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Complemented subspaces of $L_p$, which embed into $\ell_p \otimes \ell_2$


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COMPLEMENTED SUBSPACES OF $L_p$ WHICH EMBED INTO $\ell^p \oplus \ell^2$

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In this seminar we report on joint work with Ted Odell [5] concerning the isomorphic classification of complemented subspaces of $L_p$, $1 < p \neq 2 < \infty$. There are now known to exist uncountably many mutually non-isomorphic complemented subspaces of $L_p$ for each $1 < p \neq 2 < \infty$ [1]. However, there probably are only finitely many which are "small". For example, the only complemented subspace of $L_p$ which embeds into $l_p$ is $l_p$ itself [6].

The question studied in [5] is "what are the complemented subspaces of $L_p$ which embed into $l_p \oplus l_2$?" For $1 < p < 2$, the following partial answer is given:

**Theorem A:** If $X$ is a complemented subspace of $L_q$ ($1 < q < 2$) which has an unconditional basis and $X$ embeds into $l_p \oplus l_2$, then $X$ is isomorphic to $l_q$, $l_2$, or $l_q \oplus l_2$.

It is of course a major unsolved problem whether every complemented subspace of $L_p$ ($1 < p \neq 2 < \infty$) has an unconditional basis.

Theorem A is an immediate consequence of the result of [6] mentioned above and:

**Proposition B:** Let $X$ be a subspace of $L_p$ ($2 < p < \infty$) which has an unconditional basis and which is isomorphic to a quotient of $l_p \oplus l_2$. Then there is a subspace $U$ of $l_p$ (possibly $U = \{0\}$) so that $X$ is isomorphic to $U$, $l_2$, or $U \oplus l_2$. 
The classification of complemented subspaces of $L_p$ which embed into $l_p \oplus l_2$ is more complicated for $2 < p < \infty$ because of the presence of Rosenthal's space $X_p$ [11]. However, in [5] the following is proved:

**Theorem C:** If $X$ is a complemented subspace of $L_p$ ($2 < p < \infty$) which has an unconditional basis and which embeds into $l_p \oplus l_2$, then $X$ is isomorphic to $l_p$, $l_2$, $l_p \oplus l_2$, or $X_p$.

Below we give a more-or-less complete proof of Proposition B and outline the proof of Theorem C. Actually, Theorem A is also a consequence of Theorem C and the following result from [5] which will not be discussed in this seminar:

**Theorem D:** If $X$ is a subspace of $L_p$ ($2 < p < \infty$) which is isomorphic to a quotient of a subspace of $l_p \oplus l_2$, then $X$ embeds into $l_p \oplus l_2$.

**Proof of Proposition B:** Let $(x_i)$ be a normalized unconditional basis for $X$ and let $\xi$ be a norm one operator from $l_p \oplus l_2$ onto $X$.

**Claim:** There exists $\epsilon > 0$ so that for all $0 < \delta < \epsilon$, \( \delta \leq \|x_i\|_2 \leq \epsilon \) is finite. (Here $\|x\|_r = (\int_0^1 |x(t)|^r dt)^{1/r}$ for $1 \leq r < \infty$.)

If the claim is false, then there are $\epsilon_1 > \epsilon_2 > \ldots > 0$ and infinite sets $M_n$ of integers so that $\epsilon_{n+1} < \|x_i\|_2 \leq \epsilon_n$ for $i \in M_n$ and $n = 1, 2, \ldots$. Since $(x_i)$ is unconditional, it follows from the classical results of Kadec and Pelczynski [7] that $(x_i)_i \in M_n$ is equivalent to the unit vector basis for $l_2$ for each $n = 1, 2, \ldots$, hence so is...
(f_i)_1 \in M_n$, if $(f_i)$ is the sequence of biorthogonal functionals to $(x_i)$.

But this means that for each $n = 1, 2, \ldots$, the $\ell_q$ contribution to the norm of $(Q*f_i)$ tends to zero as $i \to \infty$ in $M_n$, because every operator from $l_2$ into $\ell_q$ is compact. Consequently, since $Q^*$ is an isomorphism, we can select $i_n \in M_n$ so that $(Q*f_{i_n})_{n=1}^\infty$ is equivalent to the unit vector basis of $\ell_2$, hence the same is true of $(x_{i_n})_{n=1}^\infty$. But $(x_{i_n})_{n=1}^\infty$ has a subsequence equivalent to the unit vector basis of $\ell_p$ because

$$\lim_{n \to \infty} \|x_{i_n}\|_2 = 0.$$ This completes the proof of the claim.

**Exercise:** Where was unconditionality of $(x_n)$ used in the proof of the claim?

Since for any $\epsilon > 0$, the closed linear span of $(x_i: \|x_i\|_2 > \epsilon)$ is either finite dimensional or isomorphic to $\ell_2$, we can, in view of the claim, assume that $\|x_n\|_2 \to 0$ and hence [?] that no subsequence of $(x_n)$ is equivalent to the unit vector basis for $\ell_2$. We will show that this condition implies that $X$ embeds into $\ell_p$.

Let $f_i = g_i \oplus e_i \in \ell_q \oplus \ell_2$ be a normalized sequence which is equivalent to the biorthogonal functionals to $(x_i)$. In view of Lemma 1 below, we can assume that $(g_i)$ is a monotonely unconditional basic sequence in $\ell_q'$, and $(h_i)$ is orthogonal in $\ell_2'$. Since no subsequence of $(f_i)$ is equivalent to the unit vector basis of $\ell_2$, there exists $\delta > 0$ and $n$ so that $\|g_i\| \geq \delta$ for all $i \geq n$. Letting $P$ denote the natural projection of $\ell_q \oplus \ell_2$ onto $\ell_q'$, we complete the proof by observing that $P$ is an isomorphism when restricted to $[(f_i)_{i=n}^\infty]$, the closed linear span of $(f_i)_{i=n}^\infty$. Indeed, since $(g_i)$ is monotonely unconditional, we have for all scalars $(a_i)$ that $\langle \Sigma |a_i|^2 \rangle^{1/2} \leq K_q \delta^{-1} \|P a_i g_i\|$ where $K_q$ is
Khintchine's constant for $L_q$. Hence for any $f = \sum_{i=n}^{\infty} a_i f_i \in [(f_i)_i]$, 

$$\|Pf\| \leq \|f\| = \max (\|\Sigma a_i f_i\|, \|\Sigma a_i h_i\|) \leq \max (\|Pf\|, (\sum_{i=1}^{\infty} |a_i|^2)^{1/2}) \leq K_q^{5^{-1}} \|Pf\|.$$ 

In the proof of Proposition B, we used:

**Lemma 1:** Let $i$ be an unconditional basic sequence in $\ell_p \oplus \ell_2$ $(1 < p < \infty)$. Then there is a monotonely unconditional basic sequence $(u_i)$ in $\ell_p$ and an orthogonal sequence $(v_i)$ in $\ell_2$ so that $(x_i)$ is equivalent to $(u_i \oplus v_i)$ in $\ell_p \oplus \ell_2$.

**Proof.** The proof uses an idea of Schechtman's [13]. Note that by a perturbation argument we can assume that, if $(e_n)$ denotes the natural basis for $\ell_p \oplus \ell_2$, then for any $n = 1, 2, \ldots$, only finitely many of the $x_i$'s have a non-zero nth coordinate when $x_i$ is expanded in terms of $(e_n)$. We can represent $(e_n)$ in $L_p [-1,1]$ by having $(e_{2n-1})_{n=1}^{\infty}$ be a sequence of $L_p$-normalized indicator functions of disjoint subsets of $[-1,0)$ and letting $(e_{2n-1})_{n=1}^{\infty}$ be the Rademacher functions on $[0,1]$. Write $x_i = y_i + z_i$ with $y_i \in [(e_{2n})_{n=1}^{\infty}]$ and $z_i \in [(e_{2n-1})_{n=1}^{\infty}]$. The sequence $(x_i)$ is easily seen to be equivalent to the sequence $(r_i \otimes y_i + r_i \otimes z_i)$ in $L_p ([0,1] \times [-1,1])$, where $(r_i)$ is the usual sequence of Rademacher functions. Of course, $(r_i \otimes z_i)$ is equivalent to an orthogonal sequence; the point is that the terms of the monotonely unconditional sequence $(r_i \otimes y_i)$ are measurable with respect to a purely atomic sub-sigma field of $[0,1] \times [-1,0]$ so that $[(r_i \otimes y_i)]$ embeds isometrically into $\ell_p$. □
Throughout the rest of this seminar, we let $2 < p < \infty$ and let $(e_n)$ (respectively, $(\delta_n)$) denote the unit vector basis for $l_p$ (respectively, $l_2$). Given $z = y \oplus z \in l_p \oplus l_2$, we let $|x|_p = |y|$ and $|x|_2 = |z|$. Given a sequence $w = (w_n)$ of non-negative weights, the space $X_{p,w}$ is defined to be the subspace $[e_n \oplus w_n \delta_n]$ of $l_p \oplus l_2$. We use $(b_n)$ to denote the natural basis $(e_n \oplus w_n \delta_n)$ for a generic $X_{p,w}$ space; if confusion is likely to result, we use $|\cdot|_{2,w}$ to denote the $l_2$-part of the norm in $X_{p,w}$, so that for $x = \sum a_nb_n \in X_{p,w}$, $|x|_{2,w} = (\sum |a_n w_n|^2)^{1/2}$.

No matter what the weight sequence $w$ is, the space $X_{p,w}$ is isomorphic to $l_2$, $l_p$, $l_p \oplus l_2$ or the space $X_p$ introduced by Rosenthal [11]. Rosenthal showed that $X_{p,w}$ is isomorphic to $X_p$ if and only if for each $\epsilon > 0$,

$$\sum_{n=1}^{\infty} \frac{w_n^{2p/(p-2)}}{w_n} = \infty.$$

$X_p$ is isomorphic to a complemented subspace of $l_p$ but is not isomorphic to a complemented subspace of $l_p \oplus l_2$. It has become clear during the last ten years that, rather than being a pathological example, $X_p$ plays a fundamental role in the study of $l_p$ (cf., e.g., [2], [4], and [12]).

There are three important steps in the proof of Theorem C:

**Proposition 2:** Let $X$ be a subspace of $l_p \oplus l_2$ ($2 < p < \infty$) and let $T$ be an operator from $l_p$ into $X$. Then $T$ factors through $X_p$.

**Proposition 3:** If $X$ is isomorphic to a complemented subspace of $X_p$ and $X_p$ is isomorphic to a complemented subspace of $X$, then $X$ is isomorphic to $X_p$. 
Proposition 4: Let $X$ be a subspace of $l_p \oplus l_2$ ($2 < p < \infty$) with a normalized basis $x_n = y_n \oplus z_n$, where $(y_n)$ (respectively, $(z_n)$) is a basic sequence in $l_p$ (respectively, $l_2$). Assume that $|z_n|_2 \to 0$ as $n \to \infty$. Then either $X$ embeds into $l_p$ or $X_p$ is isomorphic to a complemented subspace of $X$.

Notice that Proposition 2 implies that a complemented subspace of $l_p$ which embeds into $l_p \oplus l_2$ is isomorphic to a complemented subspace of $X_p$. Suppose now that $X$ is a complemented subspace of $l_p$ which embeds into $l_p \oplus l_2$ and $X$ has normalized unconditional basis which in $l_p \oplus l_2$ can be represented as $x_n = y_n \oplus z_n$, where by Lemma 1 we can assume that $(y_n)$ is unconditional in $l_p$ and $(z_n)$ is orthogonal in $l_2$. Suppose that

$$\text{(*) } \left\{ \begin{array}{l}
\text{There are } 1 > \epsilon_1 > \epsilon_2 > \ldots > 0 \text{ so that for } n = 1, 2, \ldots,
M_n = \{i: \epsilon_{n+1} \leq |z_i|_2 < \epsilon_n\} \text{ is infinite.}
\end{array} \right.$$  

We can then use a standard gliding hump and perturbation argument to find infinite $M'_n \subseteq M_n$ so that, setting $M = \bigcup_{n=1}^{\infty} M'_n$, we have that $(y_i)_{i \in M}$ is equivalent to the unit vector basis of $l_p$ and $(z_i)_{i \in M}$ is equivalent to an orthogonal sequence in $l_2$. Thus by Rosenthal's characterization of $X_p$ mentioned earlier, $[(x_i)_{i \in M}]$ is isomorphic to $X_p$ and is complemented in $X$ because $(x_i)$ is unconditional, hence by Propositions 2 and 3, $X$ is isomorphic to $X_p$.

If (*) is false, then there is $\epsilon > 0$ and $A \subseteq \mathbb{N}$ so that $|z_i|_2 \geq \epsilon$ for $i \notin A$ and $\lim_{i \to \infty} |z_i|_2 = 0$. 


By Proposition 4, either $X_p$ is complemented in $[(x_i)_{i \in A}]$ and hence in $X$, so that, by Proposition 3, $X$ and $X_p$ are isomorphic, or $[(x_i)_{i \in A}]$ embeds into $l^1_p$, and so is finite dimensional or isomorphic to $l^1_p$ since it embeds into $l^1_p$ as a complemented subspace. Of course, $[(x_i)_{i \notin A}]$ is isomorphic to a Hilbert space and so if $[(x_i)_{i \in A}]$ embeds into $l^1_p'$, then $X$ is isomorphic to $l^1_p$, $l^1_p \cong l^1_2$, or $l^1_2$ if, respectively, $\mathbb{N} \sim A$ is finite, $A$ and $\mathbb{N} \sim A$ are infinite, or $A$ is finite.

To indicate how to prove Proposition 2, we need to recall the concept of a blocking of a finite dimensional decomposition (f.d.d., in short).

Given an f.d.d. $(E_n)$ for some space $Z$, a blocking of $(E_n)$ is an f.d.d. for $Z$ of the form $(E'_n)$, where for $k = 1, 2, ..., E'_k = [(E'_{i})_{i = n(k)}^{n(k+1)-1}]$ for some sequence $1 = n(1) < n(2) < ...$ of integers. The simplest version of the blocking method, introduced in [6] (cf. also Proposition 1.9 in [8]) can be stated qualitatively as follows: If $Z$ has a shrinking f.d.d. $(E_n)$, $Y$ has an f.d.d. $(F_n)$, and $T: Z \to Y$ is an operator, then there are blockings $(E'_n)$ of $(E_n)$ and $(F'_n)$ of $(F_n)$ so that for all $n = 1, 2, ..., T E'_n$ is "essentially" contained in $F'_n + F'_{n+1}$. ("Essentially" means: given any $\varepsilon_n \downarrow 0$, $(E'_n)$ and $(F'_n)$ may be chosen so that for $x \in E_n$, $d(Tx, F'_n + F'_{n+1}) \leq \varepsilon_n ||x||$.) An easy consequence of this blocking principle is:

**Lemma 5:** If $(E_n)$ is a shrinking f.d.d. for $Z$, $(F_n)$ is an f.d.d. for $Y$, and $T: Z \to Y$ is an operator, then there are blockings $(E'_n)$ of $(E_n)$ and $(F'_n)$ of $(F_n)$ so that $T: (\sum_{n=1}^{\infty} E'_n)_p + (\sum_{n=1}^{\infty} F'_n)_p$ is bounded.

We are now ready to prove Proposition 2. By a change of density on the underlying measure space, we can by one of Maurey's theorems [9]
assume that $T$ is bounded as an operator from $L_2$ into $(X, \| \cdot \|_2)$, i.e., for all $x \in L_p$, $|Tx|_2 \leq K \|x\|_2$ for some constant $K$. Secondly, by Lemma 5, we can find a blocking $(H_n)$ of the Haar basis so that $T$ is bounded as an operator from $\left( \sum_{n=1}^{\infty} (H_n, \| \cdot \|_p) \right)_p$ into $(X, \| \cdot \|_p)$. (To see this, embed $(X, \| \cdot \|_p)$ into $L_p$ and block the unit vector basis for $L_p$.) Consequently, if for $x \in L_p$, $x = \sum_n (x_n \in E_n)$, we define $\|\| x \|\| = \max_1 \left( \sum \|x_n\|_p^{1/p}, \|x\|_2 \right)$ then we have that $T$ is bounded as an operator from $(L_p, \|\| \cdot \|\|)$ into $X$. The identity mapping from $L_p$ into $(L_p, \|\| \cdot \|\|)$ is bounded because the Haar basis, being unconditional, admits a lower $\ell_p$-estimate. Thus the operator $T : L_p \to X$ factors through $(L_p, \|\| \cdot \|\|)$.

To complete the proof of Proposition 2 we only need to observe that the completion of $(L_p, \|\| \cdot \|\|)$ is isomorphic to a complemented subspace of $X_{p,w}$ for some weight sequence $w$. This is done by seeing that the completion of $(L_p, \|\| \cdot \|\|) = (\sum H_n, \|\| \cdot \|\|)$ is norm one complemented in $(\sum E_n, \|\| \cdot \|\|)$ by the orthogonal projection, where for $n = 1, 2, \ldots$, $E_n = [(h_i)_{i=1}^{k(n)}]$ and $k(n)$ is chosen so that $H_n \subset E_n$. If $f_i^n \in E_n$ denotes the $L_p$-normalized indicator function of the interval $[(i-1)2^{-k(n)}, i2^{-k(n)})$ for $1 \leq i \leq \varepsilon^{k(n)}$; $n = 1, 2, \ldots$, then one can easily see that $(f_i^n)_{i=1}^{\varepsilon^{k(n)}} \in (\sum E_n, \|\| \cdot \|\|)$ is equivalent to the natural basis of $X_{p,w}$ for the weight sequence $w = (\|f_i^n\|_2)_{i=1}^{\varepsilon^{k(n)}}$.

To prove Proposition 3 we need the following:

**Lemma 6:** There exists $M_p < \infty$ so that if $T$ is an operator on $X_{p,w}$ for some weight sequence $w = (w_n)_{n=1}^{\infty}$, then there exists a weight sequence...
The lemma can be proved by embedding $X_p$ into $L_p([-1,1])$ by identifying the $n$th-unit vector of $X_p,w$ with the function $f_n = g_n + w_n r_n$, where $(g_n)$ are disjointly supported unit vectors in $L_p[-1,0]$, $\|g_n\|_2 \leq w_n$, and $(r_n)$ are the Rademacher functions on $[0,1]$. Note that $\|\cdot\|_2,w$ on $X_p,w$ is equivalent to $\|\cdot\|_2$ under this identification. Now one uses [3] to get a change of density $\phi \geq \frac{1}{2}$ on $[-1,1]$ so that $T$ is bounded when considered as an operator from $([f_n], \|\cdot\|_{L_2(\phi \, dm)})$ into itself. One can check that the weight sequence $v = (v_n)$ defined by $v_n^2 = w_n^2 + \|g^{-1/p} e_n\|_{L_2(\phi \, dm)}^2$ does the job.

We are now ready to prove Proposition 3. The idea is to use Pelczynski's classical proof [10] that every complemented subspace of $L_p$ is isomorphic to $L_p$. We need to write $X_p$ as a symmetric sum $(X_p \oplus X_p \oplus \ldots)$ in such a way that $(X \oplus X \oplus \ldots)$ is complemented in $(X_p \oplus X_p \oplus \ldots)$. The problem is that $X_p$ is not isomorphic to $(X_p \oplus X_p \oplus \ldots)_p$. However, if we represent $X_p$ as $X_p,w'$, then $X_p$ is isomorphic to $(X_p,w' \oplus X_p,w' \oplus \ldots)_p,2$ where for $x_n \in X_p,w'$, the norm in $(X_p,w' \oplus X_p,w' \oplus \ldots)_p,2$ of $y = (x_n)_{n=1}^\omega$ is given by $\|y\| = \max ((\Sigma |x_n|^p)^{1/p}, (\Sigma |x_n|^2,w')^{1/2})$. (One checks the isomorphism of $X_p$ with $(X_p,w' \oplus X_p,w' \oplus \ldots)_p,2$ by observing that $(X_p,w' \oplus X_p,w' \oplus \ldots)_p,2$ is isometric to $X_p,w'$, where the weight sequence $v$ consists of all terms of the weight sequence $w$, each repeated infinitely many times.) Unfortunately, it is not true that $(X \oplus X \oplus \ldots)$ must be complemented in $(X_p,w' \oplus X_p,w' \oplus \ldots)_p,2$ if $X$ is complemented in $X_p,w'$, so Pelczynski's argument does not apply. However, if the projection
P: $X_p \to X$ is bounded in both the $\| \cdot \|_p$ and the $\| \cdot \|_{2,w}$ norms on $X$, then $(X \oplus X \oplus \ldots)$ is complemented in $(X_{p,w} \oplus X_{p,w} \oplus \ldots)_{p,2}$ by the projection $P \oplus P \oplus \ldots$. The point of Lemma 6 is that we can assume, without loss of generality, that $\|P\|_{2,w} < \infty$. Of course, $\|P\|_p$ might be infinite, but there is by Lemma 5 a blocking $(E_n)$ of the natural basis for $X_{p,w}$ so that $P$ is bounded as an operator from $(\Sigma E_n)^{p,2}$ into itself, where each space $E_n$ has the $X_{p,w}$ norm, $\| \cdot \|$, on it. If we define

$$\| x \|_{p,w} = \left( \Sigma \| x_n \|_{p} \right)^{1/p}$$

then it is easy to check that the $X_{p,w}$ norm is equivalent to the norm $\| \|_x = \max (\|x\|_p, \|x\|_{2,w})$. Since $\|P\|_p$ and $\|P\|_2$ are both finite, $(X \oplus X \oplus \ldots)$ is complemented in $((X_{p,w}, \| \cdot \|), (X_{p,w}, \| \cdot \|), \ldots)_{p,2}$ and this letter space is easily seen to be isomorphic to $X_p$. This completes the sketch of the proof of Proposition 3.

We complete this seminar by giving a proof of Proposition 4.

If $\ell_2$ does not embed into $X$, then $X$ embeds into $\ell_p$ by a result of Johnson and Odell (or see [2]). Thus we may assume $X$ contains a copy of $\ell_2$.

Since $\|z_n\|_2 \to 0$, we can assume without loss of generality that $\|z_n\|_2 < 1$ for each $n$. For a subspace $Y$ of $X$, let $\delta(Y) = \sup \{ \|y\|_2 : \|y\| = 1 \}$. Note that since $X$ contains $\ell_2$, if $\dim X/Y < \infty$, then $\delta(Y) = 1$. By the blocking technique [6] there exists $0 = k(1) < k(2) < \ldots$ such that if $E_n = [(y_1^k k(n+1)]$ and $F_n = [(z_1^k k(n+1)]$, then $(E_n)$ is an $\ell_p$-f.d.d. for $[(y_n)]$ and $(F_n)$ is an $\ell_2$-f.d.d. for $[(z_n)]$. Thus if $u_n \in E_n$, then $\|\Sigma u_n\|_p \sim (\Sigma |u_n|^p)^{1/p}$ and a similar statement holds for $(F_n)$. Also by our above remark we can insure that
\( s([x_i^{k(n)}]_{k(n)+1}) \geq 1/2 \) for each \( n \). Since \( |z_n|_2 \to 0 \), we can find \( q(n) \) such that if \( H_n = [(x_i)]_{k(n)+1}^{q(n)} \), then

\[ l > s(H_n) > 0 \quad \text{for each} \quad n, \]

\[ \sum_{n=1}^{\infty} s(H_n)^2p/(p-2) = \infty, \quad \text{and} \quad \lim_{n \to \infty} s(H_n) = 0. \]

Let \( e_n \in H_n \) so that \( \|e_n\| = 1 \) and \( |e_n|_2 = s(H_n) \). Clearly \( (e_n) \) is isomorphic to \( X_p \). We must show it is also complemented in \( X \). Thus we wish to find \( \tilde{a} \in H \) so that \( (\tilde{a}) \) is biorthogonal to \( (e_n) \) and

\[ \tilde{a} = \sum_{n} \tilde{a}_n (x) e_n \]

is a bounded operator, and hence a projection onto \( [(e_n)] \).

Let \( f_n \) be the functional on \( H_n \) defined by \( f_n(h) = \langle h, e_n |e_n|_2^{-2} \rangle \). Then

\[ |f_n|_p = \max_{|h|_p = 1} \langle h, e_n |e_n|_2^{-2} \rangle \]

\[ \leq \max_{|h|_p = 1} |h|_2 |e_n|_2^{-1} = 1, \]

since \( |e_n|_2 = s(H_n) \) and \( \| \cdot \| = | \cdot |_p \) on \( H_n \). Thus \( f_n \) is a norm 1 functional on \( H_n \) in the \( l_p \) norm. Extend \( f_n \) to a functional \( \tilde{f}_n \) on \( X \) by letting \( \tilde{f}_n(x_i) = 0 \) if \( i < k(n) \) or \( i > q(n) \). Since \( (y_i) \) and \( (z_i) \) are basic, we have

\[ |\tilde{f}_n|_p \leq K \quad \text{and} \quad |\tilde{f}_n|_2 \leq K|f_n|_2 = K|e_n|_2^{-1}. \]
where $K$ is twice the larger basis constant of $(y_i)$ and $(z_i)$. Moreover, since $(E_n)$ and $(F_n)$ are p- and 2-f.d.d.'s, respectively, and $|e_n|_p \leq 1$, we see that $P(x) = \sum \tilde{f}_n(x) e_n$ is bounded. \hfill \Box

References


[5] W.B. Johnson and E.W. Odell, Subspaces and quotients of $l_p \oplus l_2$ and $X_p$,


