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Geometry of nuclear spaces. II - Linear topological invariants

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GEOMETRY OF NUCLEAR SPACES
II - LINEAR TOPOLOGICAL INVARIANTS

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0. The spaces of holomorphic functions $H(\mathcal{S}^k)$ and $H(\mathcal{C}^N)$ were the stimulating examples for Kolmogoroff [1] and Pełczyński [2] to construct the linear topological invariants, so-called approximative and diametral dimensions, on the class of Schwartz metric spaces. After the observation that

\[
H(\mathcal{S}^k) \simeq K(c) \ , \ c_{np} = \exp\left(-\frac{1}{p} n^{1/k}\right) \ , \ n \in \mathbb{Z}_+ \ ,
\]

and

\[
H(\mathcal{C}^k) \simeq K(d) \ , \ d_{np} = \exp(p |n|) \ , \ n \in \mathbb{Z}_+^k \ ,
\]

these invariants show in particular that all the spaces of two series (1) and (2) are pairwise-nonisomorphic (see details on [3], [4], [5]).

Recently the author [6], [7] and V. Zaharyuta [8], [9] give the further series of more general invariants and the core of this talk is to present these invariants and some concrete examples of its applications.

1. Recall that a nuclear Fréchet space (with a continuous norm) is the common domain of (monotone systems of) self-adjoint operators $\{A_p\}_{p>0}$ in a Hilbert space $H_o$, i.e. $1 = A_o \leq A_1^2 \leq A_2^2 \leq \ldots$, and $X = \bigcap_{p>0} \mathcal{S}(A_p)$. (We will denote this series of operators $A_p$, or norms $\|x\|_p = |A_p x|$, or Hilbert spaces $H_p = \mathcal{S}(A_p)$, graphically as the line with points $\{p\}$.) If two such spaces $X$ and $Y$ are isomorphic,

\[
\begin{array}{ccc}
X & \xrightarrow{J} & Y \\
\downarrow & & \downarrow \quad \downarrow \\
p_0 & \quad \quad & q_0 \\
p_1 & \quad \quad & q_1 \\
(B_q) & & (A_p)
\end{array}
\]
i.e. \( J: X \to Y \) is a linear topological isomorphism then

\[ \forall \, q \ni p, \, C' \mid \|Jx\|_q \leq C' \|x\|_p \]

\[ \forall \, p \ni q', \, C'' \mid \|J^{-1}y\|_p \leq C'' \|y\|_{q'}, \, \text{so} \quad \frac{1}{C} T_{q'} \subset J S_p \subset C T_q, \]

where \( C = \max \{ C', C'' \} \), and \( S_p = \{ x \in X : \|A_p x\| \leq 1 \} \), \( T_q = \{ y \in Y : \|B_q y\| \leq 1 \} \), or

\[ C T_{q_0} \supset J S_{p_0} \supset J S_{p_1} \supset \frac{1}{C} T_{q_1}. \]

Hence,

\[ s_k(T_{q_1}, T_{q_0}) \leq C^2 \cdot s_k(S_{p_1}, S_{p_2}) \]

where \( [s_k(\mathcal{E}_1, \mathcal{E}_2)] \) denotes the sequence of s-numbers of the identity operator \( 1: H_{\mathcal{E}_1} \to H_{\mathcal{E}_2} \) of two Hilbert spaces with the unit balls \( \mathcal{E}_1 \) and \( \mathcal{E}_2 \) correspondingly. If we put \( N_A(p_0, p_1; t) = \{ k : s_k(S_{p_1}, S_{p_0}) \geq 1/t \} \),

then by (3)

\[ \forall \, q_0 \ni p_0, \, \forall \, p_1 \supset p_0 \supset q_1, \, C N_B(q_0, q_1; t) \leq N_A(p_0, p_1; \frac{t}{C}) \]

and analogous condition \((4')\) holds if we change the places of \( A \) and \( B \).

So the system \( \{ N_A(p_0, p_1; t) \} \) is a characteristics of the space, and the systems \( \{ N_A \} \) and \( \{ N_B \} \) are equivalent in the sense (4) - (4') if the spaces \( X \) and \( Y \) are isomorphic. Hence, any scalar parameter or any functional object generated by the class of equivalent systems of functions \( \{ N_A(P; t) \} \), \( P = (p_0, p_1) \), would be a linear topological invariant.

For example, in the case (2) \( N(p_0, p_1; t) \sim \frac{\log t}{p_1 - p_0} \)

and the parameter

\[ \gamma(p_0, p_1; A) = \limsup_{t \to \infty} \frac{\log N(p_0, p_1; t)}{\log \log t} \]

is the same (by occasion ?) for different \( p_0, p_1 \) and is equal to \( k \). So the spaces (2) (and (1) also) are not isomorphic for different \( k \).

The parameters

\[ \beta(p_0, p_1; A) = \limsup_{t \to \infty} \frac{(N(p_0, p_1; t))^{1/k}}{\log t} = \frac{1}{p_1 - p_0} \]
II.3

show that $H(\mathcal{L}^k)$ and $H(\mathcal{L}^k)$ are not isomorphic for the same $k$.

2. Now we consider the more complicated invariants for the case of $K$ the spaces

$$K(a) = \{x = (x_n)_{n=0}^\infty : \sum a_{n,p}^2 |x_n|^2 < \infty, n \leq p\}$$

i.e. by [10], Theorem AB, for the case of nuclear Fréchet spaces with a basis.

Put $a_p(i) = a_{i,p}$, $P = (p_0, p_1, p_2)$, and

$$N(a; t_1, t_2) = \left\{ \left\{ \frac{a_p(i)}{a_{p_0}(i)} \geq t_1, \frac{a_{p_2}(i)}{a_{p_1}(i)} \leq t_2 \right\} \right\}.$$

If the spaces $X = K(a)$ and $Y = K(b)$ are isomorphic then

and for any $x \in E = \text{Lin. Span}[e_i : i \in I]$, $I$ being the set in the right side of (8), we have the following inequalities

$$|x|_{p_0} \leq \frac{1}{t_1} |x|_{p_1}, \quad |x|_{p_2} \leq t_2 |x|_{p_1}$$

and then for any $y = Jx \in L$, $L = JE \subset Y$, by (9) we have for some $C > 0$:

$$\|y\|_{q_0} \leq C |x|_{p_0} \leq \frac{C}{t_1} |x|_{p_1} \leq \frac{C^2}{t_1} \|y\|_{q_1}$$

$$\|y\|_{q_2} \leq C t_2 |x|_{p_1} \leq C t_2 \|y\|_{q_1}$$

Lemma CE : Let $V, W_0, W_1$ be coaxed ellipsoids in $\mathcal{C}^\infty$

$$V = \{\xi = (\xi_n) : \sum |\xi_n|^2 \leq 1\}$$
and \[ W_\varepsilon = \{ x \in C^\infty \colon \Sigma |x_i|^2/w_\varepsilon i^2 \leq 1 \} , \quad w_\varepsilon i > 0 , \quad \varepsilon = 0, 1 \],

and for some subspace \( L \), \( \dim L = k \), the inequality

\begin{equation}
\|y\|_V \geq \|y\|_{W_\varepsilon} , \quad \varepsilon = 0, 1 ,
\end{equation}

holds for any \( y \in L \).

Then there exists a coordinate \( k \)-dimensional subspace \( L^0 \)
such that \( \|y\|_V \geq \frac{1}{2} \|y\|_{W_\varepsilon} , \quad \varepsilon = 0, 1 \), for any \( y \in L^0 \), i.e. \( |K| \geq k \), where
\[ K = \{ i : w_\varepsilon i \geq \frac{1}{2} , \quad \varepsilon = 0, 1 \} . \]

**Proof:** Let us consider the coordinate subspace \( \mathbb{C}^k = \{(x_i) : x_i = 0, i \notin K \} \)
and the natural projection \( \pi_K : \mathbb{C}^\infty \to \mathbb{C}^K \). Then \( \text{Ker}(\pi_K \mid L) = \{0\} \); indeed if \( y \in L \) and \( \pi_K y = 0 \) then by (11)

\[ \|y\|_V^2 \geq \sup_{\varepsilon} \|y\|_{W_\varepsilon}^2 \geq \frac{1}{2} \left( \sum \frac{|y_i|^2}{w_\varepsilon i^2} + \sum \frac{|y_i|^2}{w_0 i^2} \right) \]
\[ \geq \frac{1}{2} \sum \frac{|y_i|^2}{2 r_i^1} = \frac{1}{2} \sum \frac{|y_i|^2}{r_i^1/2} \geq \frac{1}{2} \sum 4 |y_1|^2 = 2 \|y\|_V^2 \]

and
\[ \|y\| = 0 . \]

Hence \( \pi_K \mid L : L \to \mathbb{C}^K \) is a monomorphism and \( |K| \geq \dim \text{Im}(\pi_K \mid L) = \dim L = k \).

Now if we put \( V = T_{q_1} , \quad W = \frac{C^2}{t_1} T_{q_0} , \quad W_i = C^2 t_2 . T_{q_2} \), then by

**Lemma CE** and by (10), \( Q = (q_0, q_1, q_2) \),

\[ N_b(Q; \frac{t_1}{2C^2} , 2C^2 t_2) \overset{\text{def}}{=} |\{ j : b_{q_1} (j)/b_{q_0} (j) \geq \frac{t_1}{2C^2} \} , \]
\[ b_{q_2} (j)/b_{q_1} (j) \leq 2C^2 t_2 \} | \geq N_a(P; t_1, t_2) \quad (\text{see } (8)). \]

Hence the following statement is true.

**Theorem IN:** If the spaces \( X \) and \( Y \) in (9) are isomorphic then the systems (8) of functions \( \{ N_a(P, t) \} \) and \( \{ N_b(Q, t) \} , \quad t \in \mathbb{R}^2 \) , are equivalent.
in the following sense:

\[ \forall q_0 \nless p_0, \forall p_1 > p_0 \nless q_1, \forall q_2 > q_1 \nless p_2, \ C \ni \]

\[ N_b(Q; t') \geq N_a(P; t) \]

and

\[ N_a(Q; t') \geq N_b(P; t), \ t' = \left( \frac{t_1}{2C^2}, 2C^2t_2 \right) \]

3. These invariants have been motivated by [6] where the particular cases of (7)

\[ a_{ip} = a_i^{-1/p} \quad \text{and} \quad b) \quad a_{ip} = a_i^p \]

have been considered in detail. Remark that in (13) there is no restriction to the sequence \((a_i)_{i=0}^{\infty}\) but \(a_i \geq 1\), so it may have finite points of accumulation or take the same value infinitely many times. In the case (13.b)

\[
N_a(P; t) = \left| \left\{ i : \frac{1}{p_1-p_0} \leq a_i \leq \frac{1}{p_2-p_1} \right\} \right|
\]

\[
= \left| \left\{ i : \frac{1}{p_1-p_0} \log t_1 \leq \log a_i \leq \frac{1}{p_2-p_1} \log t_2 \right\} \right|
\]

**Example**: The space \(H(\mathbb{C}^k; V)\) of all entire vector-valued functions, \(V\) be a Hilbert space, \(\dim V = \infty\), is isomorphic to the generalized Köthe space

\[ K(a; V) = \{ x = (x_n)_{n=0}^{\infty}, x_n \in V: \sum a_n^{2p} \|x_n\|^2 < \infty, \forall p \} \]

\[ \log a_n = (1+n)^{1/k}, \ n \in \mathbb{Z}_+ \]

In this case for \(t_s = \exp \tau_s, s = 1, 2\),

\[ N_a(P; t) = \infty \text{ if } \left[ \left( \frac{\tau_1}{p_1-p_0} \right)^k, \left( \frac{\tau_2}{p_2-p_1} \right)^k \right] \cap \mathbb{Z}_+ \neq \emptyset \]

\[ = 0 \text{ otherwise} \]
The spaces \( H(\mathcal{C}^k;V) \), \( \dim V = \infty \), are isomorphic for all \( k \in \mathbb{Z}^+ \), \( k > 0 \).

However if we consider the spaces (14) for the sequences \((a_n)\)

\[
\log a_n = \lambda_n^\gamma, \quad \text{where} \quad \lambda_{n+1}/\lambda_n \to 0, \quad 0 < \gamma < \infty
\]

then the spaces \( K_Y(a;V) \) defined by (14), (15) are pairwise-nonisomorphic for a continuum \( \Gamma \subset \mathbb{R}_+ \). Indeed

\[
N_a(\gamma)(p;\xi) = \infty \quad \text{if} \quad \left\{ i: \frac{r_1}{p_1-p_0} \leq \lambda_i^\gamma \leq \frac{r_2}{p_2-p_1} \right\} \neq \emptyset
\]

= 0 otherwise,

and one can prove the following statement.

**Lemma RI** : If the systems of functions \( \{N_a(\gamma)\} \) are equivalent in the sense (12) then

\[
\lim \frac{1}{n} \log \log \lambda_n = \delta \neq 0
\]

does exist, and \( \delta/(\log Y) \) is a rational number.

Hence there are two possibilities : 1°. the spaces \( K_Y(a;V) \) are pairwise-nonisomorphic for all \( \gamma > 0 \); 2°. for a pair \( (\gamma,5) \) the spaces \( K_\gamma \) and \( K_5 \) are isomorphic and then (16) holds. In the second case we choose a continuum \( \Gamma \subset \mathbb{R}_+ \) by such a way that for any \( \gamma_1, \gamma_2 \in \Gamma \) the number \( \frac{1}{2} \log \frac{\gamma_1}{\gamma_2} \) is irrational.

4. The invariants (8) and (12) of Theorem IN can be extended essentially. Let us define for any \( n \geq 1 \) the system of functions

\[
N^n(p;x,y) = |\{ i: a_{p_{2j+1}}(i)/a_{p_{2j}}(i) \geq e^{y^{j+1}}, \quad a_{p_{2j+2}}(i)/a_{p_{2j+1}}(i) \leq e^{y^{j+1}}, \quad j = 0,1,\ldots,n-1 \}|
\]

where \( p = (p_0,p_1,\ldots,p_{2n}) \); \( x,y \in \mathbb{R}^n \).
If the spaces $X$ and $Y$ in (9) are isomorphic then the systems 
(17) $\{N^a_n\}$ and $\{N^b_n\}$, $P \in \mathbb{Z}^{2n+1}$, $x, y \in \mathbb{R}^n$, are equivalent in the following sense:

$$ (18) \quad q_0 \leq p_0, \quad q_1 \leq p_1, \quad \ldots \quad q_{2n} \leq p_{2n}, \quad T \text{ and } S \text{ subdiagonal matrices} $$

$$ \exists N^a_n(P; x, y) \leq N^b_n(Q; x - TX - SY, y + TX + SY), \quad \forall x, y \in \mathbb{R}^n. $$

This is the more general statement than Theorem IN above; the relations (17), (18) give the invariant $I_n$ for any $n \geq 0$.

**Theorem SM**: For any $n \geq 0$ one can construct such a pair of (nuclear) Köthe spaces $E_n$ and $F_n$ that the systems of functions $N^k_{E_n}$ and $N^k_{F_n}$ are equivalent in the sense (18) for $0 \leq k \leq n$, and are not equivalent for $k = n+1$.

If $N^a_{n+1}$ and $N^b_{n+1}$ are equivalent, then $N^k_a$ and $N^k_b$ are (18)-equivalent, $0 \leq k \leq n$, so by Theorem SM $\{I_n\}_0^\infty$ is a strongly monotone system of invariants on the class of Köthe spaces.

Analogous systems $\{I_n\}$ can be constructed for multiindices $P$, $|P|$ be even.

5. The spaces $E$ and $F$ in Theorem SM need the special construction. What "natural" spaces can be considered with the help of these invariants?

V. Zaharyuta [9] studied the spaces $H(G)$ of holomorphic functions in Reinhardt domains $G$, $G \subset \mathbb{C}^n$, $n \geq 2$. By the definition

$$ z \in G; \quad |w_i| < |z_i|, \quad 1 \leq i \leq n \Rightarrow w \in G, $$

and $G$ is a domain of holomorphy, so this domain is determined by the support function

$$ h_G(w) = \sup\{(x, w) : x_i = \log |z_i|, \quad 1 \leq i \leq n, z \in G\}, $$

$$ w \in \sigma^{n-1} = \{y \in \mathbb{R}^n : \sum y_i = 1\}. $$

Modified invariants of type (8), (12) are defined by the functions
and by analogous functions $M^1_{P}\alpha$ of several variables $P$ and $\tau$.

As in (5) or (6) one can define the "functional" parameter

\[
\delta(p; v) = \lim_{p_3 \to \infty} \lim_{\alpha \to 0^+} \frac{M^1(p_0, p_1, p_3; \alpha, \beta)}{N(p_1, p_2; e^v)}, \quad \beta / \alpha \to 0^+ \cap \gamma / \alpha \to 0^+.
\]

\[P = (p_0, p_1, p_2), \quad v \in \mathbb{R}_+^1.
\]

It happens that the properties of the support function $h_G$ can be described in terms of the invariant (19). Namely,

\textbf{Lemma Z} ([9], p. 29) : \forall p_0 \not\subset p_1, \forall p_2 \not\subset C \exists

\[
\frac{1}{C} \mathcal{L}_G (ct) \leq \delta(p, t) \leq C \mathcal{L}_G \left(\frac{t}{C}\right), \quad t \geq t_0
\]

where $\mathcal{L}(n) = \text{mes}\{w \in \mathbb{S}^{n-1} : u \leq h_G (w) < \infty\}$.

It implies that if for any $C_1, C_2 > 0$ the function

$\mathcal{L}_G_1 (c_1 t) / \mathcal{L}_G_2 (c_2 t)$ is unbounded, then the spaces $H(G_1)$ and $H(G_2)$ are not isomorphic.

\textbf{Corollary} : For any $n \geq 2$ there exists a continuum $G_\gamma$ of domains of holomorphy in $\mathbb{C}^n$ such that the spaces $H(G_\gamma)$ are pairwise-nonisomorphic.

For example, one can choose

\[G_\gamma = \{z \in \mathbb{C}^n : |z_i| < 1, 1 \leq i \leq n-1, \ |z_n| < \exp(\log \frac{1}{|z_1|^{1/\gamma}}) \}, \quad 0 < \gamma < 1.
\]

6. It should be mentioned that the general problem of quasi-equivalence of bases in a nuclear Fréchet space with a base motivated the construction of new invariants. Recall that two bases $(x_n)$ and $(f_n)$ in $E$ are quasiequivalent if there is a bijection $\rho: \mathbb{N} \to \mathbb{N}$ of the positive integers and a sequence of nonzero scalars $(r_n)$ such that the operator $T :$
\[ T_{f_n} = r_n x_{\sigma(n)}, \quad n \in \mathbb{N}, \]

is an automorphism of the space \( E \).

For any unconditional basis \( (x_n) \) in a Fréchet space \( E \) one can define the group

\[ G(x) = \{ \sigma: \mathbb{N} \to \mathbb{N} \; | \; (r_n), \; r_n \not= 0; \; T \in \text{Auto} \; E \; \} \]

\[ T_{x_n} = r_n x_{\sigma(n)}, \; \forall \; n \in \mathbb{N} \}

of rearrangements of \( \mathbb{N} \).

If the bases \( (x_n) \) and \( (f_n) \) are quasiequivalent, then subgroups \( G(x) \) and \( G(f) \) are isomorphic:

\[ \rho^*: G(x) \to G(f), \quad \rho^*: \sigma \mapsto \rho^{-1} \circ \sigma \circ \rho. \]

Hence, if the space \( E \) has QEP (quasiequivalence property), i.e. any two bases in \( E \) are quasiequivalent, then the groups \( G(x) \) are isomorphic, \( x \) be a basis, so this group is an invariant in the class of nuclear spaces with a basis and QEP.

The wide class of nuclear Fréchet spaces with regular basis has QEP (see [11] and references there). Nevertheless we do not know whether any Fréchet space with a base has QEP. But invariants described above are "characteristics" of non-invariant or invariant (respect to the base) group \( G(x) \) and these characteristics are invariants.

REFERENCES


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