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Hopf π-Crossed Biproduct and Related Coquasitriangular Structures

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ABSTRACT - Let π be a group and $H = (\{H_\alpha\}, \Delta, \varepsilon, S)$ a Hopf π-coalgebra (not necessarily associative), $\alpha \in \pi$. Let A be an algebra and a coalgebra. We find the necessary and sufficient conditions on the π-crossed product $A\#^\pi_\sigma H$ with suitable comultiplication and counit to be a Hopf π-coalgebra. Moreover, the necessary and sufficient conditions for a Hopf π-crossed product $A\natural^\pi_\sigma H$ to be a coquasitriangular Hopf π-coalgebra are given. In this case the category ${}^{A\natural^\pi_\sigma H}\mathcal{M}$ of the left π-comodules over $A\natural^\pi_\sigma H$ is a braided monoidal category.

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1. Introduction

Recently, Turaev introduced the notion of a modular crossed group-category ([5]). Examples of group-category can be constructed from the so-called Hopf π-coalgebras. Since then this notion has been studied extensively. Some investigation related to Hopf π-coalgebras in a purely algebraic study can be found in [6, 3, 7, 8, 11, 10].

In [3], Shen and Wang introduced the notion of π-crossed product $A\#^\pi_\sigma H$ and gave the necessary and sufficient conditions for $A\#^\pi_\sigma H$ equipped with suitable comultiplication and counit to be a Hopf π-coalgebra (we

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denote this Hopf π-coalgebra by $A\natural_\sigma^\pi H$ called Hopf π-crossed product) generalizing the π-smash product. We note that in fact the comultiplication in [3] is trivial. Instead of the trivial comultiplication by a new one induced by the coaction of H on A, it is interesting to investigate when the π-crossed product $A\#_\sigma^\pi H$ becomes a Hopf π-coalgebra with respect to the new comultiplication, which extends the Hopf π-crossed product $A\natural_\sigma^\pi H$ considered by Shen and Wang.

While in [10], Zhu, Chen and Li gave the definition of coquasitriangular Hopf π-coalgebras and showed that H is a coquasitriangular Hopf π-coalgebra if and only if ${}^H\mathcal{M}$ is a braided monoidal category and (F_α, id, id) is a braided monoidal endofunctor of ${}^H\mathcal{M}$ for any $\alpha \in \pi$. So under what conditions the Hopf π-crossed product $A\natural_\sigma^\pi H$ is coquasitriangular becomes very interesting, and in this case the category ${}^{A\natural_\sigma^\pi H}\mathcal{M}$ of the left π-comodules over $A\natural_\sigma^\pi H$ is a braided monoidal category.

In this paper we will answer the above questions. The paper is organized as follows.

In section 2, we recall some basic notions about π-crossed products and Hopf π-coalgebras. Let π be a group and $H = (\{H_\alpha\}, \Delta, \varepsilon, S)$ a Hopf π-coalgebra (not necessarily associative), $\alpha \in \pi$. Let A be an algebra and a coalgebra. In section 3, we find the necessary and sufficient conditions on the π-crossed product $A\#_\sigma^\pi H$ with suitable comultiplication induced by the coaction of H on A and counit to be a Hopf π-coalgebra, named Hopf π-crossed biproduct and denoted by $A \times^\pi H$. As applications, we get the π-coalgebra's version of the well-known Radford biproduct and the Hopf π-crossed product $A\natural_\sigma^\pi H$. Section 4 is devoted to giving the necessary and sufficient conditions for Hopf π-crossed product $A\natural_\sigma^\pi H$ (when the coaction is trivial in $A \times^\pi H$) to be a coquasitriangular Hopf π-coalgebra.

2. Preliminaries

Throughout this paper, let π be a discrete group (with neutral element 1), k will be a fixed field, and the tenser product $\otimes = \otimes_k$ is always assumed to be over k. If U and V are k-vector spaces, $T_{U,V} : U \otimes V \longrightarrow V \otimes U$ will denote the flip map defined by $T_{U,V}(u \otimes v) = v \otimes u$, for all $u \in U$ and $v \in V$.

Next we recall some useful definitions and notations from [6, 3].

Definition 2.1. A *π-coalgebra* is a family of k-spaces $C = \{C_\alpha\}_{\alpha\in\pi}$ together with a family of k-linear maps $\Delta = \{\Delta_{\alpha,\beta} : C_{\alpha\beta} \longrightarrow C_\alpha \otimes C_\beta\}_{\alpha,\beta\in\pi}$

(called a comultiplication) and a k-linear map $\varepsilon : C_1 \longrightarrow k$ (called a counit), such that Δ is coassociative in the sense that for any $\alpha, \beta, \gamma \in \pi$,

$$(\Delta_{\alpha,\beta} \otimes id_{C_\gamma})\Delta_{\alpha\beta,\gamma} = (id_{C_\alpha} \otimes \Delta_{\beta,\gamma})\Delta_{\alpha,\beta\gamma},$$
$$(id_{C_\alpha} \otimes \varepsilon)\Delta_{\alpha,1} = id_{C_\alpha} = (\varepsilon \otimes id_{C_\alpha})\Delta_{1,\alpha}.$$

We use the Sweedler's notation ([4]) for a comultiplication in the following way: for all $\alpha, \beta \in \pi$ and $c \in C_{\alpha\beta}$, we write $\Delta_{\alpha,\beta}(c) = c_{(1,\alpha)} \otimes c_{(2,\beta)}$. For $\forall\ \alpha, \beta, \gamma \in \pi$ and $c \in C_{\alpha\beta\gamma}$, the coassociativity axiom can be written as:

$$c_{(1,\alpha\beta)(1,\alpha)} \otimes c_{(1,\alpha\beta)(2,\beta)} \otimes c_{(2,\gamma)} = c_{(1,\alpha)} \otimes c_{(2,\beta\gamma)(1,\beta)} \otimes c_{(2,\beta\gamma)(2,\gamma)},$$

this element of $C_\alpha \otimes C_\beta \otimes C_\gamma$ is written as $c_{(1,\alpha)} \otimes c_{(2,\beta)} \otimes c_{(3,\gamma)}$. By iterating the procedure, we define inductively $c_{(1,\alpha_1)} \otimes c_{(2,\alpha_2)} \otimes \ldots \otimes c_{(n,\alpha_n)}$ for any $c \in C_{\alpha_1\alpha_2\ldots\alpha_n}$.

DEFINITION 2.2. A *Hopf π-coalgebra* is a π-coalgebra $H = (\{H_\alpha\}_{\alpha\in\pi}, \Delta, \varepsilon)$ endowed with a family of k-linear maps $S = \{S_\alpha : H_\alpha \longrightarrow H_{\alpha^{-1}}\}_{\alpha\in\pi}$ (called antipode) such that:

(1) each H_α is an algebra with multiplication m_α and unit element $1_\alpha \in H_\alpha$,

(2) $\varepsilon : H_1 \longrightarrow k$ and $\Delta_{\alpha,\beta} : H_{\alpha\beta} \longrightarrow H_\alpha \otimes H_\beta$ are algebra maps, for all $\alpha, \beta \in \pi$,

(3) for each $\alpha \in \pi$, $m_\alpha(S_{\alpha^{-1}} \otimes id_{H_\alpha})\Delta_{\alpha^{-1},\alpha} = \varepsilon 1_\alpha = m_\alpha(id_{H_\alpha} \otimes S_{\alpha^{-1}})\Delta_{\alpha,\alpha^{-1}}$.

If a π-coalgebra H satisfies conditions (1) and (2), we call it a *semi-Hopf π-coalgebra*.

Remark that $(H_1, m_1, 1_1, \Delta_{1,1}, \varepsilon, S_1)$ is a usual Hopf algebra. The antipode of a Hopf π-coalgebra is anti-multiplication and anti-comultiplication, i.e., for $\forall\ \alpha, \beta \in \pi$ and $h, g \in H_\alpha$,

$$S_\alpha(hg) = S_\alpha(g)S_\alpha(h), \quad S_\alpha(1_\alpha) = 1_{\alpha^{-1}},$$
$$\Delta_{\beta^{-1},\alpha^{-1}} S_{\alpha\beta} = T_{H_{\alpha^{-1}},H_{\beta^{-1}}}(S_\alpha \otimes S_\beta)\Delta_{\alpha,\beta}, \quad \varepsilon S_1 = \varepsilon.$$

DEFINITION 2.3. Let H be a Hopf π-coalgebra and A an algebra. H *acts weakly on* A if there exists a family of maps: $H_\alpha \otimes A \longrightarrow A$, $h \otimes a \longmapsto h \cdot a$, $\forall\ \alpha \in \pi$, $h \in H_\alpha$, such that

($E1$) $1_\alpha \cdot a = a, \forall\ a \in A,\ \alpha \in \pi$,

($E2$) $h \cdot (ab) = (h_{(1,\alpha)} \cdot a)(h_{(2,\beta)} \cdot b), \forall\ h \in H_{\alpha\beta},\ a, b \in A$,

($E3$) $h \cdot 1_A = \varepsilon(h)1_A, \forall\ h \in H_1$.

Furthermore, if A is an H_α-module for $\forall\ \alpha \in \pi$ satisfies ($E2$) and ($E3$), we call that A is a *π-H-module algebra*.

Definition 2.4. Let $H=\{H_\alpha\}_{\alpha\in\pi}$ be a Hopf π-coalgebra and A an algebra. H acts weakly on A. Let $\sigma: H_1\otimes H_1\longrightarrow A$ be a k-linear map. Define $A\otimes H=\{A\otimes H_\alpha\}_{\alpha\in\pi}$. For each $A\otimes H_\alpha$, we define a multiplication by

$$(a\otimes h)(b\otimes g)=a(h_{(1,1)}\cdot b)\sigma(h_{(2,1)},g_{(1,1)})\otimes h_{(3,\alpha)}g_{(2,\alpha)}.$$

If each $A\otimes H_\alpha$ is associative with $1_A\otimes 1_\alpha$ as indentity element, we call $A\otimes H$ a *π-crossed product*, denoted by $A\#_\sigma^\pi H$.

Proposition 2.5. *$A\#_\sigma^\pi H$ is a π-crossed product if for $\forall\ h,g,k\in H_1$ and $a\in A$, the following conditions*

$(F1)$ $\sigma(1_1,h)=\varepsilon(h)1_A=\sigma(h,1_1)$;

$(F2)$ $(h_{(1,1)}\cdot(g_{(1,1)}\cdot a))\sigma(h_{(2,1)},g_{(2,1)})=\sigma(h_{(1,1)},g_{(1,1)})(h_{(2,1)}g_{(2,1)}\cdot a)$;

$(F3)$ $\sigma(h_{(1,1)},g_{(1,1)})\sigma(h_{(2,1)}g_{(2,1)},k)=(h_{(1,1)}\cdot\sigma(g_{(1,1)},k_{(1,1)}))\sigma(h_{(2,1)},g_{(2,1)}k_{(2,1)})$

are satisfied.

Remark. If $\sigma(h,l)=\varepsilon(h)\varepsilon(l)1_A$, then the π-crossed product has the form of π-smash product. From Proposition 2.5, we get each $A\#H_\alpha$ forms an algebra if A is π-H-module algebra.

Definition 2.6. Let $C=(\{C_\alpha\}_{\alpha\in\pi},\Delta,\varepsilon)$ be a π-coalgebra. A *left π-comodule over C* is a family $M=\{M_\alpha\}_{\alpha\in\pi}$ of k-spaces endowed with a family $\rho^M=\{\rho^M_{\alpha,\beta}:M_{\alpha\beta}\longrightarrow C_\alpha\otimes M_\beta\}_{\alpha,\beta\in\pi}$ of k-linear maps such that for $\forall\ \alpha,\beta,\gamma\in\pi$, $(id_{C_\alpha}\otimes\rho^M_{\beta,\gamma})\rho^M_{\alpha,\beta\gamma}=(\Delta_{\alpha,\beta}\otimes id_{M_\gamma})\rho^M_{\alpha\beta,\gamma}$ and $(\varepsilon\otimes id_{M_\alpha})\rho^M_{1,\alpha}=id_{M_\alpha}$.

Definition 2.7. Let $H=(\{H_\alpha\}_{\alpha\in\pi},\Delta,\varepsilon)$ be a π-coalgebra. A *left π-H-comodule object over H* is a k-space A endowed with a family $\rho^A=\{\rho^A_\alpha:A\longrightarrow H_\alpha\otimes A\}_{\alpha\in\pi}$ of k-linear maps such that for $\forall\ \alpha,\beta\in\pi$ and $a\in A$, the following conditions

$(G1)$ $a_{(-1,\alpha\beta)(1,\alpha)}\otimes a_{(-1,\alpha\beta)(2,\beta)}\otimes a_{(0,\alpha\beta)}=a_{(-1,\alpha)}\otimes a_{(0,\alpha)(-1,\beta)}\otimes a_{(0,\alpha)(0,\beta)}$,

$(G2)$ $\varepsilon(a_{(-1,1)})a_{(0,1)}=a$

are satisfied.

Definition 2.8. Let $H=(\{H_\alpha\}_{\alpha\in\pi},\Delta,\varepsilon)$ be a Hopf π-coalgebra. Assume that A is a coalgebra and a π-H-comodule object over H with the structure maps $\rho^A=\{\rho^A_\alpha\}_{\alpha\in\pi}$. If for $\forall\ \alpha\in\pi$ and $a\in A$, the following conditions are satisfied:

$$(H1)\quad a_{1(-1,\alpha)}a_{2(-1,\alpha)} \otimes a_{1(0,\alpha)} \otimes a_{2(0,\alpha)} = a_{(-1,\alpha)} \otimes a_{(0,\alpha)1} \otimes a_{(0,\alpha)2},$$

$$(H2)\quad a_{(-1,1)}\varepsilon_A(a_{(0,1)}) = \varepsilon_A(a),$$

then we call A a *left π-H-comodule coalgebra.*

Remark that when π is trivial, a left π-H-comodule coalgebra is exactly the left H-comodule coalgebra.

3. Hopf π-Crossed Biproduct $A \times^\pi H$

In this section, we give the necessary and sufficient conditions for the π-crossed product equipped with suitable comultiplication and counit to be a Hopf π-coalgebra.

THEOREM 3.1. *Let A be an algebra and a coalgebra, $H = \{H_\alpha\}_{\alpha\in\pi}$ a Hopf π-coalgebra and act weakly on A. Let $\sigma : H_1 \otimes H_1 \longrightarrow A$ be a k-linear map and $A\#^\pi_\sigma H = \{A\#^\pi_\sigma H_\alpha\}_{\alpha\in\pi}$ a π-crossed product. Assume that A is a π-H-comodule coalgebra. Then $A\#^\pi_\sigma H$ equipped with the following comultiplication and counit*

$$\begin{array}{rcl} \Delta_{\alpha,\beta} : A\#^\pi_\sigma H_{\alpha\beta} & \longrightarrow & A\#^\pi_\sigma H_\alpha \otimes A\#^\pi_\sigma H_\beta \\ a\#_\sigma h & \longmapsto & a_1\#_\sigma a_{2(-1,\alpha)}h_{(1,\alpha)} \otimes a_{2(0,\alpha)}\#_\sigma h_{(2,\beta)} \\ \bar\varepsilon : A\#^\pi_\sigma H_1 & \longrightarrow & k \\ a\#_\sigma x & \longmapsto & \varepsilon_A(a)\varepsilon(x) \end{array}$$

is a semi-Hopf π-coalgebra if and only if the conditions below are satisfied:

(A1) $\varepsilon_A(\sigma(x,y)) = \varepsilon(x)\varepsilon(y)$;

(A2) $\varepsilon_A(x \cdot a) = \varepsilon(x)\varepsilon_A(a)$;

(A3) $\rho_\alpha(1_A) = 1_{A_{(-1,\alpha)}} \otimes 1_{A_{(0,\alpha)}} = 1_\alpha \otimes 1_A$;

(A4) $\Delta_A(1_A) = 1_A \otimes 1_A$, ε_A *is an algebra map*;

(A5) $a_1 \otimes a_{2(-1,\alpha)}h_{(1,\alpha)} \otimes a_{2(0,\alpha)} \otimes h_{(2,\beta)}$
$= a_1\sigma(a_{2(-1,\alpha)(1,1)}, h_{(1,1)}) \otimes a_{2(-1,\alpha)(2,\alpha)}h_{(2,\alpha)} \otimes a_{2(0,\alpha)} \otimes h_{(3,\beta)}$;

(A6) $(ab)_1 \otimes (ab)_{2(-1,\alpha)} \otimes (ab)_{2(0,\alpha)} = a_1(a_{2(-1,\alpha)(1,1)} \cdot b_1)\sigma(a_{2(-1,\alpha)(2,1)}, b_{2(-1,\alpha)(1,1)})$
$\otimes a_{2(-1,\alpha)(3,\alpha)}b_{2(-1,\alpha)(2,\alpha)} \otimes a_{2(0,\alpha)}b_{2(0,\alpha)}$;

(A7) $\sigma(h_{(1,1)}, g_{(1,1)})_1 \otimes \sigma(h_{(1,1)}, g_{(1,1)})_{2(-1,\alpha)}h_{(2,\alpha)}g_{(2,\alpha)} \otimes \sigma(h_{(1,1)}, g_{(1,1)})_{2(0,\alpha)}$
$\otimes h_{(3,\beta)}g_{(3,\beta)} = \sigma(h_{(1,1)}, g_{(1,1)}) \otimes h_{(2,\alpha)}g_{(2,\alpha)} \otimes \sigma(h_{(3,1)}, g_{(3,1)}) \otimes h_{(4,\beta)}g_{(4,\beta)}$;

(A8) $(h_{(1,1)} \cdot a)_1 \otimes (h_{(1,1)} \cdot a)_{2(-1,\alpha)}h_{(2,\alpha)} \otimes (h_{(1,1)} \cdot a)_{2(0,\alpha)} \otimes h_{(3,\beta)}$
$= (h_{(1,1)} \cdot a_1)\sigma(h_{(2,1)}, a_{2(-1,\alpha)(1,1)}) \otimes h_{(3,\alpha)}a_{2(-1,\alpha)(2,\alpha)} \otimes h_{(4,1)} \cdot a_{2(0,\alpha)} \otimes h_{(5,\beta)}$,

where $a, b \in A$, $x, y \in H_1$ and $h, g \in H_{\alpha\beta}$.

In this case we call this semi-Hopf π-coalgebra Hopf π-crossed biproduct *denoted by $A \times^{\pi} H$.*

Proof. ($\Longleftarrow$) Since A is a π-H-comodule coalgebra, we can get $\{A \otimes H_\alpha\}_{\alpha\in\pi}$ is a π-coalgebra with the above comultiplication and counit. And by $A\#_\sigma^\pi H = \{A\#_\sigma^\pi H_\alpha\}_{\alpha\in\pi}$ a π-crossed product we know each $A\#_\sigma^\pi H_\alpha$ ($=A \otimes H_\alpha$ as a vector space) is an algebra. Next we prove that $\Delta_{\alpha,\beta}$ and $\bar{\varepsilon}$ are algebra maps. In fact, for all $a, b \in A$ and $h, g \in H_{\alpha\beta}$, we have

$$\begin{aligned}
&\Delta((a\#_\sigma h)(b\#_\sigma g))\\
&=\Delta(a(h_{(1,1)}\cdot b)\sigma(h_{(2,1)}, g_{(1,1)})\#_\sigma h_{(3,\alpha\beta)}g_{(2,\alpha\beta)})\\
&=(a(h_{(1,1)}\cdot b)\sigma(h_{(2,1)}, g_{(1,1)}))_1\#_\sigma(a(h_{(1,1)}\cdot b)\sigma(h_{(2,1)}, g_{(1,1)}))_{2(-1,\alpha)}h_{(3,\alpha)}g_{(2,\alpha)}\\
&\quad\otimes(a(h_{(1,1)}\cdot b)\sigma(h_{(2,1)}, g_{(1,1)}))_{2(0,\alpha)}\#_\sigma h_{(4,\beta)}g_{(3,\beta)}
\end{aligned}$$

$$\begin{aligned}
&\overset{(A5)}{=}(a(h_{(1,1)}\cdot b)\sigma(h_{(2,1)}, g_{(1,1)}))_1\sigma((a(h_{(1,1)}\cdot b)\sigma(h_{(2,1)}, g_{(1,1)}))_{2(-1,\alpha)(1,1)}, h_{(3,1)}g_{(2,1)})\\
&\quad\#_\sigma(a(h_{(1,1)}\cdot b)\sigma(h_{(2,1)}, g_{(1,1)}))_{2(-1,\alpha)(2,\alpha)}h_{(4,\alpha)}g_{(3,\alpha)}\\
&\quad\otimes(a(h_{(1,1)}\cdot b)\sigma(h_{(2,1)}, g_{(1,1)}))_{2(0,\alpha)}\#_\sigma h_{(5,\beta)}g_{(4,\beta)}
\end{aligned}$$

$$\begin{aligned}
&\overset{(A6)}{=}a_1(a_{2(-1,\alpha)(1,1)}\cdot((h_{(1,1)}\cdot b)\sigma(h_{(2,1)}, g_{(1,1)}))_1)\\
&\qquad\times\sigma(a_{2(-1,\alpha)(2,1)}, ((h_{(1,1)}\cdot b)\sigma(h_{(2,1)}, g_{(1,1)}))_{2(-1,\alpha)(1,1)})\\
&\qquad\times\sigma(a_{2(-1,\alpha)(3,1)}((h_{(1,1)}\cdot b)\sigma(h_{(2,1)}, g_{(1,1)}))_{2(-1,\alpha)(2,1)}, h_{(3,1)}g_{(2,1)})\\
&\quad\#_\sigma a_{2(-1,\alpha)(4,\alpha)}((h_{(1,1)}\cdot b)\sigma(h_{(2,1)}, g_{(1,1)}))_{2(-1,\alpha)(3,\alpha)}h_{(4,\alpha)}g_{(3,\alpha)}\\
&\quad\otimes a_{2(0,\alpha)}((h_{(1,1)}\cdot b)\sigma(h_{(2,1)}, g_{(1,1)}))_{2(0,\alpha)}\#_\sigma h_{(5,\beta)}g_{(4,\beta)}
\end{aligned}$$

$$\begin{aligned}
&\overset{(F3)}{=}a_1(a_{2(-1,\alpha)(1,1)}\cdot((h_{(1,1)}\cdot b)\sigma(h_{(2,1)}, g_{(1,1)}))_1)\\
&\qquad\times(a_{2(-1,\alpha)(2,1)}\cdot\sigma(((h_{(1,1)}\cdot b)\sigma(h_{(2,1)}, g_{(1,1)}))_{2(-1,\alpha)(1,1)}, h_{(3,1)}g_{(2,1)}))\\
&\qquad\times\sigma(a_{2(-1,\alpha)(3,1)}, ((h_{(1,1)}\cdot b)\sigma(h_{(2,1)}, g_{(1,1)}))_{2(-1,\alpha)(2,1)}h_{(4,1)}g_{(3,1)})\\
&\quad\#_\sigma a_{2(-1,\alpha)(4,\alpha)}((h_{(1,1)}\cdot b)\sigma(h_{(2,1)}, g_{(1,1)}))_{2(-1,\alpha)(3,\alpha)}h_{(5,\alpha)}g_{(4,\alpha)}\\
&\quad\otimes a_{2(0,\alpha)}((h_{(1,1)}\cdot b)\sigma(h_{(2,1)}, g_{(1,1)}))_{2(0,\alpha)}\#_\sigma h_{(6,\beta)}g_{(5,\beta)}
\end{aligned}$$

$$\begin{aligned}
&\overset{(E2)}{=}a_1(a_{2(-1,\alpha)(1,1)}\cdot((h_{(1,1)}\cdot b)\sigma(h_{(2,1)}, g_{(1,1)}))_1\\
&\qquad\times\sigma(((h_{(1,1)}\cdot b)\sigma(h_{(2,1)}, g_{(1,1)}))_{2(-1,\alpha)(1,1)}, h_{(3,1)}g_{(2,1)}))\\
&\qquad\times\sigma(a_{2(-1,\alpha)(2,1)}, ((h_{(1,1)}\cdot b)\sigma(h_{(2,1)}, g_{(1,1)}))_{2(-1,\alpha)(2,1)}h_{(4,1)}g_{(3,1)})\\
&\quad\#_\sigma a_{2(-1,\alpha)(3,\alpha)}((h_{(1,1)}\cdot b)\sigma(h_{(2,1)}, g_{(1,1)}))_{2(-1,\alpha)(3,\alpha)}h_{(5,\alpha)}g_{(4,\alpha)}\\
&\quad\otimes a_{2(0,\alpha)}((h_{(1,1)}\cdot b)\sigma(h_{(2,1)}, g_{(1,1)}))_{2(0,\alpha)}\#_\sigma h_{(6,\beta)}g_{(5,\beta)}
\end{aligned}$$

$$
\begin{aligned}
&\overset{(A6)}{=} a_1(a_{2(-1,\alpha)(1,1)} \cdot (h_{(1,1)} \cdot b)_1((h_{(1,1)} \cdot b)_{2(-1,\alpha)(1,1)} \cdot \sigma(h_{(2,1)}, g_{(1,1)})_1) \\
&\quad \times \sigma((h_{(1,1)} \cdot b)_{2(-1,\alpha)(2,1)}, \sigma(h_{(2,1)}, g_{(1,1)})_{2(-1,\alpha)(1,1)})\sigma((h_{(1,1)} \cdot b)_{2(-1,\alpha)(3,1)} \\
&\quad \times \sigma(h_{(2,1)}, g_{(1,1)})_{2(-1,\alpha)(2,1)}, h_{(3,1)}g_{(2,1)}))\sigma(a_{2(-1,\alpha)(2,1)}, (h_{(1,1)} \cdot b)_{2(-1,\alpha)(4,1)} \\
&\quad \times \sigma(h_{(2,1)}, g_{(1,1)})_{2(-1,\alpha)(3,1)}h_{(4,1)}g_{(3,1)})\#_\sigma a_{2(-1,\alpha)(3,\alpha)}(h_{(1,1)} \cdot b)_{2(-1,\alpha)(5,\alpha)} \\
&\quad \times \sigma(h_{(2,1)}, g_{(1,1)})_{2(-1,\alpha)(4,\alpha)}h_{(5,\alpha)}g_{(4,\alpha)} \otimes a_{2(0,\alpha)}(h_{(1,1)} \cdot b)_{2(0,\alpha)}\sigma(h_{(2,1)}, g_{(1,1)})_{2(0,\alpha)} \\
&\quad \#_\sigma h_{(6,\beta)}g_{(5,\beta)}
\end{aligned}
$$

$$
\begin{aligned}
&\overset{(F3)}{=} a_1(a_{2(-1,\alpha)(1,1)} \cdot (h_{(1,1)} \cdot b)_1((h_{(1,1)} \cdot b)_{2(-1,\alpha)(1,1)} \cdot \sigma(h_{(2,1)}, g_{(1,1)})_1) \\
&\quad \times ((h_{(1,1)} \cdot b)_{2(-1,\alpha)(2,1)} \cdot \sigma(\sigma(h_{(2,1)}, g_{(1,1)})_{2(-1,\alpha)(1,1)}, h_{(3,1)}g_{(2,1)})) \\
&\quad \times \sigma((h_{(1,1)} \cdot b)_{2(-1,\alpha)(3,1)}, \sigma(h_{(2,1)}, g_{(1,1)})_{2(-1,\alpha)(2,1)}h_{(4,1)}g_{(3,1)})) \\
&\quad \times \sigma(a_{2(-1,\alpha)(2,1)}, (h_{(1,1)} \cdot b)_{2(-1,\alpha)(4,1)}\sigma(h_{(2,1)}, g_{(1,1)})_{2(-1,\alpha)(3,1)}h_{(5,1)}g_{(4,1)}) \\
&\quad \#_\sigma a_{2(-1,\alpha)(3,\alpha)}(h_{(1,1)} \cdot b)_{2(-1,\alpha)(5,\alpha)}\sigma(h_{(2,1)}, g_{(1,1)})_{2(-1,\alpha)(4,\alpha)}h_{(6,\alpha)}g_{(5,\alpha)} \\
&\quad \otimes a_{2(0,\alpha)}(h_{(1,1)} \cdot b)_{2(0,\alpha)}\sigma(h_{(2,1)}, g_{(1,1)})_{2(0,\alpha)}\#_\sigma h_{(7,\beta)}g_{(6,\beta)}
\end{aligned}
$$

$$
\begin{aligned}
&\overset{(E2)}{=} a_1(a_{2(-1,\alpha)(1,1)} \cdot (h_{(1,1)} \cdot b)_1((h_{(1,1)} \cdot b)_{2(-1,\alpha)(1,1)} \\
&\quad \cdot \sigma(h_{(2,1)}, g_{(1,1)})_1\sigma(\sigma(h_{(2,1)}, g_{(1,1)})_{2(-1,\alpha)(1,1)}, h_{(3,1)}g_{(2,1)})) \\
&\quad \times \sigma((h_{(1,1)} \cdot b)_{2(-1,\alpha)(2,1)}, \sigma(h_{(2,1)}, g_{(1,1)})_{2(-1,\alpha)(2,1)}h_{(4,1)}g_{(3,1)})) \\
&\quad \times \sigma(a_{2(-1,\alpha)(2,1)}, (h_{(1,1)} \cdot b)_{2(-1,\alpha)(3,1)}\sigma(h_{(2,1)}, g_{(1,1)})_{2(-1,\alpha)(3,1)}h_{(5,1)}g_{(4,1)}) \\
&\quad \#_\sigma a_{2(-1,\alpha)(3,\alpha)}(h_{(1,1)} \cdot b)_{2(-1,\alpha)(4,\alpha)}\sigma(h_{(2,1)}, g_{(1,1)})_{2(-1,\alpha)(4,\alpha)}h_{(6,\alpha)}g_{(5,\alpha)} \\
&\quad \otimes a_{2(0,\alpha)}(h_{(1,1)} \cdot b)_{2(0,\alpha)}\sigma(h_{(2,1)}, g_{(1,1)})_{2(0,\alpha)}\#_\sigma h_{(7,\beta)}g_{(6,\beta)}
\end{aligned}
$$

$$
\begin{aligned}
&\overset{(A5)}{=} a_1(a_{2(-1,\alpha)(1,1)} \cdot (h_{(1,1)} \cdot b)_1((h_{(1,1)} \cdot b)_{2(-1,\alpha)(1,1)} \cdot \sigma(h_{(2,1)}, g_{(1,1)})_1) \\
&\quad \times \sigma((h_{(1,1)} \cdot b)_{2(-1,\alpha)(2,1)}, \sigma(h_{(2,1)}, g_{(1,1)})_{2(-1,\alpha)(1,1)}h_{(3,1)}g_{(2,1)})) \\
&\quad \times \sigma(a_{2(-1,\alpha)(2,1)}, (h_{(1,1)} \cdot b)_{2(-1,\alpha)(3,1)}\sigma(h_{(2,1)}, g_{(1,1)})_{2(-1,\alpha)(2,1)}h_{(4,1)}g_{(3,1)}) \\
&\quad \#_\sigma a_{2(-1,\alpha)(3,\alpha)}(h_{(1,1)} \cdot b)_{2(-1,\alpha)(4,\alpha)}\sigma(h_{(2,1)}, g_{(1,1)})_{2(-1,\alpha)(3,\alpha)}h_{(5,\alpha)}g_{(4,\alpha)} \\
&\quad \otimes a_{2(0,\alpha)}(h_{(1,1)} \cdot b)_{2(0,\alpha)}\sigma(h_{(2,1)}, g_{(1,1)})_{2(0,\alpha)}\#_\sigma h_{(6,\beta)}g_{(5,\beta)}
\end{aligned}
$$

$$
\begin{aligned}
&\overset{(A7)}{=} a_1(a_{2(-1,\alpha)(1,1)} \cdot (h_{(1,1)} \cdot b)_1((h_{(1,1)} \cdot b)_{2(-1,\alpha)(1,1)} \cdot \sigma(h_{(2,1)}, g_{(1,1)})) \\
&\quad \times \sigma((h_{(1,1)} \cdot b)_{2(-1,\alpha)(2,1)}, h_{(3,1)}g_{(2,1)})) \\
&\quad \times \sigma(a_{2(-1,\alpha)(2,1)}, (h_{(1,1)} \cdot b)_{2(-1,\alpha)(3,1)}h_{(4,1)}g_{(3,1)})\#_\sigma a_{2(-1,\alpha)(3,\alpha)}(h_{(1,1)} \cdot b)_{2(-1,\alpha)(4,\alpha)} \\
&\quad \times h_{(5,\alpha)}g_{(4,\alpha)} \otimes a_{2(0,\alpha)}(h_{(1,1)} \cdot b)_{2(0,\alpha)}\sigma(h_{(6,1)}, g_{(5,1)})\#_\sigma h_{(7,\beta)}g_{(6,\beta)}
\end{aligned}
$$

$$
\begin{aligned}
&\overset{(F3)}{=} a_1(a_{2(-1,\alpha)(1,1)} \cdot (h_{(1,1)} \cdot b)_1\sigma((h_{(1,1)} \cdot b)_{2(-1,\alpha)(1,1)}, h_{(2,1)}) \\
&\quad \times \sigma((h_{(1,1)} \cdot b)_{2(-1,\alpha)(2,1)}h_{(3,1)}, g_{(1,1)})) \\
&\quad \times \sigma(a_{2(-1,\alpha)(2,1)}, (h_{(1,1)} \cdot b)_{2(-1,\alpha)(3,1)}h_{(4,1)}g_{(2,1)})\#_\sigma a_{2(-1,\alpha)(3,\alpha)}(h_{(1,1)} \cdot b)_{2(-1,\alpha)(4,\alpha)} \\
&\quad \times h_{(5,\alpha)}g_{(3,\alpha)} \otimes a_{2(0,\alpha)}(h_{(1,1)} \cdot b)_{2(0,\alpha)}\sigma(h_{(6,1)}, g_{(4,1)})\#_\sigma h_{(7,\beta)}g_{(5,\beta)}
\end{aligned}
$$

$$\begin{aligned}
&\overset{(E2)}{=} a_1(a_{2(-1,\alpha)(1,1)} \cdot (h_{(1,1)} \cdot b)_1 \sigma((h_{(1,1)} \cdot b)_{2(-1,\alpha)(1,1)}, h_{(2,1)}))\\
&\quad \times (a_{2(-1,\alpha)(2,1)} \cdot \sigma((h_{(1,1)} \cdot b)_{2(-1,\alpha)(2,1)} h_{(3,1)}, g_{(1,1)}))\\
&\quad \times \sigma(a_{2(-1,\alpha)(3,1)}, (h_{(1,1)} \cdot b)_{2(-1,\alpha)(3,1)} h_{(4,1)} g_{(2,1)}) \#_\sigma a_{2(-1,\alpha)(4,\alpha)} (h_{(1,1)} \cdot b)_{2(-1,\alpha)(4,\alpha)}\\
&\quad \times h_{(5,\alpha)} g_{(3,\alpha)} \otimes a_{2(0,\alpha)} (h_{(1,1)} \cdot b)_{2(0,\alpha)} \sigma(h_{(6,1)}, g_{(4,1)}) \#_\sigma h_{(7,\beta)} g_{(5,\beta)}
\end{aligned}$$

$$\begin{aligned}
&\overset{(A5)}{=} a_1(a_{2(-1,\alpha)(1,1)} \cdot (h_{(1,1)} \cdot b)_1)(a_{2(-1,\alpha)(2,1)} \cdot \sigma((h_{(1,1)} \cdot b)_{2(-1,\alpha)(1,1)} h_{(2,1)}, g_{(1,1)}))\\
&\quad \times \sigma(a_{2(-1,\alpha)(3,1)}, (h_{(1,1)} \cdot b)_{2(-1,\alpha)(2,1)} h_{(3,1)} g_{(2,1)}) \#_\sigma a_{2(-1,\alpha)(4,\alpha)} (h_{(1,1)} \cdot b)_{2(-1,\alpha)(3,\alpha)}\\
&\quad \times h_{(4,\alpha)} g_{(3,\alpha)} \otimes a_{2(0,\alpha)} (h_{(1,1)} \cdot b)_{2(0,\alpha)} \sigma(h_{(5,1)}, g_{(4,1)}) \#_\sigma h_{(6,\beta)} g_{(5,\beta)}
\end{aligned}$$

$$\begin{aligned}
&\overset{(F3)}{=} a_1(a_{2(-1,\alpha)(1,1)} \cdot (h_{(1,1)} \cdot b)_1) \sigma(a_{2(-1,\alpha)(2,1)}, (h_{(1,1)} \cdot b)_{2(-1,\alpha)(1,1)} h_{(2,1)})\\
&\quad \times \sigma(a_{2(-1,\alpha)(3,1)} (h_{(1,1)} \cdot b)_{2(-1,\alpha)(2,1)} h_{(3,1)}, g_{(1,1)}) \#_\sigma a_{2(-1,\alpha)(4,\alpha)} (h_{(1,1)} \cdot b)_{2(-1,\alpha)(3,\alpha)}\\
&\quad \times h_{(4,\alpha)} g_{(2,\alpha)} \otimes a_{2(0,\alpha)} (h_{(1,1)} \cdot b)_{2(0,\alpha)} \sigma(h_{(5,1)}, g_{(3,1)}) \#_\sigma h_{(6,\beta)} g_{(4,\beta)}
\end{aligned}$$

$$\begin{aligned}
&\overset{(A8)}{=} a_1(a_{2(-1,\alpha)(1,1)} \cdot (h_{(1,1)} \cdot b_1) \sigma(h_{(2,1)}, b_{2(-1,\alpha)(1,1)})) \sigma(a_{2(-1,\alpha)(2,1)}, h_{(3,1)} b_{2(-1,\alpha)(2,1)})\\
&\quad \times \sigma(a_{2(-1,\alpha)(3,1)} h_{(4,1)} b_{2(-1,\alpha)(3,1)}, g_{(1,1)}) \#_\sigma a_{2(-1,\alpha)(4,\alpha)} h_{(5,\alpha)} b_{2(-1,\alpha)(4,\alpha)} g_{(2,\alpha)}\\
&\quad \otimes a_{2(0,\alpha)} (h_{(6,1)} \cdot b_{2(0,\alpha)}) \sigma(h_{(7,1)}, g_{(3,1)}) \#_\sigma h_{(8,\beta)} g_{(4,\beta)}
\end{aligned}$$

$$\begin{aligned}
&\overset{(E2)}{=} a_1(a_{2(-1,\alpha)(1,1)} \cdot (h_{(1,1)} \cdot b_1))(a_{2(-1,\alpha)(2,1)} \cdot \sigma(h_{(2,1)}, b_{2(-1,\alpha)(1,1)}))\\
&\quad \times \sigma(a_{2(-1,\alpha)(3,1)}, h_{(3,1)} b_{2(-1,\alpha)(2,1)}) \sigma(a_{2(-1,\alpha)(4,1)} h_{(4,1)} b_{2(-1,\alpha)(3,1)}, g_{(1,1)})\\
&\quad \#_\sigma a_{2(-1,\alpha)(5,\alpha)} h_{(5,\alpha)} b_{2(-1,\alpha)(4,\alpha)} g_{(2,\alpha)} \otimes a_{2(0,\alpha)} (h_{(6,1)} \cdot b_{2(0,\alpha)}) \sigma(h_{(7,1)}, g_{(3,1)})\\
&\quad \#_\sigma h_{(8,\beta)} g_{(4,\beta)}
\end{aligned}$$

$$\begin{aligned}
&\overset{(F3)}{=} a_1(a_{2(-1,\alpha)(1,1)} \cdot (h_{(1,1)} \cdot b_1)) \sigma(a_{2(-1,\alpha)(2,1)}, h_{(2,1)})\\
&\quad \times \sigma(a_{2(-1,\alpha)(3,1)} h_{(3,1)}, b_{2(-1,\alpha)(1,1)}) \sigma(a_{2(-1,\alpha)(4,1)} h_{(4,1)} b_{2(-1,\alpha)(2,1)}, g_{(1,1)})\\
&\quad \#_\sigma a_{2(-1,\alpha)(5,\alpha)} h_{(5,\alpha)} b_{2(-1,\alpha)(3,\alpha)} g_{(2,\alpha)} \otimes a_{2(0,\alpha)} (h_{(6,1)} \cdot b_{2(0,\alpha)}) \sigma(h_{(7,1)}, g_{(3,1)})\\
&\quad \#_\sigma h_{(8,\beta)} g_{(4,\beta)}
\end{aligned}$$

$$\begin{aligned}
&\overset{(F2)}{=} a_1 \sigma(a_{2(-1,\alpha)(1,1)}, h_{(1,1)})(a_{2(-1,\alpha)(2,1)} h_{(2,1)} \cdot b_1) \sigma(a_{2(-1,\alpha)(3,1)} h_{(3,1)}, b_{2(-1,\alpha)(1,1)})\\
&\quad \times \sigma(a_{2(-1,\alpha)(4,1)} h_{(4,1)} b_{2(-1,\alpha)(2,1)}, g_{(1,1)}) \#_\sigma a_{2(-1,\alpha)(5,\alpha)} h_{(5,\alpha)} b_{2(-1,\alpha)(3,\alpha)} g_{(2,\alpha)}\\
&\quad \otimes a_{2(0,\alpha)} (h_{(6,1)} \cdot b_{2(0,\alpha)}) \sigma(h_{(7,1)}, g_{(3,1)}) \#_\sigma h_{(8,\beta)} g_{(4,\beta)}
\end{aligned}$$

$$\begin{aligned}
&\overset{(F3)}{=} a_1 \sigma(a_{2(-1,\alpha)(1,1)}, h_{(1,1)})(a_{2(-1,\alpha)(2,1)} h_{(2,1)} \cdot b_1)\\
&\quad \times (a_{2(-1,\alpha)(3,1)} h_{(3,1)} \cdot \sigma(b_{2(-1,\alpha)(1,1)}, g_{(1,1)})) \sigma(a_{2(-1,\alpha)(4,1)} h_{(4,1)}, b_{2(-1,\alpha)(2,1)} g_{(2,1)})\\
&\quad \#_\sigma a_{2(-1,\alpha)(5,\alpha)} h_{(5,\alpha)} b_{2(-1,\alpha)(3,\alpha)} g_{(3,\alpha)} \otimes a_{2(0,\alpha)} (h_{(6,1)} \cdot b_{2(0,\alpha)}) \sigma(h_{(7,1)}, g_{(4,1)})\\
&\quad \#_\sigma h_{(8,\beta)} g_{(5,\beta)}
\end{aligned}$$

$$\overset{(E2)}{=} a_1\sigma(a_{2(-1,\alpha)(1,1)}, h_{(1,1)})(a_{2(-1,\alpha)(2,1)}h_{(2,1)} \cdot b_1\sigma(b_{2(-1,\alpha)(1,1)}, g_{(1,1)}))$$
$$\times\, \sigma(a_{2(-1,\alpha)(3,1)}h_{(3,1)}, b_{2(-1,\alpha)(2,1)}g_{(2,1)})\#_\sigma a_{2(-1,\alpha)(4,\alpha)}h_{(4,\alpha)}b_{2(-1,\alpha)(3,\alpha)}g_{(3,\alpha)}$$
$$\otimes\, a_{2(0,\alpha)}(h_{(5,1)} \cdot b_{2(0,\alpha)})\sigma(h_{(6,1)}, g_{(4,1)})\#_\sigma h_{(7,\beta)}g_{(5,\beta)}$$

$$\overset{(A5)}{=} a_1(a_{2(-1,\alpha)(1,1)}h_{(1,1)} \cdot b_1\sigma(b_{2(-1,\alpha)(1,1)}, g_{(1,1)}))$$
$$\times\, \sigma(a_{2(-1,\alpha)(2,1)}h_{(2,1)}, b_{2(-1,\alpha)(2,1)}g_{(2,1)})\#_\sigma a_{2(-1,\alpha)(3,\alpha)}h_{(3,\alpha)}b_{2(-1,\alpha)(3,\alpha)}g_{(3,\alpha)}$$
$$\otimes\, a_{2(0,\alpha)}(h_{(4,1)} \cdot b_{2(0,\alpha)})\sigma(h_{(5,1)}, g_{(4,1)})\#_\sigma h_{(6,\beta)}g_{(5,\beta)}$$

$$\overset{(A5)}{=} a_1(a_{2(-1,\alpha)(1,1)}h_{(1,1)} \cdot b_1)\sigma(a_{2(-1,\alpha)(2,1)}h_{(2,1)}, b_{2(-1,\alpha)(1,1)}g_{(1,1)})$$
$$\#_\sigma a_{2(-1,\alpha)(3,\alpha)}h_{(3,\alpha)}b_{2(-1,\alpha)(2,\alpha)}g_{(2,\alpha)} \otimes a_{2(0,\alpha)}(h_{(4,1)} \cdot b_{2(0,\alpha)})\sigma(h_{(5,1)}, g_{(3,1)})$$
$$\#_\sigma h_{(6,\beta)}g_{(4,\beta)}$$
$$= ((a_1\#_\sigma a_{2(-1,\alpha)}h_{(1,\alpha)})(b_1\#_\sigma b_{2(-1,\alpha)}g_{(1,\alpha)})) \otimes ((a_{2(0,\alpha)}\#_\sigma h_{(2,\beta)})(b_{2(0,\alpha)}\#_\sigma g_{(2,\beta)}))$$
$$= \Delta(a\#_\sigma h)\Delta(b\#_\sigma g)$$

and for all $h, g \in H_1$, we compute

$$\begin{aligned}
& \overline{\varepsilon}((a\#_\sigma h)(b\#_\sigma g)) \\
= \;& \overline{\varepsilon}(a(h_{(1,1)} \cdot b)\sigma(h_{(2,1)}, g_{(1,1)})\#_\sigma h_{(3,1)}g_{(2,1)}) \\
= \;& \varepsilon_A(a(h_{(1,1)} \cdot b)\sigma(h_{(2,1)}, g_{(1,1)}))\varepsilon(h_{(3,1)}g_{(2,1)}) \\
= \;& \varepsilon_A(a)\varepsilon_A(h_{(1,1)} \cdot b)\varepsilon_A(\sigma(h_{(2,1)}, g)) \\
\overset{(A1)(A2)}{=} \;& \varepsilon_A(a)\varepsilon(h)\varepsilon_A(b)\varepsilon(g) \\
= \;& \overline{\varepsilon}(a\#_\sigma h)\overline{\varepsilon}(b\#_\sigma g)
\end{aligned}$$

Necessity is straightforward. □

We note that Theorem 3.1 is the Wang-Jiao-Zhao's crossed product in the Hopf π-coalgebra setting, but here the conditions is different from the ones in [9] and the condition $(A5)$ is necessary for Hopf π-crossed biproduct $A \times^\pi H$ to become a semi-Hopf π-coalgebra.

Definition 3.2. Let $H = \{H_\alpha\}_{\alpha\in\pi}$ be a semi-Hopf π-coalgebra, A an algebra and a coalgebra, $\sigma : H_1 \otimes H_1 \longrightarrow A$ and $S_\alpha : H_\alpha \longrightarrow H_{\alpha^{-1}}$ be linear maps. Then $S = \{S_\alpha\}_{\alpha\in\pi}$ is called a *σ-antipode* of H if

$(B1)$ $\sigma(h_{(1,\alpha)(1,1)}, S_{\alpha^{-1}}(h_{(2,\alpha^{-1})})_{(1,1)}) \otimes h_{(1,\alpha)(2,\alpha)}S_{\alpha^{-1}}(h_{(2,\alpha^{-1})})_{(2,\alpha)} = (1_A \otimes 1_\alpha)\varepsilon(h)$;

$(B2)$ $\sigma(S_{\alpha^{-1}}(h_{(1,\alpha^{-1})})_{(1,1)}, h_{(2,\alpha)(1,1)}) \otimes S_{\alpha^{-1}}(h_{(1,\alpha^{-1})})_{(2,\alpha)}h_{(2,\alpha)(2,\alpha)} = (1_A \otimes 1_\alpha)\varepsilon(h)$,

for $\forall\ h \in H_1$. In this case, we say that H is a *σ-Hopf π-coalgebra.*

THEOREM 3.3. *Let A be an algebra and a coalgebra, the linear map $S_A : A \longrightarrow A$ satisfy $S_A(a_1)a_2 = a_1S_A(a_2) = \varepsilon_A(a)1_A$, $H = (\{H_\alpha\}, \Delta, \varepsilon, S)$ a Hopf π-coalgebra. If $A \times^\pi H$ is a Hopf π-crossed biproduct (semi-Hopf π-coalgebra), then $A \times^\pi H$ is a Hopf π-coalgebra with antipode $\bar{S} = \{\bar{S}_\alpha\}_{\alpha\in\pi}$ defined as:*

$$\bar{S}_\alpha(a\#_\sigma h) = (1_A\#_\sigma S_\alpha(a_{(-1,\alpha)}h))(S_A(a_{(0,\alpha)})\#_\sigma 1_{\alpha^{-1}}),\ \forall\ a \in A,\ h \in H_\alpha.$$

if and only if H is a σ-Hopf π-coalgebra.

PROOF. ($\Longleftarrow$) We prove $\bar{S} = \{\bar{S}_\alpha\}_{\alpha\in\pi}$ is the antipode of $A \times^\pi H$. For all $a \in A$, $h \in H_1$ (Here $*$ denote the convolution product), we compute as follows:

$$\begin{aligned}
&(\bar{S}_{\alpha_{-1}} * id_{A\times^\pi H})(a\#_\sigma h)\\
&= \bar{S}_{\alpha^{-1}}(a_1\#_\sigma a_{2(-1,\alpha^{-1})}h_{(1,\alpha^{-1})})(a_{2(0,\alpha^{-1})}\#_\sigma h_{(2,\alpha)})\\
&= (1_A\#_\sigma S_{\alpha^{-1}}(a_{1(-1,\alpha^{-1})}a_{2(-1,\alpha^{-1})}h_{(1,\alpha^{-1})}))(S_A(a_{1(0,\alpha^{-1})})\#_\sigma 1_\alpha)(a_{2(0,\alpha^{-1})}\#_\sigma h_{(2,\alpha)})\\
&\overset{(H1)}{=} (1_A\#_\sigma S_{\alpha^{-1}}(a_{(-1,\alpha^{-1})}h_{(1,\alpha^{-1})}))(S_A(a_{(0,\alpha^{-1})1})\#_\sigma 1_\alpha)(a_{(0,\alpha^{-1})2}\#_\sigma h_{(2,\alpha)})\\
&= (1_A\#_\sigma S_{\alpha^{-1}}(a_{(-1,\alpha^{-1})}h_{(1,\alpha^{-1})}))(S_A(a_{(0,\alpha^{-1})1})a_{(0,\alpha^{-1})2}\#_\sigma h_{(2,\alpha)})\\
&= (1_A\#_\sigma S_{\alpha^{-1}}(a_{(-1,\alpha^{-1})}h_{(1,\alpha^{-1})}))(1_A\#_\sigma h_{(2,\alpha)})\varepsilon_A(a_{(0,\alpha^{-1})})\\
&\overset{(H2)}{=} (1_A\#_\sigma S_{\alpha^{-1}}(h_{(1,\alpha^{-1})}))(1_A\#_\sigma h_{(2,\alpha)})\varepsilon_A(a)\\
&\overset{(B2)}{=} (1_A\#_\sigma 1_\alpha)\varepsilon_A(a)\varepsilon(h)
\end{aligned}$$

while

$$\begin{aligned}
&(id_{A\times^\pi H} * \bar{S}_{\alpha_{-1}})(a\#_\sigma h)\\
&= (a_1\#_\sigma a_{2(-1,\alpha)}h_{(1,\alpha)})\bar{S}_{\alpha^{-1}}(a_{2(0,\alpha)}\#_\sigma h_{(2,\alpha^{-1})})\\
&= (a_1\#_\sigma a_{2(-1,\alpha)}h_{(1,\alpha)})(1_A\#_\sigma S_{\alpha^{-1}}(a_{2(0,\alpha)(-1,\alpha^{-1})}h_{(2,\alpha^{-1})}))(S_A(a_{2(0,\alpha)(0,\alpha^{-1})})\#_\sigma 1_\alpha)\\
&\overset{(G1)}{=} (a_1\#_\sigma a_{2(-1,1)(1,\alpha)}h_{(1,\alpha)})(1_A\#_\sigma S_{\alpha^{-1}}(a_{2(-1,1)(2,\alpha^{-1})}h_{(2,\alpha^{-1})}))(S_A(a_{2(0,1)})\#_\sigma 1_\alpha)\\
&= (a_1\#_\sigma(a_{2(-1,1)}h)_{(1,\alpha)})(1_A\#_\sigma S_{\alpha^{-1}}((a_{2(-1,1)}h)_{(2,\alpha^{-1})}))(S_A(a_{2(0,1)})\#_\sigma 1_\alpha)\\
&= (a_1\sigma((a_{2(-1,1)}h)_{(1,\alpha)(1,1)}, S_{\alpha^{-1}}((a_{2(-1,1)}h)_{(2,\alpha^{-1})})_{(1,1)})\#_\sigma(a_{2(-1,1)}h)_{(1,\alpha)(2,\alpha)}\\
&\times\ S_{\alpha^{-1}}((a_{2(-1,1)}h)_{(2,\alpha^{-1})})_{(2,\alpha)})(S_A(a_{2(0,1)})\#_\sigma 1_\alpha)\\
&\overset{(B1)}{=} (a_1\#_\sigma 1_\alpha)(S_A(a_{2(0,1)})\#_\sigma 1_\alpha)\varepsilon(a_{2(-1,1)}h)\\
&= (a_1\#_\sigma 1_\alpha)(S_A(a_2)\#_\sigma 1_\alpha)\varepsilon(h)\\
&= (1_A\#_\sigma 1_\alpha)\varepsilon_A(a)\varepsilon(h)
\end{aligned}$$

Thus $\bar{S}$ is the antipode of $A \times^{\pi} H$.

($\Longrightarrow$) is straightforward. □

Let σ be trivial in Theorem 3.1, then we can get the following result.

COROLLARY 3.4. *Let $H = \{H_\alpha\}_{\alpha\in\pi}$ be a Hopf π-coalgebra and A a π-H-module algebra and π-H-comodule coalgebra. Then the π-smash product $A\#^{\pi}H = \{A\#H_\alpha\}_{\alpha\in\pi}$ equipped with the comultiplication and counit in Theorem 3.1 is a semi-Hopf π-coalgebra if and only if the following conditions hold:*

(C1) $\varepsilon_A(x \cdot a) = \varepsilon(x)\varepsilon_A(a)$;

(C2) $\rho_\alpha(1_A) = 1_{A_{(-1,\alpha)}} \otimes 1_{A_{(0,\alpha)}} = 1_\alpha \otimes 1_A$;

(C3) ε_A *is an algebra map,* $\Delta(1_A) = 1_A \otimes 1_A$;

(C4)
$$\begin{aligned}&(ab)_1 \otimes (ab)_{2(-1,\alpha)} \otimes (ab)_{2(0,\alpha)}\\ &= a_1(a_{2(-1,\alpha)(1,1)} \cdot b_1) \otimes a_{2(-1,\alpha)(2,\alpha)}b_{2(-1,\alpha)} \otimes a_{2(0,\alpha)}b_{2(0,\alpha)};\end{aligned}$$

(C5)
$$\begin{aligned}&(h_{(1,1)} \cdot a)_1 \otimes (h_{(1,1)} \cdot a)_{2(-1,\alpha)}h_{(2,\alpha)} \otimes (h_{(1,1)} \cdot a)_{2(0,\alpha)} \otimes h_{(3,\beta)}\\ &= (h_{(1,1)} \cdot a_1) \otimes h_{(2,\alpha)}a_{2(-1,\alpha)} \otimes h_{(3,1)} \cdot a_{2(0,\alpha)} \otimes h_{(4,\beta)},\end{aligned}$$

where $a, b \in A$, $h \in H_{\alpha\beta}$ and $x \in H_1$. This semi-Hopf π-coalgebra is called Radford π-biproduct denoted by $A \star H$.

PROOF. Let σ be trivial in Theorem 3.1. □

COROLLARY 3.5. *Let A be an algebra and a coalgebra, the linear map $S_A : A \longrightarrow A$ satisfy $S_A(a_1)a_2 = a_1S_A(a_2) = \varepsilon_A(a)1_A$, $H = (\{H_\alpha\}, \Delta, \varepsilon, S)$ a Hopf π-coalgebra. If $A \star H$ is a Radford π-biproduct (semi-Hopf π-coalgebra), then $A \star H$ is a Hopf π-coalgebra with antipode $\bar{S}' = \{\bar{S}_\alpha'\}_{\alpha\in\pi}$ described by*

$$\bar{S}_\alpha'(a\#^{\pi}h) = (1_A\#^{\pi}S_\alpha(a_{(-1,\alpha)}h))(S_A(a_{(0,\alpha)})\#^{\pi}1_{\alpha^{-1}}),$$

for all $a \in A$ and $h \in H_\alpha$.

When ρ is trivial in Theorem 3.1, i.e., $\rho(a) = a_{(-1,\alpha)} \otimes a_{(0,\alpha)} = 1_\alpha \otimes 1_A$, we have

COROLLARY 3.6 ([3]). *Let $H = \{H_\alpha\}_{\alpha\in\pi}$ be a semi-Hopf π-coalgebra, A a bialgebra. Suppose that $A\#^{\pi}_{\sigma}H = \{A\#^{\pi}_{\sigma}H_\alpha\}_{\alpha\in\pi}$ is a π-crossed product. Then $A\#^{\pi}_{\sigma}H$ equipped with the following comultiplication and counit*

$$\Delta_{\alpha,\beta}(a\#_\sigma h) = a_1\#_\sigma h_{(1,\alpha)} \otimes a_2\#_\sigma h_{(2,\beta)},\ \forall\ a \in A,\ h \in H_{\alpha\beta},$$
$$\tilde{\varepsilon}(a\#_\sigma x) = \varepsilon_A(a)\varepsilon(x),\ \forall\ x \in H_1$$

is a semi-Hopf π-coalgebra if and only if the following conditions hold

$(D1)\quad \varepsilon_A(x\cdot a)=\varepsilon(x)\varepsilon_A(a);$

$(D2)\quad \varepsilon_A(\sigma(x,y))=\varepsilon(x)\varepsilon(y);$

$$(D3)\quad \sigma(h_{(1,1)},g_{(1,1)})_1\otimes h_{(2,\alpha)}g_{(2,\alpha)}\otimes\sigma(h_{(1,1)},g_{(1,1)})_2\otimes h_{(3,\beta)}g_{(3,\beta)} \\ =\sigma(h_{(1,1)},g_{(1,1)})\otimes h_{(2,\alpha)}g_{(2,\alpha)}\otimes\sigma(h_{(3,1)},g_{(3,1)})\otimes h_{(4,\beta)}g_{(4,\beta)};$$

$$(D4)\quad (h_{(1,1)}\cdot a)_1\otimes h_{(2,\alpha)}\otimes(h_{(1,1)}\cdot a)_2\otimes h_{(3,\beta)} \\ =(h_{(1,1)}\cdot a_1)\otimes h_{(2,\alpha)}\otimes h_{(3,1)}\cdot a_2\otimes h_{(4,\beta)},$$

where $x,y\in H_1$, $h,g\in H_{\alpha\beta}$ and $a\in A$. This semi-Hopf π-coalgebra is called a Hopf π-crossed product and denoted by $A\natural_\sigma^\pi H$.

Corollary 3.7. *Let A be a Hopf algebra with antipode S_A, $H=(\{H_\alpha\},\Delta,\varepsilon,S)$ a Hopf π-coalgebra. If $A\natural_\sigma^\pi H$ is a Hopf π-crossed product (semi-Hopf π-coalgebra), then $A\natural_\sigma^\pi H$ is a Hopf π-coalgebra with antipode $\bar{S}''=\{\bar{S}_\alpha''\}_{\alpha\in\pi}$ described by*

$$\bar{S}_\alpha''(a\#_\sigma h)=(1_A\#_\sigma S_\alpha(h))(S_A(a)\#_\sigma 1_{\alpha^{-1}}),$$

for all $a\in A$ and $h\in H_\alpha$ if and only if H is a σ-Hopf π-coalgebra.

We also call this Hopf π-coalgebra *Hopf π-crossed product* and also denote by $A\natural_\sigma^\pi H$.

4. The Category ${}^{A\natural_\sigma^\pi H}\mathcal{M}$ Is Braided

This section is devoted to discussing when the category ${}^{A\natural_\sigma^\pi H}\mathcal{M}$ of the left comodules over the Hopf π-crossed product $A\natural_\sigma^\pi H$ is braided, following the idea of [2].

Definition 4.1. Let $H=(\{H_\alpha\},\Delta,\varepsilon,S)$ be a Hopf π-coalgebra. H is called a coquasitriangular Hopf π-coalgebra if there exists a linear map $\theta:H_1\otimes H_1\longrightarrow k$ such that for all $\alpha\in\pi$, $x,y\in H_\alpha$ and $h,l,z\in H_1$, satisfying the following conditions:

$(CQT1)\quad \theta(h,zl)=\theta(h_{(1,1)},l)\theta(h_{(2,1)},z);$

$(CQT2)\quad \theta(hz,l)=\theta(h,l_{(1,1)})\theta(z,l_{(2,1)});$

$(CQT3)\quad \theta(x_{(1,1)},y_{(1,1)})x_{(2,\alpha)}y_{(2,\alpha)}=y_{(1,\alpha)}x_{(1,\alpha)}\theta(x_{(2,1)},y_{(2,1)});$

$(CQT4)\quad \theta(1_1,h)=\theta(h,1_1)=\varepsilon(h).$

REMARK. When $H = (\{H_\alpha\}, \Delta, \varepsilon, S)$ is a Hopf π-coalgebra, Definition 4.1 is equivalent to the Definition 5 in [10] since $(H_1, \Delta_{1,1}, \varepsilon, S_1)$ is a usual Hopf algebra (see [1]).

DEFINITION 4.2. Let A be a Hopf algebra and $H = \{H_\alpha\}_{\alpha\in\pi}$ a Hopf π-coalgebra. Assume that there exist a map $\sigma : H_1 \otimes H_1 \longrightarrow A$ and two bilinear forms $\beta : A \otimes A \longrightarrow k$, $\omega : A \otimes H_1 \longrightarrow k$ such that the following conditions

$(DP1)$ $\omega(ab, h) = \omega(a, h_{(1,1)})\omega(b, h_{(2,1)})$;
$(DP2)$ $\omega(a_1, g_{(2,1)}h_{(2,1)})\beta(a_2, \sigma(g_{(1,1)}, h_{(1,1)})) = \omega(a_1, h)\omega(a_2, g)$;
$(DP3)$ $\omega(1_A, h) = \varepsilon(h)$, $\omega(a, 1_1) = \varepsilon_A(a)$

are satisfied for all $h, g \in H_1$ and $a, b \in A$. Then we call (A, H, ω, σ) a *skew* (ω, σ, β)*-Hopf* π*-coalgebra quadruple*.

DEFINITION 4.3. Let A be a Hopf algebra and $H = \{H_\alpha\}_{\alpha\in\pi}$ a Hopf π-coalgebra. Assume that there exist a map $\sigma : H_1 \otimes H_1 \longrightarrow A$ and two bilinear forms $\beta : A \otimes A \longrightarrow k$, $\gamma : H_1 \otimes A \longrightarrow k$. Then (H, A, γ, σ) is called an *anti-skew* (γ, σ, β)*-Hopf* π*-coalgebra quadruple* if for all $h, g \in H_1$ and $a, b \in A$, the following conditions hold:

$(ADP1)$ $\beta(\sigma(h_{(1,1)}, g_{(1,1)}), a_1)\gamma(h_{(2,1)}g_{(2,1)}, a_2) = \gamma(h, a_1)\gamma(g, a_2)$;

$(ADP2)$ $\gamma(h, ab) = \gamma(h_{(1,1)}, b)\gamma(h_{(2,1)}, a)$;

$(ADP3)$ $\gamma(1_1, a) = \varepsilon_A(a)$, $\gamma(h, 1_A) = \varepsilon(h)$.

DEFINITION 4.4. Let A be a Hopf algebra and $H = \{H_\alpha\}_{\alpha\in\pi}$ a Hopf π-coalgebra. Assume that there exist a family of maps $\sigma_\alpha : H_\alpha \otimes H_\alpha \longrightarrow A \otimes H_\alpha$ and two bilinear forms $\omega : A \otimes H_1 \longrightarrow k$, $\gamma : H_1 \otimes A \longrightarrow k$. A *coquasitriangular-like Hopf* π*-coalgebra associated to* $(\omega, \gamma, \sigma_\alpha)$ is a pair (H, φ) where $\varphi : H_1 \otimes H_1 \longrightarrow k$ is a bilinear form satisfying

$(CQTL1)$ $\omega(\sigma(h_{(1,1)}, g_{(1,1)}), l_{(1,1)})\varphi(h_{(2,1)}g_{(2,1)}, l_{(2,1)}) = \varphi(h, l_{(1,1)})\varphi(g, l_{(2,1)})$;

$(CQTL2)$ $\varphi(h_{(1,1)}, g_{(2,1)}l_{(2,1)})\gamma(h_{(2,1)}, \sigma(g_{(1,1)}, l_{(1,1)})) = \varphi(h_{(1,1)}, l)\varphi(h_{(2,1)}, g)$;

$(CQTL3)$
$$\varphi(x_{(1,1)}, y_{(1,1)})(\sigma(x_{(2,1)}, y_{(2,1)}) \otimes x_{(3,\alpha)}y_{(3,\alpha)}) = (\sigma(y_{(1,1)}, x_{(1,1)}) \otimes y_{(2,\alpha)}x_{(2,\alpha)})\varphi(x_{(3,1)}, y_{(3,1)});$$

$(CQTL4)$ $\varphi(1_1, h) = \varphi(h, 1_1) = \varepsilon(h)$,

for all $x, y \in H_\alpha$, $h, g, l \in H_1$.

EXAMPLE 4.5. Let A be a Hopf algebra and $H = \{H_\alpha\}_{\alpha\in\pi}$ a Hopf π-coalgebra. If the map σ_α is trivial and β, ω, γ satisfies $(CQT4)$, $(DP3)$,

$(ADP3)$, respectively, then the coquasitriangular-like Hopf π-coalgebra (H, φ) associated to $(\omega, \gamma, \sigma_\alpha)$ is exactly the coquasitriangular Hopf π-coalgebra which means Definition 4.4 is a generalization of the Definition 4.1.

In the following, we describe the coquasitriangular Hopf π-coalgebra structures over the Hopf π-crossed product $A\natural_\sigma^\pi H$.

The following is obvious:

PROPOSITION 4.6. *Let $A\natural_\sigma^\pi H$ be a Hopf π-crossed product Hopf π-coalgebra. Define maps as follows:*

$$i : A \longrightarrow A\#_\sigma H_1, \quad i(a) \longrightarrow a\#_\sigma 1_1; \quad j : H_1 \longrightarrow A\#_\sigma H_1, \quad j(h) \longrightarrow 1_A\#_\sigma h,$$

for all $a \in A$, $h \in H_1$.

Then i is a bialgebra map.

Let $A\natural_\sigma^\pi H$ be a Hopf π-crossed product, and $\tau : A\#_\sigma H_1 \otimes A\#_\sigma H_1 \longrightarrow k$ a bilinear form. Define:

$$\begin{aligned}
&\varphi : H_1 \otimes H_1 \longrightarrow k,\ \varphi(h, g) = \tau(j \otimes j)(h \otimes g);\\
&\beta : A \otimes A \longrightarrow k,\ \beta(a, b) = \tau(i \otimes i)(a \otimes b);\\
&\omega : A \otimes H_1 \longrightarrow k,\ \omega(a, h) = \tau(i \otimes j)(a \otimes h);\\
&\gamma : H_1 \otimes A \longrightarrow k, \gamma(h, a) = \tau(j \otimes i)(h \otimes a).
\end{aligned}$$

The following result is straightforward.

PROPOSITION 4.7. *With notation as above. Let $A\natural_\sigma^\pi H$ be a Hopf π-crossed product Hopf π-coalgebra. If τ satisfies $(CQT4)$, then for $a \in A$ and $h \in H_1$, we have*

$$\begin{aligned}
&(1)\quad \varphi(1_1, h) = \varphi(h, 1_1) = \varepsilon(h);\\
&(2)\quad \omega(1_A, h) = \varepsilon(h), \quad \omega(a, 1_1) = \varepsilon_A(a);\\
&(3)\quad \gamma(1_1, a) = \varepsilon_A(a), \quad \gamma(h, 1_A) = \varepsilon(h);\\
&(4)\quad \beta(1_A, a) = \beta(a, 1_A) = \varepsilon_A(a).
\end{aligned}$$

PROPOSITION 4.8. *Let $A\natural_\sigma^\pi H$ be a Hopf π-crossed product Hopf π-coalgebra and $\tau : A\#_\sigma H_1 \otimes A\#_\sigma H_1 \longrightarrow k$ a bilinear form. If $(A\natural_\sigma^\pi H, \tau)$ is a coquasitriangular Hopf π-coalgebra, then we have:*

$$\tau(a \otimes h, b \otimes g) = \omega(a_1, g_{(1,1)})\beta(a_2, b_1)\varphi(h_{(1,1)}, g_{(2,1)})\gamma(h_{(2,1)}, b_2) \quad (*)$$

for all $a, b \in A$ and $h, g \in H_1$.

PROOF. For all $a, a', b, b' \in A$ and $h, h', g, g' \in H_1$, by $(CQT1)$ and $(CQT2)$, we have

$$\begin{aligned}
&\tau((a \otimes h)(a' \otimes h'), (b \otimes g)(b' \otimes g')) \\
\overset{(CQT1)}{=} &\tau((a_1 \otimes h_{(1,1)})(a'_1 \otimes h'_{(1,1)}), b' \otimes g')\tau((a_2 \otimes h_{(2,1)})(a'_2 \otimes h'_{(2,1)}), b \otimes g) \\
\overset{(CQT2)}{=} &\tau(a_1 \otimes h_{(1,1)}, b'_1 \otimes g'_{(1,1)})\tau(a'_1 \otimes h'_{(1,1)}, b'_2 \otimes g'_{(2,1)})\tau(a_2 \otimes h_{(2,1)}, b_1 \otimes g_{(1,1)}) \\
&\times \tau(a'_2 \otimes h'_{(2,1)}, b_2 \otimes g_{(2,1)}). \quad (\star)
\end{aligned}$$

Letting $a' = b' = 1_A$ and $h = g = 1_1$ in Eq. $(\star)$, we obtain

$$\tau(a \otimes h, b \otimes g) = \omega(a_1, g_{(1,1)})\varphi(h_{(1,1)}, g_{(2,1)})\beta(a_2, b_1)\gamma(h_{(2,1)}, b_2).$$

□

PROPOSITION 4.9. *Let $A\natural^{\pi}_{\sigma}H$ be a Hopf π-crossed product Hopf π-coalgebra. Assume that the formula $(*)$ defines a coquasitriangular structure over $A\natural^{\pi}_{\sigma}H$. Then we have the following identities:*

$(BC1)$ $\beta(h_{(1,1)} \cdot a, b_1)\gamma(h_{(2,1)}, b_2) = \gamma(h, b_1)\beta(a, b_2)$;

$(BC2)$ $\omega(a_1, h_{(2,1)})\beta(a_2, h_{(1,1)} \cdot b) = \beta(a_1, b)\omega(a_2, h)$;

$(BC3)$ $\omega(h_{(1,1)} \cdot a, g_{(1,1)})\varphi(h_{(2,1)}, g_{(2,1)}) = \varphi(h, g_{(1,1)})\omega(a, g_{(2,1)})$;

$(BC4)$ $\varphi(h_{(1,1)}, g_{(2,1)})\gamma(h_{(2,1)}, g_{(1,1)} \cdot a) = \gamma(h_{(1,1)}, a)\varphi(h_{(2,1)}, g)$;

$(BC5)$ $\omega(a_1, x_{(1,1)})(a_2 \otimes x_{(2,\alpha)}) = ((x_{(1,1)} \cdot a_1) \otimes x_{(2,\alpha)})\omega(a_2, x_{(3,1)})$;

$(BC6)$ $\gamma(x_{(1,1)}, a_1)((x_{(2,1)} \cdot a_2) \otimes x_{(3,\alpha)}) = (a_1 \otimes x_{(1,\alpha)})\gamma(x_{(2,1)}, a_2)$,

where $a, b \in A$, $h, g \in H_1$ and $x \in H_\alpha$.

PROOF. By $(CQT1)$, one has:

(4.1) $$\tau(a \otimes h, (b \otimes g)(c \otimes l)) = \tau(a_1 \otimes h_{(1,1)}, c \otimes l)\tau(a_2 \otimes h_{(2,1)}, b \otimes g).$$

By $(CQT2)$, we have:

(4.2) $$\tau((a \otimes h)(b \otimes g), c \otimes l) = \tau(a \otimes h, c_1 \otimes l_{(1,1)})\tau(b \otimes g, c_2 \otimes l_{(2,1)})$$

and by $(CQT3)$ one knows:

(4.3) $$\tau(a_1 \otimes x_{(1,1)}, b_1 \otimes y_{(1,1)})(a_2 \otimes x_{(2,\alpha)})(b_2 \otimes y_{(2,\alpha)}) = (b_1 \otimes y_{(1,\alpha)})(a_1 \otimes x_{(1,\alpha)})\tau(a_2 \otimes x_{(2,1)}, b_2 \otimes y_{(2,1)})$$

Let $a = 1_A$, $g = l = 1_1$ in Eq. (4.2), the one obtains $(BC1)$ by the formula of τ. Similarly by letting $a = c = 1_A$, $g = 1_1$ in Eq. (4.2) and by the formula of τ, then $(BC3)$ holds.

Letting $b = 1_A,\ h = l = 1_1$ and $a = b = 1_A,\ l = 1_1$ in Eq. (4.1) respectively, then by the formula of τ we have $(BC2)$ and $(BC4)$.

Let $x = 1_\alpha,\ b = 1_A$ in Eq. (4.3), we have $(BC5)$, and letting $a = 1_A,\ y = 1_\alpha$ in Eq. (4.3), we can get $(BC6)$. □

PROPOSITION 4.10. *Let $A\natural_\sigma^\pi H$ be a Hopf π-crossed product Hopf π-coalgebra. Assume that the formula $(*)$ defines a coquasitriangular structure over $A\natural_\sigma^\pi H$. Then we have*

(1) *(A,β) is a coquasitriangular Hopf algebra.*

(2) *(A,H,ω,σ) is a skew (ω,σ,β)-Hopf π-coalgebra quadruple.*

(3) *(H,A,γ,σ) is an anti-skew(γ,σ,β)-Hopf π-coalgebra quadruple.*

(4) *(H,φ) is a coquasitriangular-like Hopf π-coalgebra associated to $(\omega,\gamma,\sigma_\alpha)$.*

PROOF. It follows from Proposition 4.7 that φ,β,ω and γ respectively satisfy $(CQTL4)$, $(CQT4)$, $(DP3)$ and $(ADP3)$.

(1) Since $i : A \longrightarrow A\#_\sigma H_1$ is a bialgebra map, and $(A\natural_\sigma^\pi H,\ \tau)$ is a coquasitriangular Hopf π-coalgebra, (A,β) is a coquasitriangular Hopf algebra.

(2) It is obvious by letting $h = g = 1_1$ and $c = 1_A$ in Eq. (4.2) and letting $h = 1_1$ and $b = c = 1_A$ in Eq. (4.1).

(3) It is easy to be seen by setting $l = 1_1$ and $a = b = 1_A$ in Eq. (4.2) and setting $g = l = 1_1$ and $a = 1_A$ in Eq. (4.1).

(4) Let $a = b = c = 1_A$ in Eq. (4.2), one gets $(CQTL1)$; and by letting $a = b = c = 1_A$ in Eq. (4.1) one has $(CQTL2)$; by letting $a = b = 1_A$ in Eq. (4.3), we can complete the proof that (H,φ) is a coquasitriangular-like Hopf π-coalgebra associated to $(\omega,\gamma,\sigma_\alpha)$. □

THEOREM 4.11. *Let $A\natural_\sigma^\pi H$ be a Hopf π-crossed product Hopf π-coalgebra. If there exist forms $\beta : A\otimes A \longrightarrow k,\ \omega : A\otimes H_1 \longrightarrow k$, $\gamma : H_1\otimes A \longrightarrow k$, $\varphi : H_1\otimes H_1 \longrightarrow k$ such that the following conditions hold:*

(1) *(A,β) is a coquasitriangular Hopf algebra.*

(2) *(A,H,ω,σ) is a skew (ω,σ,β)-Hopf π-coalgebra quadruple*

(3) *(H,A,γ,σ) is an anti-skew (γ,σ,β)-Hopf π-coalgebra quadruple.*

(4) *(H,φ) is a coquasitriangular-like Hopf π-coalgebra associated to $(\omega,\gamma,\sigma_\alpha)$.*

(5) *The conditions $(BC1)\sim(BC6)$ in Proposition 4.9 hold.*

Then $(A\natural_\sigma^\pi H, \tau)$ *is a coquasitriangular Hopf* π*-coalgebra with the coquasitriangular structure given by the formula* $(*)$.

PROOF. It is obvious that τ satisfies $(CQT4)$.

In what follows, we show that $(CQT1)$ hold. Similarly we can get $(CQT2)$. For all $a, b, c \in A$ and $h, g, l \in H_1$, we have

$$
\begin{aligned}
&\tau((a \otimes h), (b \otimes g)(c \otimes l)) \\
&= \tau(a \otimes h, b(g_{(1,1)} \cdot c)\sigma(g_{(2,1)}, l_{(1,1)}) \otimes g_{(3,1)} l_{(2,1)}) \\
&= \omega(a_1, (g_{(3,1)} l_{(2,1)})_{(1,1)}) \beta(a_2, (b(g_{(1,1)} \cdot c)\sigma(g_{(2,1)}, l_{(1,1)}))_1) \\
&\quad \times \varphi(h_{(1,1)}, (g_{(3,1)} l_{(2,1)})_{(2,1)}) \gamma(h_{(2,1)}, (b(g_{(1,1)} \cdot c)\sigma(g_{(2,1)}, l_{(1,1)}))_2) \\
&= \omega(a_1, g_{(3,1)} l_{(2,1)}) \beta(a_2, b_1 (g_{(1,1)} \cdot c)_1 \sigma(g_{(2,1)}, l_{(1,1)})_1) \varphi(h_{(1,1)}, g_{(4,1)} l_{(3,1)}) \\
&\quad \times \gamma(h_{(2,1)}, b_2 (g_{(1,1)} \cdot c)_2 \sigma(g_{(2,1)}, l_{(1,1)})_2) \\
&\overset{(D3)}{=} \omega(a_1, g_{(3,1)} l_{(2,1)}) \beta(a_2, b_1 (g_{(1,1)} \cdot c)_1 \sigma(g_{(2,1)}, l_{(1,1)})) \varphi(h_{(1,1)}, g_{(5,1)} l_{(4,1)}) \\
&\quad \times \gamma(h_{(2,1)}, b_2 (g_{(1,1)} \cdot c)_2 \sigma(g_{(4,1)}, l_{(3,1)})) \\
&\overset{(D4)}{=} \omega(a_1, g_{(3,1)} l_{(2,1)}) \beta(a_2, b_1 (g_{(1,1)} \cdot c_1) \sigma(g_{(2,1)}, l_{(1,1)})) \varphi(h_{(1,1)}, g_{(6,1)} l_{(4,1)}) \\
&\quad \times \gamma(h_{(2,1)}, b_2 (g_{(4,1)} \cdot c_2) \sigma(g_{(5,1)}, l_{(3,1)})) \\
&\overset{(CQT1)}{=} \omega(a_1, g_{(3,1)} l_{(2,1)}) \beta(a_2, \sigma(g_{(2,1)}, l_{(1,1)})) \beta(a_3, g_{(1,1)} \cdot c_1) \beta(a_4, b_1) \varphi(h_{(1,1)}, g_{(6,1)} l_{(4,1)}) \\
&\quad \times \gamma(h_{(2,1)}, b_2 (g_{(4,1)} \cdot c_2) \sigma(g_{(5,1)}, l_{(3,1)})) \\
&\overset{(ADP2)}{=} \omega(a_1, g_{(3,1)} l_{(2,1)}) \beta(a_2, \sigma(g_{(2,1)}, l_{(1,1)})) \beta(a_3, g_{(1,1)} \cdot c_1) \beta(a_4, b_1) \varphi(h_{(1,1)}, g_{(6,1)} l_{(4,1)}) \\
&\quad \times \gamma(h_{(2,1)}, \sigma(g_{(5,1)}, l_{(3,1)})) \gamma(h_{(3,1)}, g_{(4,1)} \cdot c_2) \gamma(h_{(4,1)}, b_2) \\
&\overset{(DP2)}{=} \omega(a_1, l_{(1,1)}) \omega(a_2, g_{(2,1)}) \beta(a_3, g_{(1,1)} \cdot c_1) \beta(a_4, b_1) \varphi(h_{(1,1)}, g_{(5,1)} l_{(3,1)}) \\
&\quad \times \gamma(h_{(2,1)}, \sigma(g_{(4,1)}, l_{(2,1)})) \gamma(h_{(3,1)}, g_{(3,1)} \cdot c_2) \gamma(h_{(4,1)}, b_2) \\
&\overset{(CQTL2)}{=} \omega(a_1, l_{(1,1)}) \omega(a_2, g_{(2,1)}) \beta(a_3, g_{(1,1)} \cdot c_1) \beta(a_4, b_1) \varphi(h_{(1,1)}, l_{(2,1)}) \\
&\quad \times \varphi(h_{(2,1)}, g_{(4,1)}) \gamma(h_{(3,1)}, g_{(3,1)} \cdot c_2) \gamma(h_{(4,1)}, b_2) \\
&\overset{(BC2)}{=} \omega(a_1, l_{(1,1)}) \beta(a_2, c_1) \omega(a_3, g_{(1,1)}) \beta(a_4, b_1) \varphi(h_{(1,1)}, l_{(2,1)}) \\
&\quad \times \varphi(h_{(2,1)}, g_{(3,1)}) \gamma(h_{(3,1)}, g_{(2,1)} \cdot c_2) \gamma(h_{(4,1)}, b_2) \\
&\overset{(BC4)}{=} \omega(a_1, l_{(1,1)}) \beta(a_2, c_1) \omega(a_3, g_{(1,1)}) \beta(a_4, b_1) \varphi(h_{(1,1)}, l_{(2,1)}) \\
&\quad \times \varphi(h_{(3,1)}, g_{(2,1)}) \gamma(h_{(2,1)}, c_2) \gamma(h_{(4,1)}, b_2) \\
&= \tau(a_1 \otimes h_{(1,1)}, c \otimes l) \tau(a_2 \otimes h_{(2,1)}, b \otimes g).
\end{aligned}
$$

Next we check that $(CQT3)$ for τ as follows. For all $a, b \in A$ and $x, y \in H_\alpha$, we have

$$
\begin{aligned}
&\tau(a_1 \otimes x_{(1,1)}, b_1 \otimes y_{(1,1)})(a_2 \otimes x_{(2,\alpha)})(b_2 \otimes y_{(2,\alpha)})\\
&= \omega(a_1, y_{(1,1)})\beta(a_2, b_1)\varphi(x_{(1,1)}, y_{(2,1)})\gamma(x_{(2,1)}, b_2)(a_3 \otimes x_{(3,\alpha)})(b_3 \otimes y_{(3,\alpha)})\\
&= \omega(a_1, y_{(1,1)})\beta(a_2, b_1)\varphi(x_{(1,1)}, y_{(2,1)})\gamma(x_{(2,1)}, b_2)(a_3(x_{(3,1)} \cdot b_3)\sigma(x_{(4,1)}, y_{(3,1)})\\
&\quad \otimes x_{(5,\alpha)}y_{(4,\alpha)})\\
&\overset{(BC6)}{=} \omega(a_1, y_{(1,1)})\beta(a_2, b_1)\varphi(x_{(1,1)}, y_{(2,1)})\gamma(x_{(4,1)}, b_3)(a_3b_2\sigma(x_{(2,1)}, y_{(3,1)}) \otimes x_{(3,\alpha)}y_{(4,\alpha)})\\
&\overset{(CQTL3)}{=} \omega(a_1, y_{(1,1)})\beta(a_2, b_1)\varphi(x_{(3,1)}, y_{(4,1)})\gamma(x_{(4,1)}, b_3)(a_3b_2\sigma(y_{(2,1)}, x_{(1,1)}) \otimes y_{(3,\alpha)}x_{(2,\alpha)})\\
&\overset{(CQT3)}{=} \omega(a_1, y_{(1,1)})\beta(a_3, b_2)\varphi(x_{(3,1)}, y_{(4,1)})\gamma(x_{(4,1)}, b_3)(b_1a_2\sigma(y_{(2,1)}, x_{(1,1)}) \otimes y_{(3,\alpha)}x_{(2,\alpha)})\\
&\overset{(BC5)}{=} \omega(a_2, y_{(4,1)})\beta(a_3, b_2)\varphi(x_{(3,1)}, y_{(5,1)})\gamma(x_{(4,1)}, b_3)(b_1(y_{(1,1)} \cdot a_1)\sigma(y_{(2,1)}, x_{(1,1)})\\
&\quad \otimes y_{(3,\alpha)}x_{(2,\alpha)})\\
&= (b_1(y_{(1,1)} \cdot a_1)\sigma(y_{(2,1)}, x_{(1,1)}) \otimes y_{(3,\alpha)}x_{(2,\alpha)})\tau(a_2 \otimes x_{(3,1)}, b_2 \otimes y_{(4,1)})\\
&= (b_1 \otimes y_{(1,\alpha)})(a_1 \otimes x_{(1,\alpha)})\tau(a_2 \otimes x_{(2,1)}, b_2 \otimes y_{(2,1)}).
\end{aligned}
$$

□

Thus it follows from Proposition 4.8, Proposition 4.9, Proposition 4.10 and Theorem 4.11 that we have

THEOREM 4.12. *Hopf π-crossed product Hopf π-coalgebra $A\natural_\sigma^\pi H$ is coquasitriangular if and only if there exist forms $\beta : A \otimes A \longrightarrow k$, $\omega : A \otimes H_1 \longrightarrow k$, $\gamma : H_1 \otimes A \longrightarrow k$, $\varphi : H_1 \otimes H_1 \longrightarrow k$ such that (A, β) is a coquasitriangular Hopf algebra, (A, H, ω, σ) is a skew (ω, σ, β)-Hopf π-coalgebra quadruple, (H, A, γ, σ) is an anti-skew (γ, σ, β)-Hopf π-coalgebra quadruple, (H, φ) is a coquasitriangular-like Hopf π-coalgebra associated to $(\omega, \gamma, \sigma_\alpha)$ and the conditions $(BC1) \sim (BC6)$ in Proposition 4.9 are satisfied. Moreover, the coquasitriangular structure τ on $A\natural_\sigma^\pi H$ has a decomposition*

$$\tau(a \otimes h, b \otimes g) = \omega(a_1, g_{(1,1)})\beta(a_2, b_1)\varphi(h_{(1,1)}, g_{(2,1)})\gamma(h_{(2,1)}, b_2).$$

REMARK. Theorem 4.12 shows that if A is not a coquasitriangular Hopf algebra then the Hopf π-crossed product Hopf π-coalgebra $A\natural_\sigma^\pi H$ is not coquasitriangular either.

Next by Theorem 2 in [10], we have

COROLLARY 4.13. *If there exist forms $\beta : A \otimes A \longrightarrow k$, $\omega : A \otimes H_1 \longrightarrow k$, $\gamma : H_1 \otimes A \longrightarrow k$, $\varphi : H_1 \otimes H_1 \longrightarrow k$ such that (A, β) is a coquasitriangular Hopf algebra, (A, H, ω, σ) is a skew (ω, σ, β)-Hopf π-coalgebra quadruple, (H, A, γ, σ) is an anti-skew (γ, σ, β)-Hopf π-coalgebra quadruple, (H, φ) is a coquasitriangular-like Hopf π-coalgebra associated to $(\omega, \gamma, \sigma_\alpha)$ and the conditions $(BC1) \sim (BC6)$ in Proposition 4.9 hold, then the category ${}^{A\natural^{\pi}_{\sigma}H}\mathcal{M}$ of the left comodules over the Hopf π-crossed product Hopf π-coalgebra $A\natural^{\pi}_{\sigma}H$ is braided monoidal category.*

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