots$ can be regarded as the dyadic number $x_02^0 + x_12 + x_22^2 + \ldots$.

Let us identify the root of the tree with the ring $\mathbb{Z}$, the vertices of the first level with $2\mathbb{Z}$ and $2\mathbb{Z}+1$ respectively. Again, the sons of $2\mathbb{Z}$ are $4\mathbb{Z}$ and $4\mathbb{Z}+2$, whereas the sons of $2\mathbb{Z}+1$ are $4\mathbb{Z}+1$ and $4\mathbb{Z}+3$. Actually, the $2^n$ vertices of the $n$-th level are, for all $n \geq 1$, the cosets of $2^n\mathbb{Z}$ in $\mathbb{Z}$.

In this way the boundary $\partial T$ of the tree is identified with the dyadic closure of $\mathbb{Z}$, which will be denoted by $\mathbb{Z}_2$.

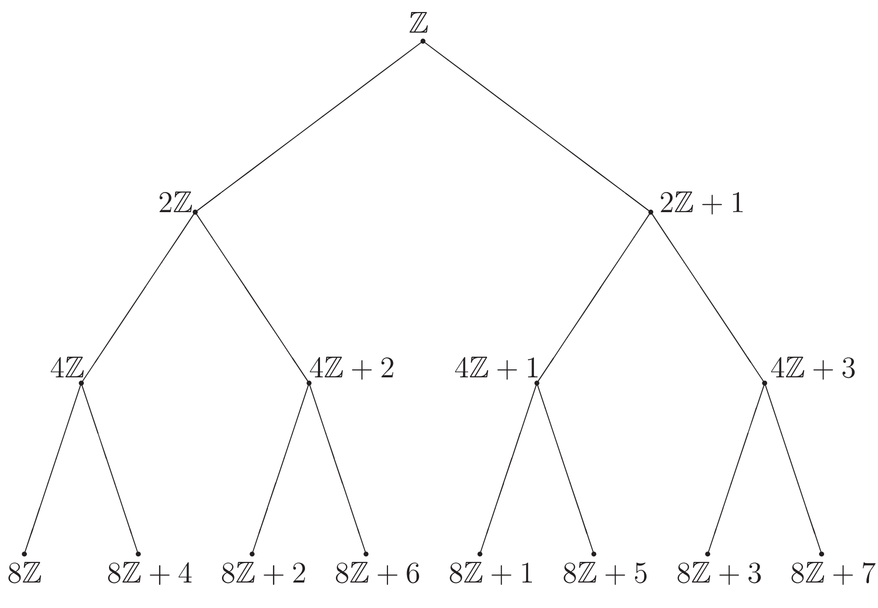


Fig. 1. – The dyadic tree.

## 3. Some automorphisms of the dyadic tree.

3.1 – *The automorphism a.*

Under this identification of the boundary of $T$ with the ring $\mathbb{Z}_2$ of dyadic integers, we introduce the automorphism $a$ of $T$ given by the translation by 1. Actually, this is the automorphism that generates the Adding Machine (see, for instance, [BGN]). If we consider the action of $a$ on the level $L_n$, then the order of $a$ is $2^n$. Therefore $a$ admits the following *self-similar* form:

$$a = (1, a)\varepsilon, \tag{1}$$

where $\varepsilon$ is the nontrivial permutation of $S_2$.

The labelling of $a$ is given in the following figure.

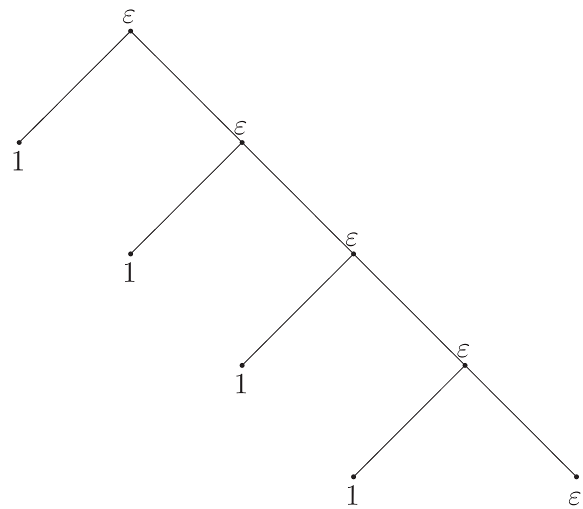


Fig. 2. – Labelling of $a$.

It is clear that the action of $a$ fixes the root and so it preserves each level $L_n$, on which it acts transitively for every $n \geq 1$.

Let us verify, by induction on $n$, that the action of the automorphism $a$ given in (1) is the translation by 1. On the first level the action of $a$ is obviously transitive, since it exchanges the vertices corresponding to the cosets $2\mathbb{Z}$ and $2\mathbb{Z}+1$, that means to perform the sum of 1 modulo $2\mathbb{Z}$.

Assume the claim true for $n-1$ and let us prove it for $n$. First consider an element of the form $2^n\mathbb{Z}+k$, with $k$ even. From the labelling of $a$ it follows that the first letter 0 is changed in 1 (that corresponds to add 1), then $a$ acts as the identity. We have:

$$2^n\mathbb{Z}+k \stackrel{\varepsilon}{\longmapsto} 2^n\mathbb{Z}+k+1 \stackrel{-1}{\longmapsto} 2^n\mathbb{Z}+k \stackrel{:2}{\longmapsto} 2^{n-1}\mathbb{Z}+\frac{k}{2}$$
$$\stackrel{id}{\longmapsto} 2^{n-1}\mathbb{Z}+\frac{k}{2} \stackrel{\cdot 2}{\longmapsto} 2^n\mathbb{Z}+k \stackrel{+1}{\longmapsto} 2^n\mathbb{Z}+k+1.$$

In fact, after applying $\varepsilon$, the element $2^n\mathbb{Z}+k$ is in the subtree $T_1$. The translation by $-1$ and the following division by 2 are made to identify the subtree $T_1$ with the whole tree $T$. Now, $a$ acts trivially. Finally, multiplication by 2 and translation by 1 give the image of the element $2^n\mathbb{Z}+k$ in $T_1$, which is $2^n\mathbb{Z}+k+1$, as desired.

A similar argument can be developed when $a$ acts on an element of the form $2^n\mathbb{Z}+k$, with $k$ odd. First we have the action of $\varepsilon$, which is the sum by $-1$, then again the action of $a$. We have:

$$2^n\mathbb{Z}+k \overset{\varepsilon}{\longmapsto} 2^n\mathbb{Z}+k-1 \overset{:2}{\longmapsto} 2^{n-1}\mathbb{Z}+\frac{k-1}{2}$$
$$\overset{a}{\longmapsto} 2^{n-1}\mathbb{Z}+\frac{k-1}{2}+1 = 2^{n-1}\mathbb{Z}+\frac{k+1}{2} \overset{\cdot 2}{\longmapsto} 2^n\mathbb{Z}+k+1.$$

In this case, after applying $\varepsilon$, the element $2^n\mathbb{Z}+k$ is in the subtree $T_0$. The division by 2 is made to identify $T_0$ with the whole tree $T$. Now $a$ acts again as $a$. By induction, this action is the translation by 1. Finally, multiplication by 2 gives the image of the element $2^n\mathbb{Z}+k$ in $T_0$, which is $2^n\mathbb{Z}+k+1$.

3.2 – *The automorphism b.*

Let us introduce the automorphism $b$ of $T$ given by multiplication by an odd number $q=2h+1$ different from 1. Since $q$ is odd, $b$ preserves each level and in particular it fixes the vertex $2^n\mathbb{Z}$ for every $n\geq 1$. Moreover $q$ has dyadic norm 1, so it is invertible in the ring of dyadic integers, but its inverse does not belong to $\mathbb{Z}$ (except in the case $q=-1$). So, it is not possible to identify the inverse of this automorphism with the multiplication by an integer. This is clear by considering that the order of the multiplication by $q$ on $L_n$ is not the same for all $n\geq 1$.

On $L_1$ multiplication by $q$ is trivial.

On $L_2$ we have $b^2=1$ for each $q$ odd.

On $L_3$ we still have $b^2=1$ for each $q$ odd. This is true because

$$8 \mid (q^2-1) \quad \text{for each } q \text{ odd.}$$

To compute the order of the multiplication by $q$ on each level $L_n$, with $n\geq 3$, we have to study the equation $q^{2^t}\equiv 1(2^n)$. Consider the following decomposition:

$$q^{2^t}-1=(q^{2^{t-1}}+1)(q^{2^{t-2}}+1)(q^{2^{t-3}}+1)\cdots(q^2+1)(q+1)(q-1).$$

Since $q^2\equiv 1(4)$ for every $q$ odd, we get that each of the factors above, except the last two, are divisible by 2 but are not divisible by higher powers of 2. Moreover, the fact that $8\mid(q^2-1)$ for each $q$ odd, implies that for all

$n \geq 3$ we have $2^n \mid q^{2^{n-2}} - 1$, and so $q^{2^{n-2}} \equiv 1(2^n)$ for all $n \geq 3$. Actually, there exists some $q$ whose order, at the level $L_n$, is strictly less then $2^{n-2}$. In fact, from the previous argument it follows that if

$$q^2 \equiv 1(2^k) \quad \text{but} \quad q^2 \not\equiv 1(2^{k+1}),$$

then, at the level $L_n$, the order of multiplication by $q$ is exactly $2^{n-k+1}$, for all $n \geq k$. Also, we have $k \geq 3$. To describe the action of $b$ on $L_n$, $n<k$, we distinguish two cases.

If $q \equiv 1(4)$, then $2^{k-1} \mid q - 1$ and so $b = 1$ on each level $L_n$, $n<k$. If $q \equiv -1(4)$, then $2^{k-1} \mid q + 1$ and so $b^2 = 1$.

As an example, let us consider the case $q = 3$. Now $q^2 - 1$ is divided by 8 but not by 16 and this implies that the order of the multiplication by 3 is $2^{n-2}$ on $L_n$ for each $n \geq 3$. In $L_1$ one has $b = 1$, in $L_2$ $b^2 = 1$.

We can now investigate the order of the orbits of vertices of $L_n$ under the action of $b$.

Let us consider the sphere of center $2^n\mathbb{Z}$ and radius $r$ with respect to the ultrametric distance already defined. It contains $2^{r-1}$ vertices for $r \geq 1$. The first vertex from the left belonging to this sphere is the element $2^n\mathbb{Z} + 2^{n-r}$. Assume that $2^l$ is the period of this element under the action of $b$, so that we get

$$2^{n-r}q^{2^l} \equiv 2^{n-r}(2^n).$$

This implies $q^{2^l} \equiv 1(2^r)$. Therefore, if the order of $b$ on $L_n$ is $2^{n-k+1}$ for every $n \geq k$, it must be

$$l = r - k + 1,$$

with $r \geq k$. So, the orbit of the element $2^n\mathbb{Z} + 2^{n-r}$ is exactly of length $2^{r-k+1}$.

It is easy to verify that the remaining vertices of $L_n$ belonging to the sphere of radius $r$ have the form

$$2^n\mathbb{Z} + 2^{n-r} + t2^{n-r+1}, \quad t = 1, \ldots, 2^{r-1} - 1.$$

We want to prove that they have the same order $2^{r-k+1}$. Consider the equation

$$(2^{n-r} + t2^{n-r+1})q^{2^l} \equiv 2^{n-r} + t2^{n-r+1}(2^n).$$

Dividing by $2^{n-r}$, we get

$$q^{2^l} + 2tq^{2^l} \equiv 1 + 2t(2^r),$$

from which

$$(q^{2^l}-1)(2t+1)\equiv 0(2^r).$$

Since $2t+1$ is odd, we have again the equation $q^{2^l}\equiv 1(2^r)$ and so $l=r-k+1$, if $q$ has order $2^{n-k+1}$ modulo $2^n$.

Hence, the sphere of radius $r$ decomposes, under the multiplication by $q$, in $2^{k-2}$ orbits of length $2^{r-k+1}$, for all $r\geq k$. For $r<k$, from what observed, the sphere of radius $r$ decomposes into orbits of length 1 or 2.

PROPOSITION 3.1. *The automorphism $b$ admits the self-similar representation*

$$b=(b,ba^h),$$

*where $q=2h+1$.*

PROOF. Let us prove it by induction on $n$.

For $n=1$, multiplication by $q$ coincides with the identity, as in the self-similar form of $b$.

Assume the result true for $n-1$ and let us show it for $n$. First consider an element of the form $2^n\mathbb{Z}+k$, with $k$ even. We have:

$$2^n\mathbb{Z}+k\overset{:2}{\longmapsto}2^{n-1}\mathbb{Z}+\frac{k}{2}\overset{b}{\longmapsto}2^{n-1}\mathbb{Z}+\frac{qk}{2}\overset{\cdot 2}{\longmapsto}2^n\mathbb{Z}+qk.$$

In fact the element $2^n\mathbb{Z}+k$ belongs to $T_0$. Division by 2 is made to identify the subtree $T_0$ with the whole tree $T$. Now, $b$ acts again as $b$ and this action is, by induction, the multiplication by $q$. Finally, multiplication by 2 gives the imagine of $2^n\mathbb{Z}+k$ in $T_0$, that is $2^n\mathbb{Z}+qk$, as desired.

A similar argument can be developed when $b$ acts on an element of the form $2^n\mathbb{Z}+k$, with $k$ odd. In this case, we have:

$$2^n\mathbb{Z}+k\overset{-1}{\longmapsto}2^n\mathbb{Z}+k-1\overset{:2}{\longmapsto}2^{n-1}\mathbb{Z}+\frac{k-1}{2}\overset{b}{\longmapsto}2^{n-1}\mathbb{Z}+\frac{q(k-1)}{2}$$

$$\overset{a^h}{\longmapsto}2^{n-1}\mathbb{Z}+\frac{q(k-1)}{2}+h=2^{n-1}\mathbb{Z}+\frac{q(k-1)}{2}+\frac{q-1}{2}=2^{n-1}\mathbb{Z}+\frac{qk-1}{2}$$

$$\overset{\cdot 2}{\longmapsto}2^n\mathbb{Z}+qk-1\overset{+1}{\longmapsto}2^n\mathbb{Z}+qk.$$

In this case, the element $2^n\mathbb{Z}+k$ is in the subtree $T_1$. The translation by $-1$ and the following division by 2 are made to identify $T_1$ with the whole tree $T$. Now, $b$ acts as $ba^h$. By induction, the action of $b$ is multiplication by $q$ and the following action of $a^h$ is the translation by $h$. The multiplication by 2 and the sum of 1 give the final image of the element $2^n\mathbb{Z}+k$ in $T_1$, that is $2^n\mathbb{Z}+qk$. □

3.3 – *The group* $G_q$.

It is possible to verify that the automorphism group $G = G_q$ of the dyadic tree generated by the translation by 1 and by the multiplication by an odd integer $q = 2h+1$ is a homomorphic image of the solvable Baumslag-Solitar group

$$BS_q = \langle s, t : t^{-1}st = s^q \rangle.$$

In fact, this relation holds for every level $L_n$, as we are going to prove.

We have

$$a = (1, a)\varepsilon, \quad b = (b, ba^h), \quad b^{-1} = (b^{-1}, a^{-h}b^{-1})$$

and so

$$b^{-1}ab = (a^h, a^{-h}b^{-1}ab)\varepsilon.$$

For $n = 1$ the relation is satisfied for any $q$ odd, because both the elements $b^{-1}ab$ and $a^q$ exchange the vertices in $L_1$.

Suppose, now, that the result is true for $n-1$ and let us show it for $n$. We have

$$b^{-1}ab = (a^h, a^{-h}b^{-1}ab)\varepsilon = (a^h, a^{-h}a^q)\varepsilon = (a^h, a^{-h+2h+1})\varepsilon =$$

$$= (a^h, a^{h+1})\varepsilon = a^{2h+1} = a^q,$$

where the second equality follows by induction, and so the relation is proved.

Actually, the two groups coincide. In fact, for $q \neq -1$, the automorphisms $a$ and $b$ that we defined have infinite order. On the other hand, the following lemma holds.

LEMMA 3.2. *Let* $BS_q = \langle s, t : t^{-1}st = s^q \rangle$ *the Baumslag-Solitar group. Then the order of at least one of* $s$ *or* $t$ *in any proper homomorphic image of* $BS_q$ *must be finite.*

PROOF. Consider the factor group $BS_q/N$, where $N$ is any proper normal subgroup of $BS_q$. If $s^k$ or $t^h$ belong to $N$ for some $k, h$ not trivial, then the claim is true. Using the following generalized relations

$$t^{-k}st^k = s^{q^k}, \quad t^{-1}s^kt = s^{kq}, \quad t^{-k}s^mt^k = s^{mq^k}$$

and suitable conjugations it is possible to write any word with occurrences of both $s$ and $t$ as a word of the form $s^mt^k$. Suppose that such an element belongs to $N$, this implies that $t^{-k}s^mt^{2k} = s^{mq^k}t^k \in N$. So the element $s^mt^k(s^{mq^k}t^k)^{-1} = s^{m(1+q^k)}$ is in $N$ and this completes the proof. □

So we get the following

THEOREM 3.3. *The automorphisms group $G = G_q$ of the dyadic tree generated by the automorphism $a$ which is the sum of* 1 *and the automorphism $b$ which is the multiplication by $q$ is isomorphic to the solvable Baumslag-Solitar group:*

$$BS_q = \langle a, b : b^{-1}ab = a^q \rangle.$$

Note that the group $G$ is a self-similar group whose action on the tree $T$ is spherically transitive. Since

$$a^2 = (a, a), \quad b = (b, ba^h), \quad ba^{-2h} = (ba^{-h}, b),$$

$G$ is a fractal group.

REMARK 3.4. The case $q = -1$ yields the infinite dihedral group. In fact, the group we get is

$$G = \langle a, b : b^{-1}ab = a^{-1} \rangle,$$

and this is exactly the presentation of the infinite dihedral group (see, for example, [BGN]). It is obvious that in this case we have $b^2 = 1$ and the generators for $G$ are

$$a = (1, a)\varepsilon, \quad b = (b, ba^{-1}).$$

REMARK 3.5. In [BŠ] L. Bartholdi and Z. Šunik obtained these groups of automorphisms as groups generated by the automaton $S_{m,n}$ in the more general case of the multiplication by $m$ in the ring $\mathbb{Z}_n$ of $n$-adic integers, with $(m, n) = 1$.

## 4. Gelfand pairs

### 4.1 – *Some definitions*

We present now some basic elements of the theory of finite Gelfand pairs that will be applied in what follows (see, for example, [CST1], [CST2] and [D] for many applications).

Let $G$ be a finite group and let $K \leq G$ be a subgroup, denote $X = G/K = \{gK : g \in G\}$ the associated homogeneous space. In this way $G$ acts transitively on $X$ and $K$ is the stabilizer of the element $x_0 \equiv K \in X$. The space $L(G) = \{f : G \longrightarrow \mathbb{C}\}$ is an algebra with respect to the convolution

$$(f_1 * f_2)(g) = \sum_{h \in G} f_1(gh)f_2(h^{-1}),$$

for all $f_1, f_2 \in L(G)$ and $g \in G$. The subspace of $L(G)$ consisting of the functions $K$-invariant to the right (i.e. $f(gk) = f(g)$ for all $g \in G$, $k \in K$) is a subalgebra that can be identified with $L(X) = \{f : X \longrightarrow \mathbb{C}\}$.

This subalgebra can be endowed with a Hilbert space structure by the scalar product $\langle f_1, f_2 \rangle = \sum_{x \in X} f_1(x)\overline{f_2(x)}$, for all $f_1, f_2 \in L(X)$. Analogously, the subspace of $L(X)$ consisting of the functions $K$-invariant is a subalgebra that we identify with $L(K \backslash G/K) = \{f : G \longrightarrow \mathbb{C} : f(kgk') = = f(g), \text{ for all } k, k' \in K, g \in G\}$. The group $G$ acts on $L(X)$ in the following way: $gf(x) = f^g(x) = f(g^{-1}x)$.

We will call $(G, K)$ a *Gelfand pair* if the algebra $L(K \backslash G/K)$ is commutative. The following are equivalent:

(1) $(G, K)$ is a Gelfand pair;

(2) the decomposition of the space $L(X)$ into irreducible submodules under the action of $G$ is multiplicity-free, i.e. each irreducible submodule occurs with multiplicity 1;

(3) given an irreducible representation $V$ of $G$, the dimension of the subspace of $K-$invariant vectors $V^K = \{v \in V : kv = v \ \ \forall k \in K\}$ is less or equal to 1, and it is 1 if and only if $V \leq L(X)$.

The reader is referred to [CST1] for details.

A particular example of Gelfand pair is given by the *symmetric Gelfand pairs*. A finite group $G$ and a subgroup $K \leq G$ constitute a symmetric Gelfand pair if for every $g \in G$ the condition $g^{-1} \in KgK$ is satisfied. In fact one can directly verify that this condition implies that the algebra

$L(K\backslash G/K)$ is commutative. Moreover, it is easy to verify that the condition $g^{-1} \in KgK$ for every $g \in G$ is equivalent to require that, for all $x, y \in X$, the pairs $(x, y)$ and $(y, x)$ are in the same orbit under the diagonal action of $G$ on $X \times X$.

For example, if we consider the finite dihedral group $D_n = C_n \rtimes C_2$, it is easy to verify that $(D_n, C_2)$ is a symmetric Gelfand pair. The condition $g^{-1} \in C_2 g C_2$, for every $g \in D_n$, is trivially satisfied for this group.

EXAMPLE 4.1. Suppose that the finite group $G$ acts on a finite metric space $(X, d)$ isometrically and with a 2-points homogeneous action, i.e. in such a way that for all $x, y, x', y' \in X$ such that $d(x, y) = d(x', y')$ there exists $g \in G$ such that $gx = x'$ and $gy = y'$. Let $K \leq G$ be the stabilizer of an element $x_0 \in X$. It follows from the previous argument that $(G, K)$ is a symmetric Gelfand pair.

We can observe that in this case the $K$-orbits (which can be identified with the double cosets of $K$ in $G$) are the spheres

$$\Lambda_j = \{x \in X : d(x_0, x) = j\}.$$

Hence, a function $f \in L(X)$ is $K$-invariant if and only if it is constant on the spheres $\Lambda_j$.

If $(G, K)$ is a Gelfand pair and $L(X) = \bigoplus_{i=0}^{n} V_i$ is a decomposition of $L(X)$ into irreducible submodules, then for each $i = 0, 1, \ldots, n$ there exists a unique (up to normalization) bi-$K$-invariant function $\phi_i$ whose $G$-translates generate the $V_i$'s. In particular, we will require that these functions take value exactly 1 on the element $x_0 \in X$ stabilized by $K$. The functions $\phi_i$, $i = 0, 1, \ldots, n$ are called *spherical functions* and they form a basis for the algebra $L(K\backslash G/K)$. So, the number of $K$-orbits under the action of $G$ on $X$ is equal to the number of spherical functions. A different basis for the algebra $L(K\backslash G/K)$ is given by the characteristic functions of the $K$-orbits.

A spherical function $\phi$ can be also defined as a bi-$K$-invariant function on $G$ satisfying the following properties:

(1) $\phi * f = ((\phi * f)(1_G))\phi$ for every $f \in L(K\backslash G/K)$;
(2) $\phi(1_G) = 1$.

As an example, the function $\phi_0 \equiv 1$ is a spherical function: this corresponds to the fact that the trivial representation always occurs in the decomposition of the space $L(X)$ into irreducible submodules.

4.2 – *Gelfand pairs associated with G.*

From now on, we will apply the theory of Gelfand pairs to the groups we introduced in Section 3.

Let $q = 2h + 1$. Then, the multiplication by $q$ has order $2^{n-k+1}$ for each level $L_n$, $n \geq k$. With $n$ fixed, the group $G_n = G/Stab_G(n)$ which acts on the $n$-th level of $T$, is given by

$$G_n = C_{2^n} \rtimes C_{2^{n-k+1}},$$

where $\langle a \rangle = C_{2^n}$ and $\langle b \rangle = C_{2^{n-k+1}}$, since $a^{2^n}, b^{2^{n-k+1}} \in Stab_G(n)$, and the action is defined as $b^{-1}ab = a^q$. Observe that $C_{2^n}$ acts transitively on $L_n$. Moreover $C_{2^{n-k+1}} = Stab_{G_n}(2^n\mathbb{Z})$ for all $n \geq k$. This parabolic subgroup will be denoted by $K_n$.

We want to show that $(G_n, K_n)$ are Gelfand pairs. In order to prove it we use the characterization of Gelfand pairs about the decomposition of the space $L(L_n)$ into irreducible submodules, by showing that this decomposition is *multiplicity-free*.

We distinguish two cases.

CASE $q \equiv 1(4)$.

For every $n<k$ we have $G_n = C_{2^n}$, $K_n = \{1\}$ and so in the decomposition of the space $L(L_n)$ all the irreducible representations (of dimension 1) of $C_{2^n}$ occur with multiplicity 1. The pairs $(C_{2^n}, 1)$ are not symmetric (except in the case $n = 1$) Gelfand pairs.

Let now $n \geq k$. We have

$$G_n = C_{2^n} \rtimes C_{2^{n-k+1}}.$$

The space $L(L_n)$ decomposes under the action of $C_{2^n}$ as

$$L(L_n) = \bigoplus_{j=0}^{2^n-1} V_j, \tag{2}$$

where the representation $V_j$ is associated with the character $\chi_j$ defined by $\chi_j(a) = \omega^j$, with $\omega = e^{\frac{2\pi i}{2^n}}$. We want to study how the automorphism $b$ acts on the $V_j$'s. By using the relations holding in the group we get

$$a(bV_j) = ba^qV_j = \omega^{qj}bV_j$$

and so $bV_j = V_{qj}$. We want to understand which of these subspaces are invariant under this action. The equation $qj \equiv j(2^n)$ gives $j(q-1) \equiv 0(2^n)$, that implies $j \equiv 0(2^{n-k+1})$. The $V_j$'s such that $j$ satisfies this property are exactly $2^{k-1}$ and they correspond to the vertices of the spheres of radius $r$, with $r<k$. They are in fact irreducible submodules of dimension 1 in the representation of $G_n$ on $L(L_n)$. More generally the $V_j$'s such that $j \equiv 0(2^{n-r})$ but $j \not\equiv 0(2^{n-r+1})$ correspond to the vertices of the sphere of radius $r$ in $L_n$ and in fact they are $2^{r-1}$ in number. They are the $V_j$'s with $j = 2^{n-r} + t2^{n-r+1}$, $t = 0, \ldots, 2^{r-1} - 1$. Since we know that the order of $b$ on the sphere of radius $r$ is $2^{r-k+1}$, we deduce that these $V_j$'s will form $2^{k-2}$ irreducible submodules for $G_n$ of dimension $2^{r-k+1}$. As an example, the orbit

$$V_{2^{n-r}} \overset{b}{\longmapsto} V_{q2^{n-r}} \overset{b}{\longmapsto} V_{q^2 2^{n-r}} \overset{b}{\longmapsto} \ldots \overset{b}{\longmapsto} V_{q^{2^{r-k+1}-1}2^{n-r}}$$

generates the irreducible representation for $G_n$

$$V_{2^{n-r}} \oplus V_{q2^{n-r}} \oplus V_{q^2 2^{n-r}} \oplus \ldots \oplus V_{q^{2^{r-k+1}-1}2^{n-r}}.$$

The matrices corresponding to this representation are

$$a \longmapsto \begin{pmatrix} \omega^{2^{n-r}} & 0 & \cdots & 0 \\ 0 & \omega^{q2^{n-r}} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \omega^{q^{2^{r-k+1}-1}2^{n-r}} \end{pmatrix},$$

$$b \longmapsto \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 1 & 0 & & 0 \\ 0 & \ddots & \ddots & \vdots \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Let us call $\psi$ the character of this representation. We have

$$\psi(a) = \chi_{2^{n-r}}(a) + \chi_{q2^{n-r}}(a) + \cdots + \chi_{q^{2^{r-k+1}-1}2^{n-r}}(a) = 0,$$

since the $2^n$-th roots of 1 which occur in the sum are pairwise opposite. Moreover $\psi(b) = 0$. Denote by $\psi_d$, $d = 1, \ldots, 2^{k-2}$ the characters of the $2^{k-2}$ irreducible representations associated with the sphere of radius $r$. In order to prove that they are pairwise non isomorphic, we present a suitable element of $G_n$ on which the characters $\psi_d$ take different values, for each $d$.

Let us consider the element $a^{2^{r-k+1}} \in G_n$. We observe that

$$(\omega^{2^{n-r}})^{2^{r-k+1}} = \omega^{2^{n-k+1}} = e^{\frac{2\pi i 2^{n-k+1}}{2^n}}.$$

Moreover we have $2^{n-k+1} \equiv q2^{n-k+1}(2^n)$, since $1 \equiv q(2^{k-1})$. The same computation holds for the following powers of $q$. Hence, all the powers of $\omega$ which occur in the diagonal of the matrix associated with $a^{r-k+1}$ correspond to the same angle $\dfrac{2\pi 2^{n-k+1}}{2^n} = \dfrac{2\pi}{2^{k-1}}$.

Let us consider now any element $2^{n-r} + t2^{n-r+1}$ in the sphere of radius $r$. We have

$$(\omega^{2^{n-r}+t2^{n-r+1}})^{2^{r-k+1}} = \omega^{2^{n-k+1}+t2^{n-k+2}}$$

and

$$q(2^{n-r} + t2^{n-r+1})2^{r-k+1} \equiv 2^{n-k+1} + t2^{n-k+2}(2^n),$$

as one can easily prove by using the property $q \equiv 1(2^{k-1})$. We want to study under which condition the elements $2^{n-r} + t2^{n-r+1}$ and $2^{n-r} + s2^{n-r+1}$ belong to the same orbit under the multiplication by $q$. We get the equation

$$2^{n-k+1} + t2^{n-k+2} \equiv 2^{n-k+1} + s2^{n-k+2}(2^n),$$

which gives $t \equiv s(2^{k-2})$.

So, the elements

$$V_{2^{n-r}}, V_{2^{n-r}+2^{n-r+1}}, V_{2^{n-r}+2\cdot 2^{n-r+1}}, \ldots, V_{2^{n-r}+(2^{k-2}-1)2^{n-r+1}}$$

belong to the $2^{k-2}$ different orbits associated to the sphere of radius $r$. The angles corresponding to the respective powers of $\omega$ are

$$\frac{2\pi}{2^{k-1}} + 2u\frac{2\pi}{2^{k-1}}, \quad u = 0, 1, \ldots, 2^{k-2} - 1.$$

For $u = 2^{k-2} - 1$ we get the angle $-\frac{2\pi}{2^{k-1}}$ and so the angles associated with these orbits are pairwise different.

We have shown that the characters $\psi_d$ take different values on the same element $a^{2^{r-k+1}}$ and so the associated representations are not isomorphic, as required.

Case $q \equiv -1(4)$.

For $n = 1$ the group $G_1$ is the cyclic group $C_2$ and we get the symmetric Gelfand pair $(C_2, 1)$.

For $2 \leq n \leq k-1$ we have $b^2 = 1$ and so

$$G_n = C_{2^n} \rtimes C_2,$$

which is the dihedral group of order $2^{n+1}$. The pairs $(G_n, K_n)$ are symmetric Gelfand pairs (as we observed above).

Let $n \geq k$. Let the $V_j$'s be the spaces given in (2), with the relation $bV_j = V_{qj}$. In this case the subspaces which are invariant under the action of $b$ are $V_0$ and $V_{2^{n-1}}$, as one can get from the equation $qj \equiv j(2^n)$ using that

$q \equiv 1(2)$ but $q \not\equiv 1(4)$. The spheres of radius $1<r<k$ decompose into $2^{r-2}$ irreducible submodules of dimension 2 and each of them is the direct sum $V_j \oplus V_{-j}$, with $j \equiv 0(2^{n-r})$ but $j \not\equiv 0(2^{n-r+1})$ and they are non isomorphic.

Let $r \geq k$. We distinguish two subcases.

Let $q = 4h - 1$ with $h = 2x + 1$. In this case necessarily $k = 3$, since $(q-1)(q+1) = 8(2x+1)(4x+1)$, and so the sphere of radius $r$ decomposes exactly into two orbits under the action of $b$. Analogously to the case $q \equiv 1(4)$, we want to write a certain power of $a$ on which the characters of these two representations take different values to deduce that they are non isomorphic.

Let us consider the element $a^{2^{r-3}}$. We observe that

$$(\omega^{2^{n-r}})^{2^{r-3}} = \omega^{2^{n-3}} = e^{\frac{2\pi i 2^{n-3}}{2^n}} = e^{\frac{i\pi}{4}}.$$

Moreover

$$(\omega^{q2^{n-r}})^{2^{r-3}} = \omega^{q2^{n-3}} = e^{\frac{2\pi i q 2^{n-3}}{2^n}} = e^{\frac{iq\pi}{4}}.$$

Now $\frac{q\pi}{4} = \frac{4h\pi}{4} - \frac{\pi}{4} \equiv \pi - \frac{\pi}{4}(2\pi)$, since $h$ is odd. The same holds for the following powers of $q$.

Viceversa, the element $\omega^{-2^{n-r}}$ raised to the power $2^{r-3}$ gives the angle $-\frac{\pi}{4}$. As before, one can show that the element $\omega^{-q2^{n-r}}$ raised to the same power $2^{r-3}$ gives the angle $\pi + \frac{\pi}{4}$. The same argument can be developed for the following powers of $q$. It is clear that the characters associated with these two representations take opposite nonzero values on the element $a^{2^{r-3}} \in G_n$ and this completes the proof in this first case.

Now let $q = 4h - 1$, with $h = 2x$. Observe that

$$q^2 - 1 = (4h-2)(4h) = 8h(2h-1)$$

and so $2^{k-3} \mid h$. In this case we consider the element $a^{2^{r-k}}$. We have

$$(\omega^{2^{n-r}})^{2^{r-k}} = \omega^{2^{n-k}} = e^{\frac{2\pi i 2^{n-k}}{2^n}} = e^{\frac{2\pi i}{2^k}}.$$

Moreover

$$(\omega^{q2^{n-r}})^{2^{r-k}} = \omega^{q2^{n-k}} = e^{\frac{2\pi i q 2^{n-k}}{2^n}} = e^{\frac{2\pi q i}{2^k}}.$$

Now observe that

$$\frac{2\pi q}{2^k} = \frac{2\pi(4h-1)}{2^k} = \frac{2\pi 4 \cdot 2^{k-3} m}{2^k} - \frac{2\pi}{2^k} = m\pi - \frac{2\pi}{2^k} \equiv \pi - \frac{2\pi}{2^k}(2\pi).$$

The last equality is true since the expression of $h$ implies that $m$ is odd. A similar argument can be developed for the following powers of $q$. So the angles $\frac{2\pi}{2^k}$ and $\pi - \frac{2\pi}{2^k}$ alternate in the orbit containing $V_{2^{n-r}}$.

Consider now any element $2^{n-r} + t2^{n-r+1}$, with $t = 0, \dots, 2^{r-1} - 1$. Its $2^{r-k}$-th power gives an angle equal to $\frac{2\pi}{2^k}$ if the equation

$$2^{n-k} \equiv 2^{n-k} + t2^{n-k+1}(2^n),$$

which implies $t \equiv 0(2^{k-1})$, is satisfied.

On the other hand the $2^{r-k}$-th power gives an angle equal to $\pi - \frac{2\pi}{2^k}$ if the equation

$$2^{n-k} + 2^{n-k} + t2^{n-k+1} \equiv 2^{n-1}(2^n),$$

which gives $1 + t \equiv 2^{k-2}(2^{k-1})$, is satisfied.

The values of $t$ modulo $2^{r-1}$ that satisfy these two equations are $\frac{2 \cdot 2^{r-1}}{2^{k-1}} = 2^{r-k+1}$, as many as the order of an orbit. So the elements whose $2^{r-k}$th power gives $\frac{2\pi}{2^k}$ or $\pi - \frac{2\pi}{2^k}$ are exactly those which satisfy the two equations.

Consider now the two equations

$$2^{n-k} + t2^{n-k+1} \equiv 2^{n-k} + s2^{n-k+1}(2^n)$$

and

$$2^{n-k} + t2^{n-k+1} + 2^{n-k} + s2^{n-k+1}(2^n) \equiv 2^{n-1}(2^n).$$

Their solutions are, respectively,

$$t \equiv s(2^{k-1}) \quad \text{and} \quad t \equiv 2^{k-2} - 1 - s(2^{k-1}).$$

So, if we fix a value of $t$ modulo $2^{r-1}$ not satisfying the first two equations, it is possible to find, by using these two new equations, $2^{r-k+1}$ values which will form a new orbit, whose corresponding angle will be different from $\frac{2\pi}{2^k}$ and $\pi - \frac{2\pi}{2^k}$.

This way we get $2^{k-2}$ orbits in the decomposition of the sphere of radius $r$, all of them corresponding to different angles and so the corresponding $2^{k-2}$ submodules have characters which take different values on $a^{2^{r-k}}$ and so they are non isomorphic.

Consider now the general case of any odd $q$ to show that the Gelfand pairs considered in this paper are non symmetric for every $n \geq k$. We recall that the pair $(G_n, K_n)$ is symmetric if the condition $g^{-1} \in K_n g K_n$ is satisfied for every $g \in G_n$.

Let us choose $g = a$ and consider $K_n = \langle b : b^{2^{n-k+1}} = 1 \rangle$. We can restrict our attention on the elements of the form $b^{-i}ab^i$. In fact the elements of the form $b^iab^j$ with $i \neq -j$ give elements with occurrences of both $a$ and $b$ which are clearly different from $a^{-1}$. Since $b^{-i}ab^i = a^{q^i}$, we have to study the equation

$$q^i \equiv -1(2^n), \tag{3}$$

with $i = 0, \ldots, 2^{n-k+1} - 1$. It is clear that in the case $i = 0$ the equation is satisfied only in the trivial case $n = 1$, when $G_n = C_2$. Also in the case already developed $q = -1$ this equation is satisfied by $i = 1$.

Consider now the equation (3). If $2^n$ divides $q^i + 1$, then $2^{n+1}$ divides $(q^i + 1)(q^i - 1) = q^{2i} - 1$. So one has $q^{2i} \equiv 1(2^{n+1})$. The smallest nonzero exponent satisfying this equation is, as we know, $2^{n-k+2}$. But it must be $2i \leq 2^{n-k+2} - 2$. This completes the proof. Collecting all the results obtained in this section we have the following

THEOREM 4.2. *Let $q$ be an odd integer such that $q^2 \equiv 1(2^k)$ but $q^2 \not\equiv 1(2^{k+1})$, then, for every $n \geq k$, $G_n = G/Stab_G(n) = C_{2^n} \rtimes C_{2^{n-k+1}}$ and $K_n = Stab_{G_n}(2^n\mathbb{Z}) = C_{2^{n-k+1}}$. The associated pairs $(G_n, K_n)$ are non symmetric Gelfand pairs.*

*For $n < k$ and $q \equiv 1(4)$ we get $G_n = C_{2^n}$ and $K_n = \{1\}$, which are non symmetric (except in the case $n = 1$) Gelfand pairs.*

*For $1 < n < k$ and $q \equiv -1(4)$ we get $G_n = D_{2^n}$ and $K_n = C_2$, which are symmetric Gelfand pairs. For $n = 1$ we have the symmetric Gelfand pair $(C_2, 1)$.*

## 4.3 – *Final remarks.*

In [CST] the authors show in a different way that the pairs $(G_n, K_n)$ introduced above are Gelfand pairs by using the general theory of the Gelfand pairs associated to the semidirect products.

Let $G = N \rtimes H$ be a finite group and let $K \leq N$ be a subgroup of $N$ invariant under the action of $H$ and such that $(N, K)$ is a Gelfand pair. Set $X = N/K$ and let $L(X) = \bigoplus_{i=0}^{n} V_i$ be the multiplicity-free decomposition of $L(X)$ into irreducible $N-$submodules and let $\phi_i \in V_i$ be the corresponding spherical functions. The map $nK \longmapsto nKH$ is a bijection between $N/K$ and

$G/KH$. The action of $G$ on $X = G/KH$ can be defined as $nh(n'KH) = nhn'h^{-1}KH$ and the induced action on $L(X)$ is, as usual, $f^g(x) = f(g^{-1}x)$. It is easy to check that if $V_i$ is any irreducible $N$-invariant submodule in $L(X)$, then $hV_i$ is still $N$-invariant and irreducible, for every $h \in H$, in other words $H$ permutes the $V_i$'s.

So the decomposition of $L(X)$ under the action of $G$ into irreducible submodules is

$$L(X) = \bigoplus_{j=0}^{r} W_j, \tag{4}$$

where $W_j = \bigoplus_{i:V_i \in \Gamma_j} V_i$ and $\Gamma_j$ denotes, for $j = 0, \dots, r$, the $H$-orbits on $\{V_0, V_1, \dots, V_n\}$. The $W_j$'s are pairwise non isomorphic because the restrictions to $N$ of the representations $W_j$ and $W_{j'}$ decompose into non isomorphic submodules for $j \neq j'$.

Hence the decomposition (4) is multiplicity-free and this implies that $(G, KH)$ is a Gelfand pair. The corresponding spherical functions can be easily computed from the $\phi_i$'s and they are given by

$$\Phi_j = \frac{1}{|\Gamma_j|} \sum_{i:V_i \in \Gamma_j} \phi_i = \frac{1}{|H|} \sum_{h \in H} h\phi_i.$$

In the case of the groups introduced above we have $N = C_{2^n}$, $K = \{1\}$ and $H = C_{2^{n-k+1}}$. The $V_i$'s are the submodules in which decomposes the space $L(L_n)$ under the action of $C_{2^n}$ whose spherical functions are exactly the characters of $C_{2^n}$. The $W_j$'s are the orbits of the spaces $V_i$'s under the automorphism $b$ with $bV_i = V_{qi}$.

The previous argument guarantees that $(C_{2^n} \rtimes C_{2^{n-k+1}}, C_{2^{n-k+1}})$ is a Gelfand pair and yields the associated spherical functions.

In Section 4.2 we preferred to give an explicit calculation of the irreducible subspaces of $L(L_n)$ under the action of $G_n$ using only congruences modulo $2^n$ and identifying the spaces $V_j$'s with the vertices of the $n$-th level of the dyadic tree.

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