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Geometry of nuclear spaces. II - Linear topological invariants

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GEOMETRY OF NUCLEAR SPACES
II - LINEAR TOPOLOGICAL INVARIANTS

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The spaces of holomorphic functions $H(\mathcal{A}^k)$ and $H(\mathfrak{c}^N)$ were the stimulating examples for Kolmogoroff [1] and Pełczynski [2] to construct the linear topological invariants, so-called approximative and diametral dimensions, on the class of Schwartz metric spaces. After the observation that

\begin{align}
(1) \quad H(\mathcal{A}^k) &\simeq K(c), \quad c_{np} = \exp\left(-\frac{1}{p} n^{1/k}\right), \quad n \in \mathbb{Z}_+^k, \\
\text{and} \quad H(\mathfrak{c}^k) &\simeq K(d), \quad d_{np} = \exp(p |n|), \quad n \in \mathbb{Z}_+^k,
\end{align}

or

\begin{align}
\simeq K(a), \quad a_{np} = \exp(p n^{1/k}), \quad n \in \mathbb{Z}_+^k,
\end{align}

these invariants show in particular that all the spaces of two series (1) and (2) are pairwise-nonisomorphic (see details on [3], [4], [5]).

Recently the author [6], [7] and V. Zaharyuta [8], [9] give the further series of more general invariants and the core of this talk is to present these invariants and some concrete examples of its applications.

1. Recall that a nuclear Fréchet space (with a continuous norm) is the common domain of (monotone systems of) self-adjoint operators $\{A_p\}_{p>0}^{\infty}$ in a Hilbert space $H_o$, i.e. $1 = A_o \leq A_1^2 \leq A_2^2 \leq \ldots$, and $X = \bigcap_{p>0} \mathcal{S}(A_p)$.

(We will denote this series of operators $A_p$, or norms $|x| = |A_p x|$, or Hilbert spaces $H_p = \mathcal{S}(A_p)$, graphically as the line with points $\{p\}$.) If two such spaces $X$ and $Y$ are isomorphic

\begin{align}
\begin{array}{c}
\begin{tikzpicture}
  \node (X) at (0,0) {$X$};
  \node (Y) at (0,-2) {$Y$};
  \node (Po) at (1,1) {$p_0$};
  \node (P1) at (2,1) {$p_1$};
  \node (Qo) at (1,-1) {$q_o$};
  \node (Q1) at (2,-1) {$q_1$};
  \node (Qp) at (3,-1) {$q'$};
  \draw (X) -- (Po) -- (P1) -- (Y);
  \draw (Qo) -- (Po) -- (Qp);
  \draw (Qo) -- (Q1);
  \draw (P1) -- (Q1);
\end{tikzpicture}
\end{array}
\end{align}

\begin{align}
(\text{A}_p) \quad (\text{B}_{q})
\end{align}
i.e. \( J : X \rightarrow Y \) is a linear topological isomorphism then

\[ \forall \ p \geq p', \ C' \quad |Jx|_q \leq C' |x|_p \]

\[ \forall \ p \geq p', C'' \quad |J^{-1}y|_p \leq C'' |y|_q', \] so

\[ \frac{1}{C} T_{q'} \supset \bigcup_{p \in C} T_q, \]

where \( C = \max \{ C', C'' \} \), and \( S_p = \{ x \in X : |A_x|_p \leq 1 \} \), \( T_q = \{ y \in Y : \|B_y\|_q \leq 1 \} \), or

\[ C \quad T_{q_0} \supset J \quad S_{p_0} \supset J \quad S_{p_1} \supset \frac{1}{C} T_{q_1}. \]

Hence,

\[ s_k(T_{q_1}, T_{q_0}) \leq C^2 \cdot s_k(S_{p_1}, S_{p_2}) \]

where \( s_k(E_1, E_o) \) denotes the sequence of s-numbers of the identity operator \( 1 : H_{E_1} \rightarrow H_{E_0} \) of two Hilbert spaces with the unit balls \( E_1 \) and \( E_0 \) correspondingly. If we put \( N_A(p_0, p_1; t) = \|k : s_k(S_{p_1}, S_{p_0}) \geq 1/t\| \), then by (3)

\[ \forall \ q_0 \geq p_0, \ q_1 \geq p_1, \ C \quad |N_B(q_0, q_1; t) \leq N_A(p_0, p_1; t) \]

and analogous condition (4') holds if we change the places of \( A \) and \( B \).

So the system \( \{ N_A(p_0, p_1; t) \} \leq p_0 < p_1 < \infty \) is a characteristics of the space, and the systems \( \{ N_A \} \) and \( \{ N_B \} \) are equivalent in the sense (4)-(4') if the spaces \( X \) and \( Y \) are isomorphic. Hence, any scalar parameter or any functional object generated by the class of equivalent systems of functions \( \{ N_A(P; t) \} \), \( P = (p_0, p_1) \), would be a linear topological invariant.

For example, in the case (2) \( N(p_0, p_1; t) \sim \frac{\log t}{(p_1 - p_0)^k} \) and the parameter

\[ \gamma(p_0, p_1; A) = \lim_{t \to \infty} \frac{\log N(p_0, p_1; t)}{\log \log t} \]

is the same (by occasion ?) for different \( p_0, p_1 \) and is equal to \( k \). So the spaces (2) (and (1) also) are not isomorphic for different \( k \). The parameters

\[ \beta(p_0, p_1; A) = \lim_{t \to \infty} \frac{(N(p_0, p_1; t))^{1/k}}{\log t} = \frac{1}{p_1 - p_0} \]
show that $H(\mathcal{D}^k)$ and $H(\mathcal{U}^k)$ are not isomorphic for the same $k$.

2. Now we consider the more complicated invariants for the case of $K\delta$ the spaces

\begin{equation}
K(a) = \{ x = (x_n)_{n=0}^{\infty} : \sum_n a_{np}^2 |x_n|^2 < \infty, \forall p \}
\end{equation}

i.e. by [10], Theorem AB, for the case of nuclear Fréchet spaces with a basis.

Put $a_p(i) = a_{1p}$, $P = (p_0, p_1, p_2)$, and

\begin{equation}
N_a(P; t_1, t_2) = \left\{ \tau : \frac{a_{p_1}(i)}{a_{p_0}(i)} \geq t_1, \frac{a_{p_2}(i)}{a_{p_1}(i)} \leq t_2 \right\}
\end{equation}

If the spaces $X = K(a)$ and $Y = K(b)$ are isomorphic then

\begin{equation}
\text{and for any } x \in E = \text{Lin. Span}[e_i : i \in I], I \text{ being the set in the right side of (8), we have the following inequalities}
\end{equation}

\begin{equation}
| x |_{p_0} \leq \frac{1}{t_1} | x |_{p_1}, \quad | x |_{p_2} \leq t_2 | x |_{p_1}
\end{equation}

and then for any $y = Jx \in L$, $L = JE \subset Y$, by (9) we have for some $C > 0$ :

\begin{equation}
\| y \|_{q_0} \leq C \left| x \right|_{p_0} \leq \frac{C}{t_1} \left| x \right|_{p_1} \leq \frac{C^2}{t_1} \| y \|_{q_1}
\end{equation}

\begin{equation}
\| y \|_{q_2} \leq C \left| x \right|_{p_2} \leq C t_2 \left| x \right|_{p_1} \leq C^2 t_2 \| y \|_{q_1}
\end{equation}

\text{Lemma CE : Let } V, W_0, W_1 \text{ be coaxed ellipsoids in } E^{\infty}

\begin{equation}
V = \{ \xi = (\xi_n) : \sum_n |\xi_n|^2 \leq 1 \}
\end{equation}
and
\[ W_\varepsilon = \{ x \in C^\infty : \sum |x_n|^{2/\varepsilon_i^2} \leq 1 \}, \quad \varepsilon_i > 0, \quad \varepsilon = 0, 1 \],

and for some subspace \( L \), \( \dim L = k \), the inequality
\[ (11) \quad \|y\|_V \geq \|y\|_{W_\varepsilon}^{\varepsilon}, \quad \varepsilon = 0, 1 \]
holds for any \( y \in L \).

Then there exists a coordinate \( k \)-dimensional subspace \( L^0 \) such that \( \|y\|_V \geq \frac{1}{2} \|y\|_{W_\varepsilon}^{\varepsilon} \), for any \( y \in L^0 \), i.e., \( |K| \geq k \), where
\[ K = \{ i : \varepsilon_i \geq \frac{1}{2}, \quad \varepsilon = 0, 1 \} \].

**Proof:** Let us consider the coordinate subspace \( C^k = \{ (\varepsilon_i) : \varepsilon_i = 0, \ i \notin K \} \) and the natural projection \( \pi_K : C^\infty \to C^K \). Then \( \text{Ker}(\pi_K | L) = \{ 0 \} \); indeed if \( y \in L \) and \( \pi_K y = 0 \) then by (11)
\[ \|y\|_V^2 \geq \sup_\varepsilon \|y\|_{W_\varepsilon}^{\varepsilon} \geq \frac{1}{2} \left( \sum \frac{|y_i|^2}{\varepsilon_i} + \sum \frac{|y_i|^2}{\varepsilon_i} \right) \]
\[ \geq \frac{1}{2} \sum \frac{|y_i|^2}{r_i^{1/2}} = \frac{1}{2} \sum_{r_i < 1/2} \frac{|y_i|^2}{r_i} \geq \frac{1}{2} \sum |y_i|^2 \geq 2 \|y\|_V^2 \]
and
\[ \|y\| = 0 . \]

Hence \( \pi_K | L : L \to C^K \) is a monomorphism and \( |K| \geq \dim \text{Im}(\pi_K | L) = \dim L = k \).

Now if we put \( V = T_{q_1} W_0 = C^2 t_1 T_{q_0}, \ W_1 = C^2 t_2 \cdot T_{q_2} \), then by Lemma CE and by (10), \( Q = (q_0, q_1, q_2) \),
\[ N_b(Q; \frac{t_1}{2C^2}, 2C^2 t_2) \overset{\text{def}}{=} \{ j : b_{q_1}(j)/b_{q_0}(j) \geq \frac{t_1}{2C^2} \} \]
\[ \geq N_a(P; t_1, t_2) \quad (\text{see } (8)). \]

Hence the following statement is true.

**Theorem IN:** If the spaces \( X \) and \( Y \) in (9) are isomorphic then the systems (8) of functions \( \{ N_a(P, t) \} \) and \( \{ N_b(Q, t) \} \), \( t \in R_+^2 \), are equivalent.
in the following sense:

$$\Psi(q_0 \not\leq p_0, \Psi p_1 \geq p_0 \not\geq q_1, \Psi q_2 > q_1 \not\leq p_2), C \geq$$

$$N_b(Q; t') \geq N_a(P; t)$$

and

$$N_a(Q; t') \geq N_b(P; t), t' = \left(\frac{t_1}{2C^2}; 2C^2t_2\right).$$

3. These invariants have been motivated by [6] where the particular cases of (7) have been considered in detail. Remark that in (13) there is no restriction to the sequence \((a_i)_{i=0}^{\infty}\) but \(a_i \geq 1\), so it may have finite points of accumulation or take the same value infinitely many times. In the case (13.b)

$$a_{i,p} = a_i^{-1/p} \quad \text{a) and b) } a_{i,p} = a_i^p$$

have been considered in detail. Remark that in (13) there is no restriction to the sequence \((a_i)_{i=0}^{\infty}\) but \(a_i \geq 1\), so it may have finite points of accumulation or take the same value infinitely many times. In the case (13.b)

$$N_a(P; t) = \left\{ i : \frac{1}{p_1-p_0} \leq a_i \leq \frac{1}{p_2-p_1} \right\} = \left\{ i : \frac{1}{p_1-p_0} \log t \leq \log a_i \leq \frac{1}{p_2-p_1} \log t \right\}.$$ 

Example: The space \(H(C^k; V)\) of all entire vector-valued functions, \(V\) be a Hilbert space, \(\dim V = \infty\), is isomorphic to the generalized Köthe space

$$K(a; V) = \{ x = (x_n)_{n=0}^{\infty}, x_n \in V : \sum a_n^{2p} \| x_n \|^2 < \infty, \Psi p \},$$

$$\log a_n = (1 + n)^{1/k}, n \in \mathbb{Z}_+.$$ 

In this case for \(t_s = \exp \frac{\tau}{s}, s = 1, 2,\)

$$N_a(P; t) = \infty \text{ if } \left[ \left(\frac{\tau_1}{p_1-p_0}\right)^k, \left(\frac{\tau_2}{p_2-p_1}\right)^k \right] \cap \mathbb{Z}_+ \neq \emptyset$$

$$= 0 \text{ otherwise }.$$
The spaces $H(C^k;V)$, $\dim V = \infty$, are isomorphic for all $k \in \mathbb{Z}_+$, $k > 0$.

However if we consider the spaces $H(\mathcal{C}^k;V)$ for the sequences $(a_n)$

\begin{equation}
\log a_n = \lambda_n^\gamma, \quad \text{where } \lambda_{n+1}/\lambda_n \to \infty, \quad 0 < \gamma < \infty
\end{equation}

then the spaces $K_Y(a;V)$ defined by (14), (15) are pairwise-nonisomorphic for a continuum $\Gamma \subseteq \mathbb{R}_+$. Indeed

\[ N_a(\gamma)(P;i) = \infty \quad \text{if} \quad \left\{ i : \frac{\tau_1}{p_1-p_0} \leq \frac{\lambda^\gamma_i}{\frac{\tau_2}{p_2-p_1}} \right\} \neq \emptyset \]

\[ = 0 \quad \text{otherwise}, \]

and one can prove the following statement.

**Lemma RI** : If the systems of functions $\{N_a(\gamma)\}$ are equivalent in the sense (12) then

\begin{equation}
\lim \frac{1}{n} \log \log \lambda_n = \ell \neq 0
\end{equation}

does exist, and $\ell/(\log \frac{\gamma_1}{\gamma_2})$ is a rational number.

Hence there are two possibilities : 1°. the spaces $K_Y(a;V)$ are pairwise-nonisomorphic for all $\gamma > 0$; 2°. for a pair $(\gamma, 5)$ the spaces $K_\gamma$ and $K_5$ are isomorphic and then (16) holds. In the second case we choose a continuum $\Gamma \subseteq \mathbb{R}_+$ by such a way that for any $\gamma_1, \gamma_2 \in \Gamma$ the number $\frac{1}{\ell} \log \frac{\gamma_1}{\gamma_2}$ is irrational.

4. The invariants (8) and (12) of Theorem IN can be extended essentially. Let us define for any $n \geq 1$ the system of functions

\begin{equation}
N^n(P;x,y) = \left| \left\{ i : a_{p_{2j+1}}(i)/a_{p_{2j}}(i) \geq e^{x^j+1}, a_{p_{2j+2}}(i)/a_{p_{2j+1}}(i) \leq e^{y^j+1}, \quad j = 0,1, \ldots, n-1 \right\} \right|,
\end{equation}

where $P = (p_0, p_1, \ldots, p_{2n})$, $x, y \in \mathbb{R}^n$. 
If the spaces $X$ and $Y$ in (9) are isomorphic then the systems (17) $\{N^n_a\}$ and $\{N^n_b\}$, $P \in \mathbb{Z}^{2n+1}_+$, $x, y \in \mathbb{R}^n$, are equivalent in the following sense:

$$(18) \quad \forall q_0 \nless p_0, \forall p_1 \nless q_1 \ldots \forall q_{2n} \nless p_{2n}, T \text{ and } S \text{ subdiagonal matrices }$$

$$\exists N^n_a(P; x, y) \leq N^n_b(Q; x - TX - SY, y + TX + SY), \forall x, y \in \mathbb{R}^n_+.$$ 

This is the more general statement than Theorem IN above; the relations (17), (18) give the invariant $I_n$ for any $n \geq 0$.

**Theorem SM**: For any $n \geq 0$ one can construct such a pair of (nuclear) Köthe spaces $E_n$ and $F_n$ that the systems of functions $N^k_{E_n}$ and $N^k_{F_n}$ are equivalent in the sense $(18_k)$ for $0 \leq k \leq n$, and are not equivalent for $k = n+1$.

If $N^{n+1}_a$ and $N^{n+1}_b$ are equivalent, then $N^k_a$ and $N^k_b$ are $(18_k)$-equivalent, $0 \leq k \leq n$, so by Theorem SM $\{I_n^k\}^\infty_0$ is a strongly monotone system of invariants on the class of Köthe spaces.

Analogous system $\{I_n^k\}$ can be constructed for multiindices $P$, $|P|$ be even.

5. The spaces $E$ and $F$ in Theorem SM need the special construction. What "natural" spaces can be considered with the help of these invariants?

V. Zaharyuta [9] studied the spaces $H(G)$ of holomorphic functions in Reinhardt domains $G$, $G \subseteq \mathbb{C}^n$, $n \geq 2$. By the definition

$$z \in G; |w_i| < |z_i|, 1 \leq i \leq n \Rightarrow w \in G,$$

and $G$ is a domain of holomorphy, so this domain is determined by the support function

$$h_G(w) = \sup\{(x, w) : x_i = \log |z_i|, 1 \leq i \leq n, z \in G\},$$

$$w \in \sigma^{n-1} = \{y \in \mathbb{R}^n_+ : \sum y_i = 1\}.$$ 

Modified invariants of type (8), (12) are defined by the functions
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\[ M^1_a(P, \tau) = \{ i : a_{p_1}(i)/a_{p_0}(i) \leq e^{\tau_1} ; a_{p_2}(i)/a_{p_1}(i) \geq e^{\tau_2} \} \]

\[ P = (p_0, p_1, p_2), \quad \tau \in \mathbb{R}^2 , \]

and by analogous functions \( M^n_a \) of several variables \( P \) and \( \tau \).

As in (5) or (6) one can define the "functional" parameter

(19) \[ \delta(P; v) = \lim_{p_3 \to \infty} \lim_{\alpha \to \infty} \frac{M^1(p_0, p_1, p_3; \alpha, \beta)}{N(p_1, p_2; e^v)} \]

where \( \beta/\alpha \to v \). \( \forall \alpha \to 1 \)

\[ P = (p_0, p_1, p_2), \quad v \in \mathbb{R}^1 . \]

It happens that the properties of the support function \( h_G \) can be described in terms of the invariant (19). Namely,

**Lemma Z** ([9], p. 29) : \( \forall p_0 \not\subseteq p_1, \forall p_2 \not\subseteq C \exists \]

\[ \frac{1}{C} \mathcal{L}_G(c t) \leq \delta(P, t) \leq C \mathcal{L}_G \left( \frac{t}{C} \right), \quad t \geq t_0 \]

where \( \mathcal{L}(n) = \text{mes}\{ w \in \mathbb{R}^{n-1} : u \leq h_G(w) < \infty \} \).

It implies that if for any \( C_1, C_2 > 0 \) the function \( \mathcal{L}_{G_1}(c_1 t)/\mathcal{L}_{G_2}(c_2 t) \) is unbounded, then the spaces \( H(G_1) \) and \( H(G_2) \) are not isomorphic.

**Corollary** : For any \( n \geq 2 \) there exists a continuum \( G_\gamma \) of domains of holomorphy in \( \mathbb{C}^n \) such that the spaces \( H(G_\gamma) \) are pairwise-nonisomorphic.

For example, one can choose

\[ G_\gamma = \{ z \in \mathbb{C}^n : |z_1| < 1, 1 \leq i \leq n-1, |z_n| < \exp(\log \frac{1}{|z_1|} \gamma) \} , \quad 0 < \gamma < 1 . \]

6. It should be mentioned that the general problem of quasi-equivalence of bases in a nuclear Fréchet space with a base motivated the construction of new invariants. Recall that two bases \( (x_n) \) and \( (f_n) \) in \( E \) are quasiequivalent if there is a bijection \( \rho : \mathbb{N} \to \mathbb{N} \) of the positive integers and a sequence of nonzero scalars \( (r_n) \) such that the operator \( T : \)
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\[ T_{n} = r_{n} x_{\rho(n)} , \quad n \in \mathbb{N} , \]

is an automorphism of the space \( E \).

For any unconditional basis \((x_{n})\) in a Fréchet space \( E \) one can define the group

\[ G(x) = \{ \sigma : \mathbb{N} \to \mathbb{N} \mid (r_{n}) , \quad r_{n} \neq 0 ; \quad T \in \text{Auto } E \} \]

\[ T_{n} = r_{n} x_{\sigma(n)} , \quad \forall n \in \mathbb{N} \}

of rearrangements of \( \mathbb{N} \).

If the bases \((x_{n})\) and \((f_{n})\) are quasiequivalent, then subgroups \( G(x) \) and \( G(f) \) are isomorphic :

\[ \rho^{*} : G(x) \to G(f) , \quad \rho^{*} : \sigma \mapsto \rho^{-1} \circ \sigma \circ \rho . \]

Hence, if the space \( E \) has QEP (quasiequivalence property), i.e. any two bases in \( E \) are quasiequivalent, then the groups \( G(x) \) are isomorphic, \( x \) be a basis, so this group is an invariant in the class of nuclear spaces with a basis and QEP.

The wide class of nuclear Fréchet spaces with regular basis has QEP (see [11] and references there). Nevertheless we do not know whether any Fréchet space with a base has QEP. But invariants described above are "characteristics" of non-invariant or invariant (respect to the base) group \( G(x) \) and these characteristics are invariants.

REFERENCES


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