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A DISJOINTNESS PROPERTY OF ℓ_p^n SEQUENCES IN L_p

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In [1] L. E. Dor proved that a subspace of $L_1(0, 1)$ which is almost isometric to a $L_1'(\mu)$ space is well complemented. The purpose of this note is to prove the analogous theorem for $1 < p < \infty$ thus solving a problem of Enflo and Rosenthal [2] and of Dor [1]. Since a detailed proof will appear shortly in [3], I'll try to give here a less formal and, hopefuly, more intuitive proof.

Theorem 1: let $1 < p < \infty$. There exist a $\lambda_0 > 1$ and a function $\varphi(\lambda)$, defined for $1 < \lambda < \lambda_0$, such that $\varphi(\lambda) \longrightarrow 1^+$ as $\lambda \longrightarrow 1^+$ and if $x_1, \dots x_n$ are functions in $L_p(0,1)$ which satisfy

$$\lambda^{-1}\left(\sum_{i=1}^{n} |a_i|^p\right)^{1/p} \leq \left\| \sum_{i=1}^{n} a_i x_i \right\| \leq \lambda\left(\sum_{i=1}^{n} |a_i|^p\right)^{1/p}$$

for all sequences $a_1, \dots, a_n$ of scalars then $[x_i]_{i=1}^{n}$ is complemented in $L_p(0,1)$ by means of a projection of norm at most $\varphi(\lambda)$.

It is well known that this implies that any $\mathcal{L}_{p,\lambda}$ subspace of $L_p(0,1)$ is complemented if λ is small enough (and the norm of the projection tends to 1 as $\lambda \longrightarrow 1$). Also, a simple perturbation argument shows that Theorem 1 is a consequence of

Theorem 2: Let $1 < p < \infty$, $p \neq 2$. There exists a function $a(\epsilon)$ such that $a(\epsilon) \longrightarrow 0$

as $\epsilon \longrightarrow 0$ *and, if* $x_1, \ldots, x_n$ *are functions in* $L_p(0,1)$ which satisfy

$$(1-\epsilon)(\sum_{i=1}^{n}|a_i|^p)^{1/p} \le \|\sum_{i=1}^{n} a_i x_i\| \le (1+\epsilon)(\sum_{i=1}^{n}|a_i|^p)^{1/p}$$

for all scalars $a_1, \ldots, a_n$, *then there exist disjoint sets* $A_1, \ldots, A_n$ *of* $[0,1]$ *such that*

$$\|\sum_{i=1}^{n} a_i(x_i - x_i|_{A_i})\| \le a(\epsilon)(\sum_{i=1}^{n}|a_i|^p)^{1/p}$$

for all scalars $a_1,\ldots,a_n$.

Indeed, if Theorem 2 is true let P be a norm one projection from $L_p(0,1)$ onto $[x_i|_{A_i}]_{i=1}^n$. The conclusion of Theorem 2 ensures that $P|_{[x_i]_{i=1}^\infty}$ is an isomorphism provided ϵ is small enough (and $\|(P|_{[x_i]_{i=1}^n})^{-1}\| \longrightarrow 1$ as $\epsilon \longrightarrow 0$). So the desired projection is given by $(P|_{[x_i]_{i=1}^n})^{-1} P$.

Theorem 2 is a stronger version of the following theorem of Dor [1] :

Let $1 \le p < \infty$, $p \ne 2$. *There exists a function* $a(\epsilon)$ such that $a(\epsilon) \to 0$ *as* $\epsilon \longrightarrow 0$ *and if* $x_1, \ldots, x_n$ *is a normalized sequence in* $L_p(0,1)$ *such that*

$$(1-\epsilon)(\sum_{i=1}^{n}|a_i|^p)^{1/p} \le \|\sum_{i=1}^{n} a_i x_i\| \le (1+\epsilon)(\sum_{i=1}^{n}|a_i|^p)^{1/p}$$

<u>for all scalars</u> $a_1, \dots, a_n$, <u>then there are disjoint sets</u> $A_1, \dots, A_n$ <u>of</u> $[0,1]$ <u>such that</u>

$$\|x_i | A_i^c\| \le a(\varepsilon), \qquad i = 1, \dots, n. \quad (A_i^c \text{ is the complement of } A).$$

The case $p = 1$ easily implies the analogue of Theorem 2 for $p = 1$. As we we'll see the proof of Theorem 2 for $1 < p < \infty$ is also based on Dor's theorem, however the deduction of Theorem 2 from Dor's theorem is much more complicated in this case.

We divide the proof into three parts. The first one is a reduction of the general case to the case where the x_i are exchangeable.

Fix n and $\varepsilon > 0$. Given normalized $x_1, \dots, x_n$ which satisfy the assumption of Theorem 2 we find disjoint sets $\{A_i\}_{i=1}^n$ as in Dor's theorem. Let Π denote the set of all permutations of $(1, \dots, n)$, let $\{I_\pi\}_{\pi\in\Pi}$ be a collection of disjoint subintervals of $[0,1]$ each of length $1/n!$ and for $\pi \in \Pi$, let φ_π be the natural linear transformation of I_π onto $[0,1]$.

Fix $\{a_i\}_{i=1}^\infty$ such that $(\sum_{i=1}^n |a_i|^p)^{1/p} = n^{1/p}$ and define a sequence $\{f_i\}_{i=1}^n$ of $L_p(0,1)$ functions by:

$$f_i(t) = a_{\pi(i)} x_{\pi(i)}(\varphi_\pi(t)) \quad \text{for } 1 \le i \le n,\ \pi \in \Pi \text{ and } t \in I_\pi$$

in a similar manner we define $\{g_i\}_{i=1}^n$ and $\{h_i\}_{i=1}^n$ using $x_i|A_i$ and $x_i - x_i|A_i$, respectively, instead of x_i.

Lemma 1: (a)

$$(1-\epsilon)(\sum_{i=1}^n |b_i|^p)^{1/p} \leq \|\sum_{i=1}^n b_i f_i\| \leq (1+\epsilon)(\sum_{i=1}^n |b_i|^p)^{1/p}$$

for every sequence $b_1, \ldots, b_n$ of scalars.

(b) $f_i = h_i + g_i$, g_i and h_i are disjointly supported for each $i=1,\ldots,n$. $g_1,\ldots, g_n$ are disjointly supported.

(c) $\|f_i\| = 1$, $\|h_i\| < a(\epsilon)$ and $\|g_i\| > (1-a(\epsilon)^p)^{1/p}$, $i = 1, \ldots, n$.

(d) $\|\sum_{i=1}^n h_i\| = \|\sum_{i=1}^n a_i z_i\|$

(e) $\{(g_i, h_i)\}_{i=1}^n$ is an exchangeable sequence; i.e., the distribution of the sequence

$$(g_1, h_1, g_2, h_2, \ldots, g_n, h_n)$$

is the same as the distribution of

$$(g_{\pi(1)}, h_{\pi(1)}, g_{\pi(2)}, h_{\pi(2)}, \ldots, g_{\pi(n)}, h_{\pi(n)})$$

for any $\pi \in \Pi$.

The proof is very simple, we'll prove only (d) and (e),

$$\|\sum_{i=1}^n h_i\| = \Big(\sum_{\pi\in\Pi}\int_{I_\pi} \Big|\sum_{i=1}^n a_{\pi(i)}\, z_{\pi(i)}\, (\varphi_\pi(t))\Big|^p dt\Big)^{1/p}$$

$$= \Big(\frac{1}{n!}\sum_{\pi\in\Pi}\int_0^1 \Big|\sum_{i=1}^n a_{\pi(i)} z_{\pi(i)}\Big|^p\Big)^{1/p}$$

$$= \Big(\frac{1}{n!}\sum_{\pi\in\Pi}\int_0^1 \Big|\sum_{i=1}^n a_i z_i\Big|^p\Big)^{1/p}$$

$$= \|\sum_{i=1}^n a_i z_i\|$$

To prove (e) we notice that for any $\pi, \rho \in \Pi$

$$\operatorname{dist}\{(g_i, h_i)_{|I_\pi}\}_{i=1}^n = \operatorname{dist}\{(g_{\rho(i)}, h_{\rho(i)})_{|I_{\pi\rho^{-1}}}\}_{i=1}^n$$

since both are equal to the distribution of

$$\{(a_{\pi(i)}\, y_{\pi(i)},\ a_{\pi(i)}\, z_{\pi(i)})\}_{i=1}^n$$

with respect to $\frac{1}{n!}\cdot$ Lebesque measure. $\square$

Lemma 1 reduces the proof of Theorem 2 to showing

$$\|\sum_{i=1}^n h_i\|^p \leq b(\varepsilon)\cdot n$$

for some function $b(\varepsilon)$, depending on ε and p alone, such that $b(\varepsilon)\longrightarrow 0$ as $\varepsilon \longrightarrow 0$.

The second step in the proof consists of the following inequality

Lemma 2: In the situation above

$$\left|\int_0^1 \left|\sum_{j=1}^n g_j + \sum_{k=1}^{\ell} h_k\right|^p - \frac{n}{\ell}\int_0^1 \left|\sum_{i=1}^{\ell} f_i\right|^p + \frac{h-\ell}{\ell}\int_0^1 \left|\sum_{k=1}^{\ell} h_k\right|^p\right| \leq \frac{n^2}{\ell}\, c(\cdot\epsilon)$$

for all $\ell = 1, \ldots, n$ where $c(\epsilon)$ depends on p and ϵ alone and

$c(\epsilon) \longrightarrow 0$ as $\epsilon \longrightarrow 0$.

We pospone the proof and continue with the third step which is the deduction of Theorem 2 from Lemma 2. As we mentioned above we'll give a heuristic proof which we hope will give the idea behind the proof. A complete formal proof, which however looks quite mysterious, can be found in [3].

The first object is to show that any two partial sums of the h_i with the same number of terms are closed each to the other.

Lemma 3: Let M_1, M_2 be two subsets of $\{1,\ldots,n\}$ of the same cardinality then

$$\left\|\sum_{k\in M_1} h_k - \sum_{k\in M_2} h_k\right\|^p \leq d(\epsilon)\cdot n$$

where $d(\epsilon) \longrightarrow 0$ as $\epsilon \longrightarrow 0$ and depends on ϵ and p alone.

Proof: First notice that it is enough to prove the lemma for $M_1 \cap M_2 = \emptyset$, and then it is enought to consider $M_1 = \{1,\ldots,\ell\}$, $M_2 = \{\ell+1,\ldots, 2\ell\}$ for some $1 \leq \ell \leq \frac{n}{2}$. Now since $\{h_i\}_{i=1}^n$ is exchangeable $h_1 - h_{\ell+1}$, $h_2 - h_{\ell+2}, \ldots, h_\ell - h_{2\ell}$ is a 1-unconditional basic sequence so if $1 < p < 2$

$$\left\|\sum_{i=1}^{\ell} h_i - \sum_{i=\ell+1}^{2\ell} h_i\right\| \leq \|h_i - h_{\ell+1}\| . \ell^{1/p} \leq 2\epsilon n^{1/p}.$$

If $p > 2$ the proof is more involved:

First notice that by Khinchine's inequality,

$$(1) \qquad \left\|\sum_{i=1}^{\ell} h_i - \sum_{i=\ell+1}^{2\ell} h_i\right\| \leq K_p \left\|\left(\sum_{i=1}^{\ell} |h_i - h_{\ell+1}|^2\right)^{1/2}\right\| \leq K_p 2^{1/2} \left\|\left(\sum_{i=1}^{2\ell} |h_i|^2\right)^{1/2}\right\|$$

for some constant K_p depending only on p.

Now, let $r_1, \ldots, r_n$ be the first n Rademacher functions, then

$$2\ell(1+\epsilon)^p \geq \int_0^1 \int_0^1 \left|\sum_{i=1}^{2\ell} r_i(t) f_i(s)\right|^p dt\,ds \geq \int_0^1 \left(\int_0^1 \left|\sum_{i=1}^{2\ell} r_i(t) f_i(s)\right|^2 dt\right)^{p/2} ds$$

$$= \int_0^1 \left(\sum_{i=1}^{2\ell} |f_i(s)|^2\right)^{p/2} ds = \int_0^1 \left(\sum_{i=1}^{2\ell} |g_i(s)|^2 + \sum_{i=1}^{2\ell} |h_i(s)|^2\right)^{p/2} ds$$

$$\geq \int_0^1 \left(\sum_{i=1}^{2\ell} |g_i(s)|^2\right)^{p/s} + \int_0^1 \left(\sum_{i=1}^{2\ell} |h_i(s)|^2\right)^{p/2}$$

$$= \int_0^1 \left|\sum_{i=1}^{2\ell} g_i(s)\right|^p + \int_0^1 \left(\sum_{i=1}^{2\ell} |h_i(s)|^2\right)^{p/2}$$

$$\geq (1-\epsilon^p)2\ell + \left\|\left(\sum_{i=1}^{2\ell} |h_i|^2\right)^{1/2}\right\|^p$$

which together with (1) finishes the proof.

We are going to use Lemma 2 only for $\ell = n/2$ and $\ell = n/4$ (assuming for simplicity that n is divisible by 4), for these values $\frac{n^2}{\ell} < 4.n$ also, using Lemma 3, we can write the conclusion of Lemma 2 as

$$(2) \qquad \left| \int_0^1 \left| \sum_{j=1}^{n} g_j + \frac{\ell}{n} \sum_{k=1}^{n} h_k \right|^p + \frac{n-\ell}{\ell} \int_0^1 \left| \frac{\ell}{n} \sum_{k=1}^{n} h_k \right|^p - n \right| \leq n.d(\varepsilon)$$

$\ell = \frac{n}{2}$ or $\ell = \frac{n}{4}$, $d(\varepsilon) \longrightarrow 0$ as $\varepsilon \longrightarrow 0$ and depends on ε and p alone.

Put $\qquad g = \sum_{i=1}^{n} g_i, \qquad h = \sum_{i=1}^{n} h_i.$

From Clarkson's inequality we get, for $1 < p < 2$,

$$\|g+\tfrac{1}{2}h\|^p + \|\tfrac{1}{2}h\|^p \leq \tfrac{1}{2}(\|g + h\|^p + \|g\|^p) \leq (1 + \tfrac{\varepsilon}{2})n$$

so (2) with $\ell = \frac{n}{2}$ says that we have an almost equality in Clarkson's inequality. The same thing holds for $p > 2$. Recall that equality in Clarkson's inequality holds if and only if the two functions are disjoint. This suggests that $g + \frac{1}{2}h$ and $\frac{1}{2}h$ are almost disjoint, that is, there exist two disjoint sets A and B such that $A \cup B = [0,1]$ and

$$(3) \qquad \|h_{|A}\|^p < e(\varepsilon).n \ , \qquad \|(g + \tfrac{1}{2}\, h)_{|B}\|^p < e(\varepsilon).n$$

for some $e(\varepsilon)$ with the same properties as the previous functions. (This can be proved using the proof of Proposition 2.1 in [1]).

Using (3) we can write (2) for $\ell = \frac{n}{4}$ in the form ($\approx$ means that the difference between the two sides is of the form $e(\varepsilon).n$ for an appropriate $e(\varepsilon)$.

$$n \approx \int_0^1 |g + \tfrac{1}{4}h|^p + 3\int_0^1 |\tfrac{1}{4}h|^p$$

$$\approx \int_A |g|^p + \int_B |g + \tfrac{1}{4}h|^p + 3\int_B |\tfrac{1}{4}h|^p$$

$$\approx \int_A |g|^p + \int_B |\tfrac{1}{2}g|^p + (2^p - 1)\int_B |\tfrac{1}{2}g|^p + (4-2^p)\int_B |\tfrac{1}{4}h|^p$$

$$\approx \int_A |g|^p - \int_B |g|^p + \frac{4 - 2^p}{4^p}\int_0^1 |h|^p$$

$$\approx n + \frac{4-2^p}{4^p}\|h\|^p$$

and this means that

$$\|h\|^p \le b(\varepsilon).n$$

We return now to the proof of Lemma 2. We first need another lemma.

Denote the support of g_i by B_i, $i = 1, \ldots, n$. By (e) of Lemma 1, whenever M_1 and M_2 are two subsets of $\{1,\ldots,n\}$ of the same cardinality and $1 \le i, j \le n$ satisfy either $i \in M_1$ and $j \in M_2$ or $i \notin M_1$ and $j \notin M_2$

$$\int_{B_i} \Big|\sum_{k\in M_1} h_k\Big|^p = \int_{B_j} \Big|\sum_{k\in M_2} h_k\Big|^p$$

and

$$\int_{B_i} \Big|g_i + \sum_{k\in M_1} h_k\Big|^p = \int_{B_j} \Big|g_j + \sum_{k\in M_2} h_k\Big|^p.$$

Indeed in each of these two cases there exists $\pi \in \Pi$ such that $\pi(M_1) = M_2$ and $\pi(i) = j$, so,

$$\mathrm{dist}(g_i, \sum_{k\in M_1} h_k) = \mathrm{dist}(g_j, \sum_{k\in M_1} h_{\pi(k)}) = \mathrm{dist}(g_j, \sum_{k\in M_2} h_k).$$

The next lemma asserts that, up to a certain error, the same is true without any restrictions on i and j.

Lemma 4: Let M_1, M_2 be subsets of $\{1,\dots,n\}$ with card M_1 = card M_2 and let i, j satisfy $1 \le i$, $j \le n$ then

(a) $$\left| \int_{B_i} \left|\sum_{k\in M_1} h_k\right|^p - \int_{B_i} \left|\sum_{k\in M_2} h_k\right|^p \right| < c(\varepsilon)$$

(b) $$\left| \int_{B_i} \left|g_i + \sum_{k\in M_1} h_k\right|^p - \int_{B_j} \left|g_j + \sum_{k\in M_2} h_k\right|^p \right| < c(\varepsilon)$$

for some function $c(\varepsilon)$ depending on p and ε alone and such that $c(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Proof: By the remark before the statement of the lemma, it is enough to assume that $M_1 = M_2$ $\{1,\dots,\ell\}$ for some $1 \le \ell < n$ and that $1 \le i \le \ell < j \le n$. First notice that, since

$$\operatorname{dist}(g_i, \sum_{k=1}^{\ell} h_k) = \operatorname{dist}(g_r, \sum_{k=1}^{\ell} h_k) \text{ for all } 1 \le r \le \ell,$$

(4) $$\left(\int_{B_i} \left|\sum_{k=1}^{\ell} h_k\right|^p\right)^{1/p} = \left(\frac{1}{\ell}\sum_{r=1}^{\ell} \int_{B_r} \left|\sum_{k=1}^{\ell} h_k\right|^p\right)^{1/p} \le \frac{1}{\ell^{1/p}} \left\|\sum_{k=1}^{\ell} h_k\right\|$$

$$\le \frac{1}{\ell^{1/p}} \left(\left\|\sum_{k=1}^{\ell} f_k\right\| + \left\|\sum_{k=1}^{\ell} g_k\right\|\right) \le 2 + \varepsilon$$

Now, since

$$\operatorname{dist}(g_j, \sum_{k=1}^{\ell} h_k) = \operatorname{dist}(g_i, \sum_{\substack{k=1\\ k\ne i}}^{\ell+1} h_k),$$

and since g_i and h_i are disjointly supported,

$$\Big(\int_{B_j} \Big|\sum_{k=1}^{\ell} h_k\Big|^p\Big)^{1/p} = \Big(\int_{B_i} \Big|\sum_{\substack{k=1\\k\neq i}}^{\ell+1} h_k\Big|^p\Big)^{1/p} = \Big(\int_{B_i} \Big|\sum_{k=1}^{\ell+1} h_k\Big|^p\Big)^{1/p}$$

$$\leq \Big(\int_{B_i} \Big|\sum_{k=1}^{\ell} h_k\Big|^p\Big)^{1/p} + \epsilon$$

and similarly

$$\Big(\int_{B_j} \Big|\sum_{k=1}^{\ell} h_k\Big|^p\Big)^{1/p} \geq \Big(\int_{B_i} \Big|\sum_{k=1}^{\ell} h_k\Big|^p\Big)^{1/p} - \epsilon$$

so that, by the mean value theorem and (1), we get (a) with $c(\epsilon) = p(2+2\epsilon)^{p-1}\epsilon$. (b) is proved in a similar way, noting that

$$\Big(\int_{B_i} \Big|g_i + \sum_{k=1}^{\ell} h_k\Big|^p\Big)^{1/p} \leq \|g_i\| + 2 + \epsilon \leq 3 + \epsilon\ .$$

<u>Proof of Lemma 2</u>: By Lemma 4(b) for each i and j,

$$\int_{B_j} \Big|g_j + \sum_{k=1}^{\ell} h_k\Big|^p \leq \int_{B_i} \Big|g_i + \sum_{k=1}^{\ell} h_k\Big|^p + c(\epsilon)$$

Summing over j we get that for every i

$$\int_0^1 \Big|\sum_{j=1}^{n} g_j + \sum_{k=1}^{\ell} h_k\Big|^p = \sum_{j=1}^{n} \int_{B_j} \Big|g_j + \sum_{k=1}^{\ell} h_k\Big|^p \leq$$

$$\leq n \int_{B_i} \Big|g_i + \sum_{k=1}^{\ell} h_k\Big|^p + n.\ c(\epsilon)$$

summing over $1 \leq i \leq \ell$ and dividing by ℓ we get

(5) $$\int_0^1 \Big|\sum_{j=1}^{n} g_j + \sum_{k=1}^{\ell} h_k\Big|^p \leq \frac{n}{\ell} \sum_{i=1}^{\ell} \int_{B_i} \Big|g_i + \sum_{k=1}^{\ell} h_k\Big|^p + n.c(\epsilon)$$

$$= \frac{n}{\ell}\int_{\bigcup_{i=1}^{\ell} B_i} \left|\sum_{i=1}^{\ell} g_i + \sum_{k=1}^{\ell} h_k\right|^p + nc(\epsilon).$$

$$= \frac{n}{\ell}\int_0^1 \left|\sum_{i=1}^{\ell} f_i\right|^p - \frac{n}{\ell}\int_{\bigcup_{i=\ell+1}^{n} B_i} \left|\sum_{k=1}^{\ell} h_k\right|^p + nc(\epsilon).$$

By Lemma 4(a), for every i and j,

$$\int_{B_i} \left|\sum_{k=1}^{\ell} h_k\right|^p \geq \int_{B_j} \left|\sum_{k=1}^{\ell} h_k\right|^p - c(\epsilon)$$

so, for every j,

$$\int_{\bigcup_{i=\ell+1}^{n} B_i} \left|\sum_{k=1}^{\ell} h_k\right|^p \geq (n-\ell)\int_{B_j} \left|\sum_{k=1}^{\ell} h_k\right|^p - (n-\ell)\, c(\epsilon),$$

summing over $1 \leq j \leq n$ we get

$$(6) \qquad \int_{\bigcup_{i=\ell+1}^{n} B_i} \left|\sum_{k=1}^{\ell} h_k\right|^p \geq \frac{n-\ell}{n}\int_0^1 \left|\sum_{k=1}^{\ell} h_k\right|^p - (n-\ell)\, c(\epsilon)$$

combining (5) and (6) we get

$$\int_0^1 \left|\sum_{j=1}^{\ell} g_j + \sum_{k=1}^{\ell} h_k\right|^p \leq \frac{n}{\ell}\int_0^1 \left|\sum_{i=1}^{\ell} f_i\right|^p - \frac{n-\ell}{\ell}\int_0^1 \left|\sum_{k=1}^{\ell} h_k\right|^p + \frac{n^2}{\ell}\, c(\epsilon).$$

The otherside inequality is proved similarly.

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