W. B. JOHNSON

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COMPLEMENTED SUBSPACES OF $L^p$ WHICH EMBED INTO $l^p \oplus l^2$

W. B. JOHNSON
(Ohio State University)
In this seminar we report on joint work with Ted Odell [5] concerning the isomorphic classification of complemented subspaces of $L_p$, $1 < p < 2 < \infty$. There are now known to exist uncountably many mutually non-isomorphic complemented subspaces of $L_p$ for each $1 < p < 2 < \infty$ [1]. However, there probably are only finitely many which are "small". For example, the only complemented subspace of $L_p$ which embeds into $l_p$ is $l_p$ itself [6].

The question studied in [5] is "what are the complemented subspaces of $L_p$ which embed into $l_p \oplus l_2$?" For $1 < p < 2$, the following partial answer is given:

**Theorem A:** If $X$ is a complemented subspace of $L_q$ ($1 < q < 2$) which has an unconditional basis and $X$ embeds into $l_q \oplus l_2$, then $X$ is isomorphic to $l_q$, $l_2$, or $l_q \oplus l_2$.

It is of course a major unsolved problem whether every complemented subspace of $L_p$ ($1 < p \neq 2 < \infty$) has an unconditional basis.

Theorem A is an immediate consequence of the result of [6] mentioned above and:

**Proposition B:** Let $X$ be a subspace of $L_p$ ($2 < p < \infty$) which has an unconditional basis and which is isomorphic to a quotient of $l_p \oplus l_2$. Then there is a subspace $U$ of $l_p$ (possibly $U = \{0\}$) so that $X$ is isomorphic to $U$, $l_2$, or $U \oplus l_2$. 


The classification of complemented subspaces of $L_p$ which embed into $L_p \oplus \ell_2$ is more complicated for $2 < p < \infty$ because of the presence of Rosenthal's space $X_p$ [11]. However, in [5] the following is proved:

**Theorem C:** If $X$ is a complemented subspace of $L_p$ ($2 < p < \infty$) which has an unconditional basis and which embeds into $L_p \oplus \ell_2$, then $X$ is isomorphic to $\ell_p$, $\ell_2$, $L_p \oplus \ell_2$, or $X_p$.

Below we give a more-or-less complete proof of Proposition B and outline the proof of Theorem C. Actually, Theorem A is also a consequence of Theorem C and the following result from [5] which will not be discussed in this seminar:

**Theorem D:** If $X$ is a subspace of $L_p$ ($2 < p < \infty$) which is isomorphic to a quotient of a subspace of $L_p \oplus \ell_2$, then $X$ embeds into $L_p \oplus \ell_2$.

**Proof of Proposition B:** Let $(x_n)$ be a normalized unconditional basis for $X$ and let $\mathcal{G}$ be a norm one operator from $L_p \oplus \ell_2$ onto $X$.

**Claim:** There exists $\varepsilon > 0$ so that for all $0 < \delta < \varepsilon$, $\{i: \delta \leq \|x_i\|_2 \leq \varepsilon\}$ is finite. (Here $\|x\|_r = (\int_0^1 |x(t)|^r dt)^{1/r}$ for $1 \leq r < \infty$.)

If the claim is false, then there are $\varepsilon_1 > \varepsilon_2 > \ldots > 0$ and infinite sets $M_n$ of integers so that $\varepsilon_{n+1} < \|x_i\|_2 \leq \varepsilon_n$ for $i \in M_n$ and $n = 1, 2, \ldots$. Since $(x_i)$ is unconditional, it follows from the classical results of Kadec and Pelczynski [7] that $(x_i)_{i \in M_n}$ is equivalent to the unit vector basis for $\ell_2$ for each $n = 1, 2, \ldots$, hence so is
(f_i)_i ∈ M_n, if (f_i) is the sequence of biorthogonal functionals to (x_i).

But this means that for each n = 1, 2, ... the ℓ_q - contribution to the norm of (Q*f_n) tends to zero as i → ∞ in M_n, because every operator from ℓ_2 into ℓ_q is compact. Consequently, since Q* is an isomorphism, we can select i_n ∈ M_n so that (Q*f_n)_{n=1} are equivalent to the unit vector basis of ℓ_2, hence the same is true of (x_n)_n=1. But (x_n)_n=1 has a subsequence equivalent to the unit vector basis of ℓ_p because

\lim_{n→∞} ∥x_n∥_2 = 0. This completes the proof of the claim.

Exercise: Where was unconditionality of (x_n) used in the proof of the claim?

Since for any ε > 0, the closed linear span of (x_i: ∥x_i∥_2 > ε) is either finite dimensional or isomorphic to ℓ_2, we can, in view of the claim, assume that ∥x_n∥_2 < 0 and hence [?] that no subsequence of (x_n) is equivalent to the unit vector basis for ℓ_2. We will show that this condition implies that X embeds into ℓ_p.

Let f_i = g_i ⊕ e_i ∈ ℓ_q ⊕ ℓ_2 (1/p + 1/q = 1) be a normalized sequence which is equivalent to the biorthogonal functionals to (x_i). In view of Lemma 1 below, we can assume that (g_i) is a monotonely unconditional basic sequence in ℓ_q', and (h_i) is orthogonal in ℓ_q. Since no subsequence of (f_i) is equivalent to the unit vector basis of ℓ_2, there exists δ > 0 and n so that ∥g_i∥ ≥ δ for all i ≥ n. Letting P denote the natural projection of ℓ_q ⊕ ℓ_2 onto ℓ_q, we complete the proof by observing that P is an isomorphism when restricted to [(f_i)_i=n], the closed linear span of (f_i)_i=n. Indeed, since (g_i) is monotonely unconditional, we have for all scalars (a_i) that (∑ |a_i|^2)^{1/2} ≤ K_q δ^{-1} ∥P a_i g_i∥ where K_q is
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Khintchine's constant for $L_q$. Hence for any $f = \sum_{i=1}^{\infty} a_i f_i \in \{(f_i)_{i=1}^{\infty}\}$,

$$\|Pf\| \leq \|f\| = \max(\|\sum a_i g_i\|, \|\sum a_i h_i\|) \leq \max(\|Pf\|, (\sum_{i=1}^{\infty} |a_i|^{1/2})^2) \leq K_q b_q^{-1} \|Pf\|. \quad \square$$

In the proof of Proposition B, we used:

**Lemma 1:** Let $(x_i)$ be an unconditional basic sequence in $\ell_p \oplus \ell_2$ ($1 < p < \infty$). Then there is a monotonely unconditional basic sequence $(u_i)$ in $\ell_p$ and an orthogonal sequence $(v_i)$ in $\ell_2$ so that $(x_i)$ is equivalent to $(u_i \oplus v_i)$ in $\ell_p \oplus \ell_2$.

**Proof.** The proof uses an idea of Schechtman's [13]. Note that by a perturbation argument we can assume that, if $(e_n)$ denotes the natural basis for $\ell_p \oplus \ell_2$, then for any $n = 1, 2, \ldots$, only finitely many of the $x_i$'s have a non-zero $n$th coordinate when $x_i$ is expanded in terms of $(e_n)$. We can represent $(e_n)$ in $\ell_p [-1,1]$ by having $(e_{2n})_{n=1}^{\infty}$ be a sequence of $\ell_p$-normalized indicator functions of disjoint subsets of $[-1,0)$ and letting $(e_{2n-1})_{n=1}^{\infty}$ be the Rademacher functions on $[0,1]$. Write $x_i = y_i + z_i$ with $y_i \in [(e_{2n})_{n=1}^{\infty}]$ and $z_i \in [(e_{2n-1})_{n=1}^{\infty}]$. The sequence $(x_i)$ is easily seen to be equivalent to the sequence $(r_i \otimes y_i + r_i \otimes z_i)$ in $\ell_p ([0,1] \times [-1,1])$, where $(r_i)$ is the usual sequence of Rademacher functions. Of course, $(r_i \otimes z_i)$ is equivalent to an orthogonal sequence; the point is that the terms of the monotonely unconditional sequence $(r_i \otimes y_i)$ are measurable with respect to a purely atomic sub-sigma field of $[0,1] \times [-1,0]$ so that $[(r_i \otimes y_i)]$ embeds isometrically into $\ell_p$. \quad \square
Throughout the rest of this seminar, we let $2 < p < \infty$ and let $(e_n)$ (respectively, $(\delta_n)$) denote the unit vector basis for $\ell_p$ (respectively, $\ell_2$). Given $z = y \oplus z \in \ell_p \oplus \ell_2$, we let $|x|_p = |y|$ and $|x|_2 = |z|$.

Given a sequence $w = (w_n)$ of non-negative weights, the space $X_{p,w}$ is defined to be the subspace $[e_n \oplus w_n \delta_n]$ of $\ell_p \oplus \ell_2$. We use $(b_n)$ to denote the natural basis $(e_n \oplus w_n \delta_n)$ for a generic $X_{p,w}$ space; if confusion is likely to result, we use $|\cdot|_{2,w}$ to denote the $\ell_2$-part of the norm in $X_{p,w}$, so that for $x = \sum a_n b_n \in X_{p,w}$, $|x|_{2,w} = (\sum |a_n w_n|^2)^{1/2}$.

No matter what the weight sequence $w$ is, the space $X_{p,w}$ is isomorphic to $\ell_2$, $\ell_p$, $\ell_p \oplus \ell_2$ or the space $X_p$ introduced by Rosenthal [11]. Rosenthal showed that $X_{p,w}$ is isomorphic to $X_p$ if and only if for each $\varepsilon > 0$,

$$\sum w_n \wedge \frac{p}{p-2} = \infty.$$  

$X_p$ is isomorphic to a complemented subspace of $\ell_p$ but is not isomorphic to a complemented subspace of $\ell_p \oplus \ell_2$. It has become clear during the last ten years that, rather than being a pathological example, $X_p$ plays a fundamental role in the study of $\ell_p$ (cf., e.g. [2], [4], and [12]).

There are three important steps in the proof of Theorem C:

**Proposition 2:** Let $X$ be a subspace of $\ell_p \oplus \ell_2$ ($2 < p < \infty$) and let $T$ be an operator from $\ell_p$ into $X$. Then $T$ factors through $X_p$.

**Proposition 3:** If $X$ is isomorphic to a complemented subspace of $X_p$ and $X_p$ is isomorphic to a complemented subspace of $X$, then $X$ is isomorphic to $X_p$. 
Proposition 4: Let $X$ be a subspace of $\ell_p \oplus \ell_2$ ($2 < p < \infty$) with a normalized basis $x_n = y_n \oplus z_n$, where $(y_n)$ (respectively, $(z_n)$) is a basic sequence in $\ell_p$ (respectively, $\ell_2$). Assume that $|z_n|_2 \to 0$ as $n \to \infty$. Then either $X$ embeds into $\ell_p$ or $X_p$ is isomorphic to a complemented subspace of $X$.

Notice that Proposition 2 implies that a complemented subspace of $\ell_p$ which embeds into $\ell_p \oplus \ell_2$ is isomorphic to a complemented subspace of $X_p$. Suppose now that $X$ is a complemented subspace of $\ell_p$ which embeds into $\ell_p \oplus \ell_2$ and $X$ has normalized unconditional basis which in $\ell_p \oplus \ell_2$ can be represented as $x_n = y_n \oplus z_n$, where by Lemma 1 we can assume that $(y_n)$ is unconditional in $\ell_p$ and $(z_n)$ is orthogonal in $\ell_2$. Suppose that

\[
(*) \quad \begin{cases} 
M_n = \{i: \varepsilon_{n+1} \leq |z_i|_2 < \varepsilon_n\} \text{ is infinite.} 
\end{cases}
\]

We can then use a standard gliding hump and perturbation argument to find infinite $M'_n \subseteq M_n$ so that, setting $M = \bigcup_{n=1}^{\infty} M'_n$, we have that

$(y_i)_{i \in M}$ is equivalent to the unit vector basis of $\ell_p$ and $(z_i)_{i \in M}$ is equivalent to an orthogonal sequence in $\ell_2$. Thus by Rosenthal's characterization of $X_p$ mentioned earlier, $[(x_i)_{i \in M}]$ is isomorphic to $X_p$ and is complemented in $X$ because $(x_1)$ is unconditional, hence by Propositions 2 and 3, $X$ is isomorphic to $X_p$.

If $(*)$ is false, then there is $c > 0$ and $A \subseteq \mathbb{N}$ so that

$|z_i|_2 \geq c$ for $i \notin A$ and $\lim_{i \to \infty} |z_i|_2 = 0$. 

$\bigcup_{i \in A}$
By Proposition 4, either $X_p$ is complemented in $[(x_i)_{i \in A}]$ and hence in $X$, so that, by Proposition 3, $X$ and $X_p$ are isomorphic, or $[(x_i)_{i \in A}]$ embeds into $\ell_p$, and so is finite dimensional or isomorphic to $\ell_p$ since it embeds into $L_p$ as a complemented subspace. Of course, $[(x_i)_{i \notin A}]$ is isomorphic to a Hilbert space and so if $[(x_i)_{i \in A}]$ embeds into $\ell_p'$, then $X$ is isomorphic to $\ell_p$, $\ell_p \cong \ell_2$, or $\ell_2$ if, respectively, $\mathbb{N} \sim A$ is finite, $A$ and $\mathbb{N} \sim A$ are infinite, or $A$ is finite.

To indicate how to prove Proposition 2, we need to recall the concept of a blocking of a finite dimensional decomposition (f.d.d., in short). Given an f.d.d. $(E_n)$ for some space $Z$, a blocking of $(E_n)$ is an f.d.d. for $Z$ of the form $(E'_n)$, where for $k = 1, 2, \ldots$, $E'_k = [(E_i)_{i = n(k)}^{n(k+1)-1}]$ for some sequence $1 = n(1) < n(2) < \ldots$ of integers. The simplest version of the blocking method, introduced in [6] (cf. also Proposition 1.g.4 in [8]) can be stated qualitatively as follows: If $Z$ has a shrinking f.d.d. $(E_n)$, $Y$ has an f.d.d. $(F_n)$, and $T: Z \to Y$ is an operator, then there are blockings $(E'_n)$ of $(E_n)$ and $(F'_n)$ of $(F_n)$ so that for all $n = 1, 2, \ldots$, $T E'_n$ is "essentially" contained in $F'_n + F'_{n+1}$. ("Essentially" means: given any $\varepsilon_n \downarrow 0$, $(E'_n)$ and $(F'_n)$ may be chosen so that for $x \in E'_n$, $d(Tx, F'_n + F'_{n+1}) \leq \varepsilon_n \|x\|$. An easy consequence of this blocking principle is:

**Lemma 5:** If $(E_n)$ is a shrinking f.d.d. for $Z$, $(F_n)$ is an f.d.d. for $Y$, and $T: Z \to Y$ is an operator, then there are blockings $(E'_n)$ of $(E_n)$ and $(F'_n)$ of $(F_n)$ so that $T: (\sum_{n=1}^{\infty} E'_n)_p \to (\sum_{n=1}^{\infty} F'_n)_p$ is bounded.

We are now ready to prove Proposition 2. By a change of density on the underlying measure space, we can by one of Maurey's theorems [9]
assume that $T$ is bounded as an operator from $L^2$ into $(X, \| \cdot \|_2)$, i.e., for all $x \in L^p$, $|Tx|_2 \leq K \|x\|_2$ for some constant $K$. Secondly, by Lemma 5, we can find a blocking $(H_n)$ of the Haar basis so that $T$ is bounded as an operator from $(\sum_{n=1}^{\infty} (H_n, \| \cdot \|_p))_p$ into $(X, \| \cdot \|_p)$. (To see this, embed $(X, \| \cdot \|_p)$ into $L^p$ and block the unit vector basis for $L^p$.) Consequently, if for $x \in L^p$, $x = \sum x_n (x_n \in E_n)$, we define $\| x \| = \max (\| x_n \|_p^{1/p}, \| x \|_2)$ then we have that $T$ is bounded as an operator from $(L^p, \| \cdot \|)$ into $X$. The identity mapping from $L^p$ into $(L^p, \| \cdot \|)$ is bounded because the Haar basis, being unconditional, admits a lower $L^p$-estimate. Thus the operator $T: L^p \rightarrow X$ factors through $(L^p, \| \cdot \|)$. To complete the proof of Proposition 2 we only need to observe that the completion of $(L^p, \| \cdot \|)$ is isomorphic to a complemented subspace of $X_{p,w}$ for some weight sequence $w$. This is done by seeing that the completion of $(L^p, \| \cdot \|) = (\sum H_n, \| \cdot \|)$ is norm one complemented in $(\sum E_n, \| \cdot \|)$ by the orthogonal projection, where for $n = 1, 2, \ldots$, $E_n = \{(h_1)_{i=1}^{2^k(n)}\}$ and $k(n)$ is chosen so that $H_n \subseteq E_n$. If $f^m_i \in E_n$ denotes the $L^p$-normalized indicator function of the interval $[(i-1)2^{-k(n)}, i2^{-k(n)}]$ for $1 \leq i \leq 2^k(n)$; $n = 1, 2, \ldots$, then one can easily see that $(f^m_i)_{i=1}^{n} \in (\sum E_n, \| \cdot \|)$ is equivalent to the natural basis of $X_{p,w}$ for the weight sequence $w = (\| f^m_i \|_2)_{i=1}^{2^k(n)}_{n=1}^{\infty}$.

To prove Proposition 3 we need the following:

Lemma 6: There exists $M_p < \infty$ so that if $T$ is an operator on $X_{p,w}$ for some weight sequence $w = (w_n)_{n=1}^{\infty}$, then there exists a weight sequence...
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\[ \|T\|_{\ell^2,v} \leq M_p\|T\| \quad \text{and} \quad \|x\| = \max (\|x\|_p, \|x\|_{\ell^2,v}) \quad \text{is} \quad M_p\text{-equivalent to the usual norm on} \quad X_p^\ast. \]

The lemma can be proved by embedding \( X_p \) into \( L_\infty[-1,1] \) by identifying the nth-unit vector of \( X_p \) with the function \( f_n = g_n + w_n r_n \), where \( (g_n) \) are disjointly supported unit vectors in \( L_\infty[-1,0] \), \( \|g_n\|_{\ell^2} \leq w_n \), and \( (r_n) \) are the Rademacher functions on \([0,1]\). Note that \( \|\cdot\|_{\ell^2,w} \) on \( X_p \) is equivalent to \( \|\cdot\|_2 \) under this identification. Now one uses [3] to get a change of density \( \phi \geq \frac{1}{2} \) on \([-1,1]\) so that \( T \) is bounded when considered as an operator from \( (f_n, \|\cdot\|_{\ell^2(\phi dm)}) \) into itself. One can check that the weight sequence \( v = (v_n) \) defined by \( v_n^2 = w_n^2 + \|g_n\|_{\ell^2(\phi dm)}^2 \) does the job.

We are now ready to prove Proposition 3. The idea is to use Pelczynski's classical proof [10] that every complemented subspace of \( A_p \) is isomorphic to \( A_p \). We need to write \( X_p \) as a symmetric sum \( (X_p \oplus X_p \oplus ...) \) in such a way that \( (X \oplus X \oplus ...) \) is complemented in \( (X_p \oplus X_p \oplus ...) \). The problem is that \( X_p \) is not isomorphic to \( (X_p \oplus X_p \oplus ...) \). However, if we represent \( X_p \) as \( X_{p,w'} \), then \( X_p \) is isomorphic to \( (X_{p,w} \oplus X_{p,w} \oplus ...) \) where for \( x_n \in X_{p,w'} \) the norm in \( (X_{p,w} \oplus X_{p,w} \oplus ...) \) of \( y = (x_n)_{n=1}^{\infty} \) is given by \( \|y\| = \max ((\sum |x_n|_p^{1/p})^{1/p}, (\sum |x_n|_{\ell^2,w}^2)^{1/2}) \). (One checks the isomorphism of \( X_p \) with \( (X_{p,w} \oplus X_{p,w} \oplus ...) \) by observing that \( (X_{p,w} \oplus X_{p,w} \oplus ...) \) is isometric to \( X_{p,w'} \) where the weight sequence \( v \) consists of all terms of the weight sequence \( w \), each repeated infinitely many times.) Unfortunately, it is not true that \( (X \oplus X \oplus ...) \) must be complemented in \( (X_{p,w} \oplus X_{p,w} \oplus ...) \) if \( X \) is complemented in \( X_{p,w'} \) so Pelczynski's argument does not apply. However, if the projection
P: $X_p \to X$ is bounded in both the $|\cdot|_p$ and the $|\cdot|_{2,w}$ norms on $X$, then $(X \oplus X \oplus \ldots)$ is complemented in $(X_{p,w} \oplus X_{p,w} \oplus \ldots)_{p,2}$ by the projection $P \oplus P \oplus \ldots$. The point of Lemma 6 is that we can assume, without loss of generality, that $|P|_{2,w} < \infty$. Of course, $|P|_p$ might be infinite, but there is by Lemma 5 a blocking $(E_n)$ of the natural basis for $X_{p,w}$ so that $P$ is bounded as an operator from $(\sum E_n)_{p,2}$ into itself, where each space $E_n$ has the $X_{p,w}$ norm, $||\cdot||$, on it. If we define $||\cdot||'$ on $X_{p,w}$ by $|x|_p' = (\sum ||x_n||_p^p)^{1/p}$ (where $x = \sum x_n$, $x_n \in E_n$, $n \geq 1$), then it is easy to check that the $X_{p,w}$ norm is equivalent to the norm $||x|| = \max(|x|_p', |x|_{2,w})$. Since $|P|_p'$ and $|P|_2$ are both finite, $(X \oplus X \oplus \ldots)$ is complemented in $(X_{p,w}', ||\cdot||') \oplus (X_{p,w}, ||\cdot||) \oplus \ldots)_{p,2}$ and this letter space is easily seen to be isomorphic to $X_p$. This completes the sketch of the proof of Proposition 3.

We complete this seminar by giving a proof of Proposition 4.

If $\ell_2$ does not embed into $X$, then $X$ embeds into $\ell_p$ by a result of Johnson and Odell (or see [2]). Thus we may assume $X$ contains a copy of $\ell_2$.

Since $|z_n|_2 \to 0$, we can assume without loss of generality that $|z_n|_2 < 1$ for each $n$. For a subspace $Y$ of $X$, let $\delta(Y) = \sup \{ |y|_2: ||y|| = 1 \}$. Note that since $X$ contains $\ell_2$, if $\dim X/Y < \infty$, then $\delta(Y) = 1$. By the blocking technique [6] there exists $0 = k(1) < k(2) < \ldots$ such that if $E_n = [(y_1, k(n+1)]$ and $F_n = [(z_1, k(n+1)]$, then $(E_n)$ is an $\ell_p$-f.d.d. for $[(y_n)]$ and $(F_n)$ is an $\ell_2$-f.d.d. for $[(z_n)]$. Thus if $u_n \in E_n$, then $|\Sigma u_n|_p \leq (\Sigma |u_n|_p^p)^{1/p}$ and a similar statement holds for $(F_n)$. Also by our above remark we can insure that
\[ \delta([x_i]^{k(n+1)}) \geq 1/2 \] for each \( n \). Since \( |z_n|_2 \to 0 \), we can find \( q(n) \) such that if \( H_n = [(x_i)]_{k(n)+1}^{q(n)} \) then

\[ 1 > \delta(H_n) > 0 \] for each \( n \),

\[ \sum_{n=1}^{\infty} \delta(H_n)^{2p/(p-2)} = \infty, \quad \text{and} \quad \lim_{n \to \infty} \delta(H_n) = 0. \]

Let \( e_n \in H_n \) so that \( \|e_n\| = 1 \) and \( |e_n|_2 = \delta(H_n) \). Clearly \( [(e_n)] \) is isomorphic to \( X_p \). We must show it is also complemented in \( X \). Thus we wish to find \( \tilde{f}_n \in X^* \) so that \( \tilde{f}_n \) is biorthogonal to \( (e_n) \) and

\[ F(x) = \sum \tilde{f}_n(x) e_n \] is a bounded operator, and hence a projection onto \( [(e_n)] \).

Let \( f_n \) be the functional on \( H_n \) defined by \( f_n(h) = \langle h, e_n \rangle |e_n|^{-2} \). Then

\[ |f_n|_p = \max \left\{ \frac{\langle h, e_n \rangle |e_n|^{-2}}{|h|_p = 1} \right\}_{h \in H_n} \]

\[ \leq \max \left\{ \frac{|h|_2 |e_n|^{-1}}{|h|_p = 1} \right\}_{h \in H_n} = 1, \]

since \( |e_n|_2 = \delta(H_n) \) and \( \|\cdot\| = \|\cdot\|_p \) on \( H_n \). Thus \( f_n \) is a norm 1 functional on \( H_n \) in the \( \ell_p \) norm. Extend \( f_n \) to a functional \( \tilde{f}_n \) on \( X \) by letting \( \tilde{f}_n(x_i) = 0 \) if \( i < k(n) \) or \( i > q(n) \). Since \( (y_i) \) and \( (z_i) \) are basic, we have

\[ |\tilde{f}_n|_p \leq K \] and \( |\tilde{f}_n|_2 \leq K |f_n|_2 = K |e_n|^{-1} \]
where $K$ is twice the larger basis constant of $(y_i)$ and $(z_i)$. Moreover, since $(E_n)$ and $(F_n)$ are $p$- and $2$-f.d.d.'s, respectively, and $|e_n|_p \leq 1$, we see that $P(x) = \sum \tilde{f}_n(x) e_n$ is bounded. □

References


[6] W.B. Johnson and M. Zippin, On subspaces and quotients of $(\mathcal{C}_p \mathcal{A}G_n)_{l_p}$ and $(\mathcal{C}_p \mathcal{A}G_n)_{c_0}$, Israel J. Math. 13 (1972), 311-316.


