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Volume estimates and nearly euclidean decompositions for normed spaces


<http://www.numdam.org/item?id=SAF_1979-1980____A22_0>
VOLUME ESTIMATES AND NEARLY EUCLIDEAN
DECOMPOSITIONS FOR NORMED SPACES

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The purpose of this talk is to present a new isomorphic invariant of a finite dimensional normed space, so called "volume ratio" (introduced in [8]). We set 

\[ \text{vr}(E) = \left( \frac{\text{vol } B_E}{\text{vol } \mathcal{E}} \right)^{1/n}, \]

where \( B_E \) is the unit ball of an \( n \)-dimensional real normed space \( E \), \( \mathcal{E} \) - the ellipsoid of maximal volume contained in \( B_E \) (so called "John's ellipsoid of \( E \)) and \( \text{vol } A \) stands for volume of a set \( A \).

It follows directly from the definition that

\[ \text{vr}(E) \leq \text{vr}(E) \, d(E,F), \]

where \( E \) and \( F \) are normed spaces of the same dimension, \( d \) - the Banach-Mazur distance.

To explain the motivation for introducing such an invariant let us mention the following:

**Theorem 1** (Kashin [6]) : There is a universal constant \( C \) such that, given \( n \), there exist two \( n \)-dimensional subspaces \( E_1, E_2 \) of \( L^1_{2n} \), orthogonal (in \( L^2_{2n} \)) satisfying

\[ d(E_i, L^2_{2n}) \leq C \quad \text{for } i = 1, 2. \]

**Proposition 2** : \( \text{vr}(L^1_n) \leq \sqrt{2} e/\pi \) for \( n = 1, 2, \ldots \)

**Proposition 3** : Let \( C \) and \( \theta < 1 \) be positive constants. Then, for any normed space \( E \) with \( \text{vr}(E) < C \) and positive integer \( k \leq \theta \dim E \), "most of" \( k \)-dimensional subspaces \( F \) of \( E \) satisfy

\[ d(F, L^2_k) \leq C', \]
where $C'$ depends only on $C$ and $\theta$. More precisely: if $G = G(k, n)$ is the Grassmann manifold of $k$-dimensional subspaces of $E$, $\mu$ - a normalized invariant measure on $G$, generated by the John's ellipsoid. Then

$$\mu(\{F \in G : F \text{ satisfies (2)}\}) > \frac{1}{2} . \quad \blacksquare$$

Deducing Th. 1 from Prop. 2 and Prop. 3 is immediate, one must only remember that the map $F \rightarrow F^\perp$ (the orthogonal complement of $F$), acting on $G(n, 2n)$, is measure-preserving.

**Proof of Prop. 2:** By direct computation. \quad \blacksquare

**Proof of Prop. 3:** Let $E = (\mathbb{R}^n, \|\cdot\|)$. We may assume that the John's ellipsoid of $E$ is equal to the Euclidean unit ball $B^n = \{\|x\| \leq 1\}$. Denote by $m$ the normalized Haar measure on $S^{n-1}$. Then

$$C^n > v\nu(E) = \int_{S^{n-1}} \|x\|^{-n} m(dx)$$

(one gets the equality by representing $\text{vol } A$ as $\int_{\mathbb{R}^n} \chi_A$ and passing to polar coordinates).

Given $r \in (0, 1)$ define $A_r = \{x \in S^{n-1} : \|x\| < r\}$. Then one gets from (4) that

$$m(A_r) < (Cr)^n .$$

On the other hand, we have

$$m(A_r) = \int_{S^{n-1}} \chi_{A_r} d\mu = \int_{G} \mu(dF) \int_{S_F} \chi_{A_r \cap F} d\mu_F =$$

$$= \int_{G} m_F(A_r \cap F) \mu(dF) ,$$

where $m_F$ is the normalized Haar measure on $S_F = F \cap S^{n-1}$. The last two formulae show that

$$\mu(\{F \in G : m_F(A_r \cap F) < 2(\text{Cr})^n\}) > \frac{1}{2} ;$$

in other words, for "most of" $F \in G$ we have

$$m_F(\{x \in S_F : \|x\| < r\}) < 2(\text{Cr})^n \leq (2 \text{Cr})^n .$$
We show that every such $F$ is "close" to $\ell^2_k$ in the Banach-Mazur sense, thus proving Prop. 3.

Indeed, since, for given $x_0 \in S_F$ and $\delta \leq \frac{1}{2}$,

$$m_F(\{x \in S_F : \|x - x_0\|_2 \leq \delta\}) \geq \left(\frac{\delta}{4}\right)^k,$$

the previous estimate shows (remember that $k \leq \Theta n$) that $S_{F \setminus A_r}$ is an $r/2$-net (in $\ell^2_n$ metric) for $S_F$, provided $r = r(\Theta, C)$ is small enough (precisely, if $r \leq (2^{3\Theta + 1} C)^{1/(1-\Theta)}$. Fix such $r$. Then, for any $y \in S_F$, there is a $y_0 \in S_{F \setminus A_r}$ (i.e. $\|y_0\| \geq r$) such that $\|y - y_0\|_2 \leq r/2$. Since (by $B^n \subset B_E$), $\|x\|_2 \geq \|x\|$ for all $x \in E$, we have also $\|y - y_0\| \leq \frac{r}{2}$. Therefore

$$\|y\| \geq \|y_0\| - \|y - y_0\| \geq r - \frac{r}{2} = \frac{r}{2}.$$

So, by homogeneity,

$$\frac{r}{2} \|y\|_2 \leq \|y\| \leq \|y\|_2$$

for all $y \in F$. Hence $d(F, \ell^2_k) \leq 2r^{-1} = 2r(\Theta, C)^{-1}$. This ends the proof of Prop. 3. ■

In the sequel, we shall frequently use the following concepts.

We say that $(e_i)$ is an unconditional basis of a B-space $E$ provided

$$\text{ubc}(e_i) \overset{\text{def}}{=} \sup_{|\varepsilon_i| \leq 1, \|\sum_i t_i e_i\| \leq 1} \left\|\sum_i \varepsilon_i t_i e_i\right\| < \infty.$$ 

We say that a B-space $E$ is of cotype $q$ ($q \geq 2$) if there is a constant $K$ such that, for every finite sequence $x_1, x_2, \ldots \in E$, we have

$$\int \left\|\sum_i r_i x_i\right\| \geq K^{-1} \left(\sum_i \|x_i\|^q\right)^{1/q},$$

where $(r_i)$ is the sequence of Rademacher functions. The smallest such constant $K$ is called the cotype $q$ constant of $E$ and denoted by $K_q(E)$.

It was proved in [4] that given $K$ there exist $C$, $\Theta > 0$ such that, for every finite dimensional $E$ with $K_2(E) \leq K$, one can find a subspace of $E$, say $F$, with $\dim F = k \geq \Theta \dim E$ and $d(F, \ell^2_k) \leq C$. Prop. 2 and Prop. 3 strengthen this result in the special case $E = \ell^1_n$. This raises
the following problems:

Problem 4: Given \( \theta \in (0, 1) \), does every normed space \( E \) contain a \( [\theta \dim E] \)-dimensional subspace \( F \) with \( d(\ell^2_{\dim F}, F) \leq C \), where \( C \) depends only on \( K_2(E) \)?

Problem 5: Does there exist a function \( C(.) \) such that \( \nu r(E) \leq C(K_2(E)) \) for every \( E \)?

Of course a positive solution of Problem 5 implies a positive solution of Problem 4. We have two partial results in this direction.

Theorem 6 [8]: Let \( E \) be a finite dimensional space, \((e_i)\) -its basis. Then

\[ \nu r(E) \leq C_k(E) \text{ubc}(e_i) \]

where \( C \) is a universal constant.

Theorem 7 [8]: There is a universal constant \( C \) such that

\[ \nu r(\ell^2_n \otimes \ell^2_n) \leq C \text{ for all } n. \]

Recall that \( \ell^2_n \otimes \ell^2_n \) is the tensor product \( \ell^2_n \otimes \ell^2_n \) equipped with the largest tensor norm (in other words: the space of nuclear operators on \( \ell^2_n \)). It is known that \( \text{ubc}(w) \) is of order \( \sqrt{n} \) for every basis \((w_i)\) of \( \ell^2_n \), while \( K_2(\ell^2_n \otimes \ell^2_n) \leq K \), where \( K \) does not depend on \( n \).

Theorem 7 can be generalized to a large class of tensor products and unitary ideals. In particular, a unitary ideal \( \mathcal{U} \) on \( \ell^2_n \) has "small" volume ratio if the associated \( n \)-dimension symmetric space \( \ell^1_\mathcal{U} \) has (in the case of Th. 7 we have \( \ell^1_n = \ell^1_n \); see e.g. [5] for definitions).

Now I present a sketch of the proof of Th. 6. We shall need two lemmas.

Lemma A: Let \((E, \| \cdot \|)\) be a \( B \)-space of cotype 2 with an unconditional basis \((e_i)\). Then there exists a norm \( \| \cdot \|(1) \) such that

a) \( \| x \| \leq \| x \|(1) \leq C K_2(E) \text{ubc}(e_i) \| x \| \) for \( x \in E \) (\( C \) is an absolute constant).

b) \( \text{ubc}(e_i) = 1 \) in \((E, \| \cdot \|(1))\)
c) the dual norm $\|\cdot\|$ on $E^*$ is 2-convex; in other words a functional
defined by $\|x_j\| = (\sum_{j=1}^{\infty} x_j e_j^{*}(1))^{2}$ is a basic sequence
$E^*$ dual to $(e_j)$ is a norm (then, of course, unconditional).

Lemma A is well known (see e.g. [1]).

**Lemma B:** Let $(F,\|\cdot\|)$ be an $n$-dimensional normed space, $(f_i)$ -its basis
with $\text{ub}c(f_i) = 1$. Then there exists a sequence of positive numbers
$\beta_1, \beta_2, \ldots, \beta_n$ such that, for all $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{R}$,

$$n \sum_i^{\infty} |\lambda_i| \leq \| \sum_i^{\infty} \beta_i \lambda_i f_i \| \leq \max_i |\lambda_i| .$$

**Proof of lemma B:** Some variants of lemma B are known in a more general
setting of B-lattices. I present a proof, which is essentially due to
T.K. Carne.

Consider $f : B_E \to R$ defined by $f(\sum b_i f_i) = \prod b_i$. Let $B = \sum \beta_i f_i$
be a point, where $f$ attains its maximum. Of course one can choose $B$ to
satisfy $\beta_i \geq 0$ for $i = 1, 2, \ldots, n$. Clearly $\|B\| = 1$; this implies immediately
the right hand inequality of (5), because $\text{ub}c(f_i) = 1$. By the same reason,
to prove the left hand inequality of (5) it is enough to show that the
functional $\varphi: \sum \lambda_i \beta_i f_i - \frac{1}{n} \sum \lambda_i$ is of norm at most 1.

It is easy to see that $\varphi$ is the only functional satisfying

(i) $\varphi(B) = 1,$
and (ii) $\varphi(x) \geq 1$ if $x \in Q \equiv \{ x = \sum_i^{\infty} x_i f_i : x_i \geq 0$ for $i = 1, 2, \ldots, n$
and $\prod_i^{\infty} x_i > \prod_i^{\infty} \beta_i \}$. But it is clear that the functional $\psi$
separating disjoint (by definition
of $\beta$) and convex sets $B_E$ and $Q$ (i.e. $\psi(B_E) \leq 1, \psi(Q) > 1$) satisfies (i)
and (ii); hence $\varphi = \psi$ and $\varphi(B_E) \leq 1$, in other words $\|\varphi\| \leq 1$. This proves
lemma B.

Now we shall derive th. 6 from lemmas A and B.

Clearly, by lemma A and (1), it is enough to prove that if
$(E,\|\cdot\|^{(1)})$ satisfies conditions (b) and (c) of lemma A, then $\varphi(E) \leq C$,
where $C$ is a universal constant. On the other hand, this estimate will
immediately follow from existence of a sequence $(\alpha_k)$ such that

$(\ast) \sum_k |x_k| \leq \| \sum_k \alpha_k x_k e_k \|^{(1)} \leq \sqrt{n} (\sum_k |x_k|^2)^{1/2}$
for all \(x_1, x_2, \ldots, x_n \in \mathbb{R}\). Indeed, defining an ellipsoid

\[ \mathcal{E} = \left\{ x = \sum_{k} \alpha_k x_k e_k : \sqrt{n} \sqrt{\sum_{k} |x_k|^2} \leq 1 \right\} \]

we get \( \mathcal{E} \subset B = B_{\mathbb{R}^n} \). Hence

\[ \operatorname{vr}(E) \leq \left( \frac{\operatorname{vol} B}{\operatorname{vol} E} \right)^{1/n} \leq \left( \frac{\operatorname{vol} B_{\mathbb{R}^n}}{\operatorname{vol} E} \right)^{1/n} = \frac{\operatorname{vr}(E)}{\sqrt{2n}} \]

by proposition 2.

To show \(*\) consider its dual version

\[ \max_k |y_k| \geq \left\| \sum_k \frac{y_k}{\lambda_k} e_k^* \right\| \geq \frac{1}{\sqrt{n}} \left( \sum_k |y_k|^2 \right)^{1/2} \]

Of course it is enough to prove \((***)\) for nonnegative sequences \((y_k)\) only. Substituting \(y_k = \sqrt{\lambda_k}\) and \(\alpha_k = 1/\sqrt{\lambda_k}\) one gets

\[ \frac{1}{n} \sum_k \lambda_k \leq \left( \sum_k \sqrt{\lambda_k} e_k^* \right)^2 \leq \max_k \lambda_k \]

Now existence of \((\beta_k)\) satisfying \((***)\) follows immediately from condition (c) of lemma A (i.e. the fact that the term in the centre of \((***)\) is equal to \(\| (\beta_k \lambda_k) e_k^* \| \) for some unconditional norm \(\| . \|\) ) and lemma B. \(\blacksquare\)

Let us introduce another invariant:

\[ \operatorname{hvr}(E) \overset{\text{def}}{=} \sup_{F \subset E, \dim F < \infty} \operatorname{vr}(E) \]

where \(E\) is a Banach space, not necessarily of finite dimension. Using some methods from \([4]\), one can easily derive from Prop. 3 the following:

**Theorem 8**: If \(\operatorname{hvr}(E) < \infty\), then \(E\) is of cotype \(2 + \varepsilon\) for every \(\varepsilon > 0\).

In general \(\varepsilon\) cannot be omitted. \(\blacksquare\)

Finally I am going to present:

**Theorem 9**: There exists a function \((0,1) \ni \varepsilon \rightarrow C(\varepsilon)\) such that for any \(k\)-dimensional subspace \(E\) of \(\ell_{\infty}^n\) we have

\[ d(E, \ell_{k}^2) > C(k/n) \sqrt{k} \]

\(\blacksquare\)
Remark : Our proof gives \( C(\emptyset) = \sqrt{\frac{\pi}{2}} e^3 \). Recently Figiel and Johnson proved th. 9 with \( C(\emptyset) = \sqrt{\frac{\pi}{2}} / 2 \). $

**Proof of theorem 9 :** Since \( d(E, \ell_k^2) = d(E^*, \ell_k^2) \), it is enough to prove (\(+\)) with \( E \) replaced by \( E^* \).

To say that \( E \) is a subspace of \( \ell_n^\infty \) is the same as to say that the unit ball of \( E^* \) has at most \( 2n \) extreme points, say \( x_1, x_2, \ldots, x_n, -x_1, -x_2, \ldots, -x_n \). Let \( \mathcal{E} \) be an ellipsoid contained in the unit ball of \( E^* \). We must show that, for some \( i \), \( x_i \in C(k/n) \forall \mathcal{E} \). Thus the proof reduces to the following fact :

Let \( B = \text{abs conv}(y_i)_{i=1}^n \subset \mathbb{R}^k \) and let the Euclidean unit ball \( B_k^k = \{ x \in \mathbb{R}^k : \| x \|_2 < 1 \} \) be contained in \( B \). Then \( \max_{1 \leq i \leq n} \| y_i \|_2 \geq k \cdot C(k/n) \).

To see the above consider all sets of the form \( B_A = \text{abs conv}(y_i)_{i \in A} \), \( A \subset \{ 1, 2, \ldots, n \} \), \( \text{card } A = k \).

Clearly \( \bigcup_A B_A = B \). Choose \( A \) so that \( \text{vol } B_A \) is maximal. Then

\[
\binom{n}{k} \cdot \text{vol } B_A \geq \text{vol } B \geq \text{vol } B_k^k .
\]

On the other hand

\[
\text{vol } B_A \leq \prod_{i \in A} \| y_i \|_2 \cdot \text{vol } B_k^k .
\]

Combining these two estimates one gets

\[
\prod_{i \in A} \| y_i \|_2 \geq \left( \frac{n}{k} \right)^{-1} \frac{\text{vol } B_k^k}{\text{vol } B_k^k} = \left( \frac{n}{k} \right)^{-1} \left( \sqrt{k} \right)^{-k} \left( \frac{\text{vol } B_k^k}{\text{vol } B_k^k} \right) = \left( \frac{n}{k} \right)^{-1} \left( \sqrt{k} \right)^{-k} \left[ \text{vr}(B_k^k) \right]^{-k} .
\]

Hence

\[
\max_{i \in A} \| y_i \|_2 \geq \left[ \left( \frac{n}{k} \right)^{1/k} \text{vr}(B_k^k) \right]^{-1} \left( \sqrt{k} \right) \geq \frac{k}{\sqrt{n}} \sqrt{\frac{\pi}{2e}} \sqrt{k} .
\]

This ends the proof of theorem 9. $

Let us mention finally some easy observations, which may indicate another application of concepts introduced here. Namely, we have

\[
\left[ \text{vr}(E) \right]^\theta \left[ \text{vr}(F) \right]^{1-\theta} = \text{vr}(E \oplus \ell_2 F)
\]
where \( \theta = \dim E/(\dim E + \dim F) \) and \( E \oplus_{\ell^2} F \) is a direct sum of \( E \) and \( F \) in the sense of \( \ell^2 \).

One can hope that this may help in investigating complemented subspaces of a normed space.

REFERENCES


