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Uniform homeomorphisms between Banach spaces

Séminaire d'analyse fonctionnelle (Polytechnique) (1975-1976), exp. nº 18, p. 1-6
<http://www.numdam.org/item?id=SAF_1975-1976____A14_0>

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SEMINAIRE MAUREY-SCHWARTZ 1975-1976

UNIFORM HOMEOMORPHISMS BETWEEN BANACH SPACES

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Exposé No XVIII

6 Avril 1976

One of the most interesting questions in uniform topology on infinite-dimensional spaces is the following problem : if two Banach spaces are uniformly homeomorphic (i.e. if there is a bijection f between them s.t. f and f^{-1} are uniformly continuous) are they (linearly) isomorphic ? (See Bessaga [1].) ... (1)

In this seminar we will be mainly concerned with some recent development on this question. Before going into this we will, however, first point out some related results and questions on uniform topology. The question above could naturally be studied also in the more general contexts of Fréchet spaces, topological linear spaces or even commutative topological groups. It is also natural to study simultaneously problems about Lipchitz homeomorphisms between spaces. Two metric spaces are called Lipschitz equivalent if there is a bijection f between them s.t. f and f^{-1} satisfy a first order Lipschitz condition with a uniform constant. One important reason for this is the simple fact that a uniformly continuous map g from a convex set in a Banach space into a metric space satisfies a first order Lipschitz condition for large distances in the following sense : for every $\delta > 0$ $\exists$ K_δ s.t. if $\|x - y\| \geq \delta$ then $d(g(x), g(y)) \leq K_\delta \|x - y\|$.

One of the first observations in this more general context is the following result (see [1]) :

1. If a locally convex space A is uniformly homeomorphic to a Banach space B then A is itself a Banach space [i.e. its topology is given by a norm].

<u>Proof</u> : By Kolmogoroff's characterization of normed spaces we need only to prove that A has a bounded neighbourhood of 0 [i.e. a neighbourhood U of 0 s.t. for every neighbourhood V of 0 there is an n s.t. $U \subset nV$]. Let E be the unit ball in B and let $f : B \to A$ be a uniform homeomorphism. Then for every convex neighbourhood V of 0 in A there is an $\frac{1}{n}$- ball W around 0 in B s.t.

$$f(E) = f(\underbrace{W + W + \ldots W}_{n \text{ terms}}) \subset \underbrace{V + V + \ldots V}_{n \text{ terms}} = nV \quad .$$

Thus f(E) is bounded and the proof is complete.

From Enflo [3] we mention the following results :

2. If a locally bounded space A [i.e. a topological linear space with a bounded neighbourhood of 0] is uniformly homeomorphic to a Banach space which is p-smooth, then A is itself a Banach space.

3. If a uniformly convex Banach space is given a commutative group structure $(x,y) \to xy$ s.t. $\exists$ K s.t. for all x, y, z $\|xz - yz\| \le K\|x - y\|$ and in addition $x^2 = y^2 \Rightarrow x = y$, then the group is isomorphic to (the additive group of) a Banach space.

Mankiewisz [8] [9] has studied uniform homeomorphisms and Lipschitz homeomorphisms between many classes of Fréchet spaces. Let s be the countable product of real lines. We mention here

4. If a Fréchet space is uniformly homeomorphic to s then it is isomorphic to s.

The problem (1) in this more general context is also open, for example we do not even know the answers to the following questions : If a a Banach space B is given a commutative group structure $(x,y) \to xy$ s.t. $(x,y) \to xy^{-1}$ is uniformly continuous, must the group be isomorphic to the additive group of B ? Can the group have the property $x^2 = e$ (the unit element) for all x ?

Another type of question that has been studied is the problem of uniform embedding. The results that we will mention also shed some light on the question (1).

5. (See [5]). c_o is not uniformly homeomorphic to any subset of Hilbert space.

6. (Aharony) C(0,1) is uniformly homeomorphic to a subset of c_o.

We do not know whether $L_p(\mu)$ is uniformly homeomorphic to some subset of some $L_q(\nu)$, $1 \le p < q < \infty$. In Mazur [10] the following result was proved :

7. The unit ball in $L_p(0,1)$ is uniformly homeomorphic to the unit ball in $L_q(0,1)$, $1 \le p \le q < \infty$.

Of these results 5 reminds more of the results 1-4 and 6 and 7 reminds more about results on infinite-dimensional topology which say that many different objects are homeomorphic. See Bessaga and Pełczynski [2]. We now go back to the question (1).

The first results were obtained by Lindenstrauss [7] who proved that many separable Banach spaces are not uniformly homeomorphic. Together with the results in [4] they give the following :

8. $L_p(\mu)$ is not uniformly homeomorphic to $L_q(\nu)$ for $1 \leq p < q < \infty$.

More examples of non uniformly homeomorphic Banach spaces were given in Henkin [6].

In [3] the following result was proved :

9. If a Banach space is uniformly homeomorphic to a Hilbert space then it is isomorphic to the Hilbert space.

The methods to obtain the results 4 and 9 (and many of Mankiewisz's other results)make use of the connection between uniformly continuous maps and Lipschitz maps. They also make use of Fréchet derivatives of certain Lipschitz maps in order to get linear maps from Lipschitz maps. The special symmetry properties of the spaces s and Hilbert space then give the results 4 and 9. A different approach to obtain linear maps from uniformly continuous maps has recently been found by Ribe [11] who proved the following result :

10. If two Banach spaces are uniformly homeomorphic then there is a number $C > 0$ s.t. every finite-dimensional subspace of one of the spaces is embeddable into the other by a linear map T with $\|T\| \, \|T^{-1}\| \leq C$. Another way of stating 10 would be to say that two uniformly homeomorphic Banach spaces are (crudely) locally equivalent. As is easily seen 10 implies the results 8 and 9. However, it still leaves the following problems open : Is ℓ_p uniformly homeomorphic to $L_p(0,1)$? For $p = 1$ the answer is known to be no. If a Banach space is uniformly homeomoprhic to a $\mathcal{L}_p$-space must it be a $\mathcal{L}_p$-space ?

We will spend the rest of this seminar trying to give the main ideas of the proof of 10. For details we refer to [11]. We say that G is a δ-net around H if for every $y \in H \ \exists\ z \in G$ with $d(z,y) \leq \delta$.

Let A and B be uniformly homeomorphic Banach spaces and $f : A \to B$ a uniform homeomorphism from A onto B. Fix an $\varepsilon > 0$ and let K_ε be the

Lipschitz constant of f corresponding to distances $\geq \varepsilon$. Take an n-dimensional subspace E of A with a basis $e_1,\dots,e_n$, not necessarily normalized. Choose a N and consider the set M of lattice points $\alpha_1 e_1 + \dots + \alpha_n e_n$, α_j integer, $|\alpha_j| \leq N$. By changing the scale we now assume

$$\min_{\substack{x,x'\in M \\ x\neq x'}} \|x - x'\| \geq \varepsilon \quad .$$

f is linearized in the following way. Put $h(y) = \frac{1}{m} \sum_{\substack{x,x'\in M \\ x'-x=y}} (f(x') - f(x))$

where m is the number of such pairs. It is easy to see that as N gets bigger h approximates better and better a linear map h_L in the following sense : Given $\delta > 0$ $\exists$ ω s.t. if $N > \omega$ and h_L is defined by $h_L(e_j) = h(e_j)$, $j = 1,2,\dots,n$ and h_L is extended by linearity then there is a δ-net around the unit sphere in E s.t. $\|h_L(y) - h(y)\| \leq \delta$ for every y in the δ-net. Moreover $\|h_L\| \leq 2K_\varepsilon$. The main problem is to ensure that h_L^{-1} exists and does not have too large a norm. The method of doing that is to pass to a sublattice of M since M itself might not work. Pick a vector y in the lattice M. We will show the following: $\exists$ numbers K_f, m_o, N_1 and N_2 and a $u \in E^*$, $\|u\| = 1$, and a sublattice of M of the form $w + 2^{m_o} \Sigma \alpha_j e_j$ $|\alpha_j| \leq N_1$ s.t. in this sublattice

$u(f(x') - f(x)) \geq K_f \,.\, k \,.\, 2^{m_o} \|y\|$ for every pair x, x', $x' - x = k \,.\, 2^{m_o} y$,....(1)

Here K_f depends only on f. N_1 and N_2 tend to infinity with N uniformly in y if y runs through a bounded set. 2^{m_o} is a scale increase of the distances in M which we get from the construction of the sublattice. Obviously $\|h(k\, 2^{m_o} y)\| \geq K_f \,.\, k\, 2^{m_o} \|y\|$, $k = 1,2,\dots,N_2$ if h is defined from this sublattice. So by now picking a new vector and passing to a sublattice of the sublattice we can ensure that h is big also on multiples of another vector. By repeating this process for sufficiently many vectors in E we get $\|h_L y\| \geq \frac{K_f}{2} \|y\|$ for every y in E.

Now we show (2). Given a vector y and a lattice M with a large N consider for every m

$$\sup_{\substack{z+2^m y\in M \\ z\in M}} \left\| \frac{f(z + 2^m y) - f(z)}{2^m} \right\| \quad .$$

By the triangle inequality this is a decreasing function of m and since it is bounded away from 0 by say $2K_f\|y\|$, there must be arbitrarily long sequeces of m's for which it is almost constant (the lengths tending to infinity with N). Assume that this happens for $m_o, m_o+1, \dots m_o+j$. Take a functional $u \in E^*$, $\|u\| = 1$ and a z_o such that

$$u\left[\frac{f(z_o + 2^{m_o+j}y) - f(z_o)}{2^{m_o+j}}\right] = \sup_z \left\|\frac{f(z + 2^{m_o+j}y) - f(z)}{2^{m_o+j}}\right\| .$$

Since this sup is almost constant and $\geq 2K_f\|y\|$ we get

$$u\left[\frac{f(z_o + (k+1)2^{m_o}y) - f(z_o + k\,2^{m_o}y)}{2^{m_o}}\right] \geq K_f\|y\|$$

for all k, $0 \leq k < 2^j$ just by the triangle inequality. For every $v = 2^{m_o}(\alpha_1 e_1 + \dots + \alpha_n e_n)$ s.t. $z_o + v$ and $z_o + v + 2^{m_o+j}y$ are in M we consider for every k, $0 \leq k \leq 2^j$

$$u\left[\frac{f(z_o + v + (k+1)2^{m_o}y) - f(z_o + v + k\,2^{m_o}y)}{2^{m_o}}\right] .$$

If $\|v\|$ is sufficiently small compared to $2^{m_o+j}\|y\|$, then the number of k's for which this last expression is $< \|K_f\|y\|$ is small compared to 2^j. Call these k's exceptional. So by removing form every "line" $z_o + v + k.2^{m_o}y$, $0 \leq k < 2^j$ all the exceptional k's we will still be left with an arbitrary big **sublattice** consisting of all points

$$z_o + v + (k_o+k')2^{m_o}y \quad , \quad k' = 1,2,\dots,N_2' \quad ,$$

v has small norm compared to $2^{m_o+j}\|y\|$ and k_o+k' is not exceptional for any v. It is easy to see that this lattice can be written in the form (2).

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