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REPORT ON M. GROMOV'S ALMOST FLAT MANIFOLDS (*)

by Hermann KARCHER

1. Introduction

A basic theme in Riemannian geometry is the following question : To what extend do assumptions on local invariants determine global properties ? Very important such assumptions are bounds for the curvature of the metric - recall that in Riemann's normal coordinates the curvature tensor is obtained as the second derivative of the metric. Examples of known results are :

(i) The only surfaces which carry positive curvature metrics are S^2 and $P^2(\mathbb{R})$, because $2\pi \cdot \chi(M) = \int_M K d\theta$.

(ii) A complete simply connected Riemannian manifold M^n of nonpositive curvature is diffeomorphic to $\mathbb{R}^n$, because the Riemannian exponential map $\exp_p$ has maximal rank on the tangent space T_pM and is in fact a covering map.

(iii) More specifically, if M^n has zero curvature ("flat") then $\exp_p$ is an isometric covering map, i.e. the fundamental group $\pi_1(M,p)$ operates as a discrete - and for compact M : uniform - group of isometries on $\mathbb{R}^n$. From Bieberbach's classification of such groups it follows that compact flat manifolds are covered by flat tori.

(iv) If M^n is complete, noncompact and has positive curvature then convexity arguments show that M^n is diffeomorphic to $\mathbb{R}^n$.

(v) If M^n is simply connected, complete and has curvature bounds $\frac{1}{4} < K \le 1$ then M^n is homeomorphic to S^n . For even dimensions ≥ 4 the result is sharp since $P^n(\mathbb{C})$ carries a metric with $\frac{1}{4} \le K \le 1$.

(vi) If M^n is complete and has curvature bounds $0.7 \le K \le 1$ then the following holds : The universal covering $\tilde{M}^n$ is diffeomorphic to S^n in such a way that the action of $\pi_1(M,p)$ on $\tilde{M}$ is conjugate to an orthogonal action on S^n , i.e. M^n is diffeomorphic to a space of constant curvature.

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(vii) In principle similar results hold if the model space S^n is replaced by any of the other symmetric spaces of compact type, but the precise formulation is more elaborate.

The purpose of this lecture is to explain the proof of the following theorem of M. Gromov [6] which differs from all the previous results by the fact that the model space is not known a priori but has to be constructed in the proof. ([6] is a general reference throughout the paper.)

1.1. THEOREM.- Let M be a compact n-dimensional Riemannian manifold, assume that the sectional curvatures K of M are bounded in terms of the diameter $d(M)$:

$$|K| \leq \varepsilon \cdot d(M)^{-2} \quad \text{with} \quad \varepsilon \leq \exp(-\exp(\exp(2n^2))) \text{ (present estimate).}$$

Then there is a finite - at most $2 \cdot (6\pi)^{\frac{1}{2}n(n-1)}$ fold - covering of M which is diffeomorphic to a compact quotient of a nilpotent Lie group.

There are many more manifolds than the compact flat ones which allow for every $\varepsilon > 0$ an ε-flat metric, i.e. one which satisfies $|K| \leq \varepsilon \cdot d(M)^{-2}$.

1.2. <u>Example</u>.- On the nilpotent Lie algebra $\mathfrak{g} = \left\{ A = \begin{pmatrix} 0 & \cdot & & a_{ij} \\ & & \cdot & \\ & & & \cdot \\ 0 & & & 0 \end{pmatrix} ; a_{ij} \in \mathbb{R} , 1 \leq i < j \leq n \right\}$ define the following family of scalar products :

$$\|A\|_q^2 = \sum_{i<j} a_{ij}^2 \cdot q^{2(j-i)}$$

and extend them by left translation to the corresponding nilpotent Lie group G of upper triagular matrices. From the estimate $\|[A,B]\|_q \leq 2 \cdot (n-2) \cdot \|A\|_q \cdot \|B\|_q$ one derives the following q-independant bound for the curvature tensors R_q of these left invariant metrics

$$\|R_q(A,B)\, C\|_q \leq 24(n-2)^2 \cdot \|A\|_q \cdot \|B\|_q \cdot \|C\|_q ,$$

or $\|R_q\|_q \leq 24 \cdot (n-2)^2$.

Each compact quotient $\Gamma\backslash G$ can be given an arbitrarily small diameter by appropriate choice of q ; therefore $\Gamma\backslash G$ is ε-flat for each $\varepsilon > 0$. If one takes for Γ the integer subgroup of G , then Γ is not a Bieberbach group since the rank of its free Abelian subgroups is too small and therefore $\Gamma\backslash G$ does not carry any flat metric.

1.3. The first steps of Gromov's proof. Because of the strong curvature assumptions the maximal rank radius r_m of the Riemannian exponential map is much larger than the diameter of M . Therefore many short geodesic loops exist and Gromov defines a product between short loops at p which satisfies the relations of a group where it is defined. From this torso one can generate the fundamental group $\pi_1(M,p)$ abstractly : by generators and relations. Each short loop at p is mapped onto its holonomy motion and this map is almost compatible with the Gromov product since small curvature implies that parallel translation varies only slightly with the change of the path. Therefore commutators of loops almost behave as commutators of motions, i.e. iterated commutators converge to the identity if the rotational part of the corresponding holonomy motion is small ($\leq \frac{1}{3}$). Every set consisting of loops with rotational parts $\leq \frac{1}{3}$ will therefore generate a nilpotent subgroup of $\pi_1(M,p)$ if the homotopy errors are not too large. Moreover the degree of nilpotency of all such subgroups has the a priori bound $d = (\frac{40}{13})^{\frac{1}{2}n(n+1)}$ which is derived by a counting argument in the group of motions. - We continue this summary in 2.15 after the more detailled explanations of chapter 2 have been given.

2. Products of short loops

From Riemannian geometry we have

2.1. Rauch's THEOREM [5].- Curvature bounds $-\lambda^2 \leq K \leq \Lambda^2$ imply for the Riemannian exponential map exp at p (for $v, w \in T_pM$)

$$|w| \cdot \frac{\sin \Lambda|tv|}{\Lambda|tv|} \leq |(d\exp)_{tv} \cdot w| \leq |w| \cdot \frac{\sinh \lambda|tv|}{\lambda|tv|} ,$$

$(d\exp)_{tv}$ has maximal rank if $|tv| < \pi \cdot \Lambda^{-1}$ $(\leq \pi \cdot \varepsilon^{-1/2} \cdot d(M)$ in 1.1).

2.2 Klingenberg's Long-Homotopy-lemma [5].- Let r_m be the maximal rank radius of $\exp_p$; assume $\exp_p v = \exp_p w$. Then any homotopy which joins the geodesic arcs $\exp tv$ and $\exp tw$ $(0 \leq t \leq 1)$ contains a curve of length $\geq r_m$.

2.3. DEFINITION.- A homotopy which contains only curves shorter than the maximal rank radius r_m of the exponential map is called a short homotopy. The corresponding equivalence classes are called short homotopy classes.

From 2.2 and the standard shortening process by geodesic segments we have

2.4. Every short homotopy class of closed curves at p contains exactly one geodesic loop at p .

2.5. DEFINITION.- Let α and β be geodesic loops at p ; assume that the sum of their lengths is less than the maximal rank radius r_m , e.g. $|\alpha| + |\beta| < \pi \cdot \varepsilon^{-1/2} \cdot d(M)$. Let $\beta \cdot \alpha$ be the closed curve "first α then β ", as usual. Gromov's product $\beta * \alpha$ is the unique (!) geodesic loop in the short homotopy class of $\beta \cdot \alpha$.

If one lifts the curve $\beta \cdot \alpha$ to T_pM by $\exp_p$, then the ray to the endpoint of this curve is mapped by $\exp_p$ onto the loop $\beta * \alpha$. Clearly α^{-1} is the loop α parametrized backwards and associativity holds as long as the sum of the lengths of the factors is $< r_m (\geq \pi \ \varepsilon^{-1/2} \ d(M))$. Every closed curve can be decomposed (in $\pi_1(M,p)$) into a product of curves shorter than $2d(M) + \eta$ ($\eta > 0$ chosen) ; therefore $\pi_1(M,p)$ is generated by geodesic loops $\leq 2d(M) + \eta$. Under the mild additional condition $5 \leq \pi \cdot \varepsilon^{-1/2}$ it can already be proved that all relations in $\pi_1(M,p)$ are products of relations which are given by short homotopies between loops of length $< 5 \cdot d(M)$. Therefore the short loops $(< 5 \cdot d(M))$ with Gromov product generate a group isomorphic to $\pi_1(M,p)$.

2.6. DEFINITION.- Let c be a curve and let a vectorfield X along c satisfy the differential equation $\frac{D}{dt} X(t) = \dot{c}(t)$. The map $m(c) : T_{c(0)} \longrightarrow T_{c(1)}M$ given by $X(0) \longrightarrow X(1)$ is called affine translation [10] along c . $m(c)$ is a motion, since its linear part is Levi-Civita translation along c .

2.7. Path dependence of translations [2]. Let c_1 , c_2 be two curves from $c_i(0) = p$ to $c_i(1) = q$; assume the existence of a smooth homotopy from c_1 to c_2 with area $\leq F$ and longest curve $\leq L$. Let $X_i(t)$ be Levi-Civita parallel along c_i and $X_1(0) = X_2(0)$; let $Y_i(t)$ be affine parallel along c_i with $Y_i(0) = 0$. Let $\|R\|$ be a bound for the curvature tensor along the homotopy. Then

$$|\sphericalangle(X_1(1), X_2(1))| \leq \|R\| \cdot F ,$$

$$|Y_i(1)| \leq \text{length } (c_i) , \quad |Y_1(1) - Y_2(1)| \leq L \cdot \|R\| \cdot F .$$

Our most important application of 2.7 is to homotopies which are given by geodesic segments spanned in geodesic triangles. L is the sum of two edgelengths and F is obtained from

2.8. Aleksandrow's area comparison [1]. Consider a geodesic triangle and span any surface with geodesic segments. Assume a curvature bound $K \leq \Lambda^2$ along the surface. Consider a triangle with the same edgelengths as the given one in the plane of constant curvature Λ^2 (if $\Lambda^2 > 0$ this requires a circumference $< 2\pi\Lambda^{-1}$). Then the area of the spanned surface is not larger than the area of this constant curvature triangle. In particular, if two edgelengths, a , b are $\leq \pi \cdot (3\Lambda)^{-1}$ then $F \leq 0.7ab$ ($\leq 0.5ab$ if $\Lambda = 0$).

To conveniently express how closely the Gromov product $\beta * \alpha$ and the composition of the holonomy motions $m(\beta) \circ m(\alpha)$ are related we use the following Finsler metrics :

2.9. DEFINITION.- For A , $B \in SO(n)$ define $d(A,B) = \max\{ |\sphericalangle(B^{-1}AX,X)| ; 0 \neq X \in R^n\}$; the corresponding norm in the tangent space $T_{id}SO(n)$ of skew symmetric matrices is $|S| = \max\{ |SX| ; X \in R^n , |X| = 1\}$. For motions $\tilde{A}_i(X) = A_i \cdot X + a_i$ define $\tilde{d}(\tilde{A}_1, \tilde{A}_2) = \max(d(A_1, A_2), 3\Lambda \cdot |a_1 - a_2|)$. (Λ^2 should be thought of as a curvature bound ; the factor 3Λ makes the definition independent of renormalizations of the metric of M ; it is also convenient in 2.12.) Abbreviate $d(A, id) = \|A\|$; $\tilde{d}(\tilde{A}, id) = \|\tilde{A}\|$.

We summarize 2.5 - 2.9 (note $|K| \leq \Lambda^2 \implies \|R\| \leq \frac{4}{3}\Lambda^2$) :

2.10. Homotopy errors. Let α , β be geodesic loops with $\beta * \alpha$ defined. Let $r(\alpha)$ and $t(\alpha)$ be rotational and translational part of the holonomy motion 2.6. Assume curvature bounds $|K| \leq \Lambda^2$. Then

$$d(r(\beta * \alpha), r(\beta) \circ r(\alpha)) \leq \Lambda^2 |t(\alpha)| \cdot |t(\beta)| ,$$

$$|r(\beta) \cdot t(\alpha) + t(\beta) - t(\beta * \alpha)| \leq (|t(\alpha)| + |t(\beta)|) \cdot \Lambda^2 |t(\alpha)| \cdot |t(\beta)| .$$

For commutators better estimates are true than follow from 2.10. One needs

2.11. Comparison of Riemannian and Euclidean translation [9]. Let $w(t)$ be a parallel vector field along the geodesic $c(t) = \exp tv$. Assume $|K| \leq \Lambda^2$. Then

$$d(\exp(v+w(0)),\exp_{c(1)} w(1)) \le \frac{1}{3}\Lambda(|v||w| \sinh \Lambda(|v| + |w|) .$$

First the translational part of the commutator $[\beta,\alpha] = \beta^{-1} * \alpha^{-1} * \beta * \alpha$ is estimated directly with 2.1 and 2.11 ; then this information is used to get a good bound on the homotopy error of the rotational part from 2.7 and 2.8. Gromov does not seem to use 2.11.

2.12. <u>Commutator estimates</u> [2]. Let α , β be short geodesic loops (2.5) at p and assume $|K| \le \Lambda^2$. Then

$$|t([\beta,\alpha])| \le \frac{2}{3}|t(\alpha)||t(\beta)|\cdot \Lambda \sinh \Lambda(|t(\alpha)| + |t(\beta)|) + 2\sin(\tfrac{1}{2}\|r(\beta)\|)\cdot|t(\alpha)| + $$
$$+ 2\sin(\tfrac{1}{2}\|r(\alpha)\|)\cdot|t(\beta)| ,$$

$$d(r([\beta,\alpha]), r(\beta)^{-1}\cdot r(\alpha)^{-1}\cdot r(\beta)\cdot r(\alpha)) \le \Lambda^2(2|t(\alpha)||t(\beta)| + (|t(\alpha)| + |t(\beta)|)\cdot|t[\beta,\alpha]|).$$

Assume in addition $|m(\alpha)|$, $|m(\beta)| \le \frac{1}{3}$ (hence $|t(\alpha)|$, $|t(\beta)| \le (9\Lambda)^{-1}$ (2.9)), then $\|m([\beta,\alpha])\| \le 2.4\,\|m(\alpha)\|\cdot\|m(\beta)\| \le 0.8 \min(\|m(\alpha)\| , \|m(\beta)\|)$.

This result is very powerful. It shows that - after handling the homotopy errors - one can work with commutators of loops almost in the same way as with commutators of motions (we recall $\|[\tilde{A},\tilde{B}]\| \le 2\|\tilde{A}\|\cdot\|\tilde{B}\|$). This use of commutators seems to go back to Margulis who derived from 2.12-type estimates a lower bound for the volume of a compact negatively curved Riemannian manifold. Gromov uses 2.12 to generate nilpotent subgroups of the fundamental group. Very surprisingly the following holds :

2.13. <u>A priori estimate</u> [2]. The <u>degree of nilpotency</u> of all subgroups of $\pi_1(M,p)$ which are generated from sets of loops which satisfy $\|m(\alpha)\| \le \frac{1}{3}$ has a bound

$$d \le \left(\frac{40}{13}\right)^{\frac{1}{2}n(n+1)} \le 1.76^{n(n+1)} .$$

<u>Proof</u>. Choose economic generators as follows : α_1 is such that $\|m(\alpha_1)\|$ is minimal (in the generating set U). If $\alpha_1,\dots,\alpha_j$ are already chosen, then consider the set U_j of Gromov-products of these and choose α_{j+1} in $U\setminus U_j$ such that $\|m(\alpha_{j+1})\|$ is minimal. After finitely many steps one has a so called short basis $\alpha_1,\dots,\alpha_k$ for U . Because of 2.12 one can show by induction that the degree of nilpotency of the generated group $\langle\alpha_1,\dots,\alpha_k\rangle$ cannot be larger than k . From the construction follows

$$\|m(\alpha_i^{-1} * \alpha_j)\| \ge \max(\|m(\alpha_i)\| , \|m(\alpha_j)\|) ,$$

and with 2.10

$$\|m(\alpha_i)^{-1} \circ m(\alpha_j)\| \geq \max(\| m(\alpha_i)\| , \|m(\alpha_j)\|) - \frac{1}{9}\|m(\alpha_i)\| \cdot \| m(\alpha_j)\| \geq$$
$$\geq \max(\| m(\alpha_i)\| - \frac{1}{27} \|m(\alpha_j)\| , \|m(\alpha_j)\| - \frac{1}{27}\|m(\alpha_i)\|) .$$

There are at most as many motions which pairwise satisfy these inequalities as there are unit vectors (Finsler length) in the tangent space of this group which satisfy $|w_i - w_j| \geq \frac{26}{27}$. The balls of radius $\frac{13}{27}$ around such w_i are disjoint and contained in a ball of radius $\frac{40}{27}$. The volume ratio $\left(\frac{40}{13}\right)^{\frac{1}{2} n(n+1)}$ of the balls gives an upper bound for the number of vectors w_i .

2.14. We have formulated 2.13 for the generated group. It is important to observe, that the inductive proof in fact shows : if d is the length of a short basis, and if a d-fold commutator of loops is defined in the sense of 2.5, then this d-fold commutator is already 0 as a loop (while 2.13 only says that this loop is 0 in $\pi_1(M)$).

2.15. The next steps of Gromov's proof. We have constructed nilpotent subgroups of $\pi_1(M)$; next, one has to find one such subgroup which can be embedded as a uniform discrete subgroup Γ into an n-dimensional nilpotent Lie group G . Observe that such a Lie group can be identified with $\mathbb{R}^n$ such that the product is given by Malcev's polynomials [11] of degree $\leq n$. These polynomials are uniquely determined if one knows their values on sufficiently many points of an uniform discrete subgroup of G . Gromov shows that a selected set of short loops, called Γ_{ρ_1} , can be found and (in 3.4) be identified with so large a ball of an integer lattice in $\mathbb{R}^n$ that the products of these loops determine Malcev polynomials [11] which define a product on $\mathbb{R}^n$ turning it into a nilpotent Lie group G . The mentioned set Γ_{ρ_1} of loops is such that the Gromov product behaves almost as the translational parts of the loops do (3.2.5). Therefore one can choose a basis in the same way as in a translational group and express the short loops in Γ_{ρ_1} as words in the basis elements ; these words allow the identification of the short loops with the lattice points of a large ball, even in such a way that loop length and lattice length almost coincide (3.4.2). - The set Γ_{ρ_1} of loops is constructed in 3.2 ; this construction requires curvature assumptions (see 3.2.3) which are so strong that homotopy errors at all other parts of the proof turn out to be almost neglegible.

3. Small rotational parts

3.1. A Dirichlet choice. We have to find a radius ρ_o with the following properties : for every $v \in T_pM$, $|v| = 3\rho_o$, one has a loop α with $|t(\alpha) - v| \le \rho_o + d(M)$ and $\|r(\alpha)\| \le \eta_1 = (2.6\pi)^{-d}$ (recall $d = 1.76^{n(n+1)}$ from 2.13).

The smallness of η_1 is explained in 3.2. To estimate the index of the constructed subgroup in $\pi_1(M)$ one needs $\rho_o \ge 2 \cdot (6\pi)^{\frac{1}{2}n(n-1)} \cdot d(M)$ (see 3.3). One can find $\rho_o \le 4^N \cdot 2(6\pi)^{\frac{1}{2}n(n-1)} \cdot d(M)$ with $N \le \exp(\exp(n^2))$.

Proof. First, a lifting argument shows that the translational parts of loops at p are $d(M)$-dense within the ball of radius r_m $(\ge \pi\varepsilon^{-1/2} d(M))$ in which $\exp_p$ has maximal rank. However the nearest loop to a given $v \in T_pM$ need not have small rotational part, but it suffices if its rotational part occurs η_1-almost among loops of length $\le \rho_o$. (Homotopy errors are neglected since they cause a neglegible contribution.) Let $B_{\frac{1}{2}\eta_1}$ be a (Finsler-) ball of radius $\frac{1}{2}\eta_1$ in $SO(n)$; there are at most $N = \operatorname{vol} O(n) / \operatorname{vol} B_{\frac{1}{2}\eta_1} \le 2\left(\frac{2\pi}{\eta_1}\right)^{\dim SO(n)} < \exp(\exp n^2)$ rotations in $O(n)$ with pairwise distance $\ge \eta_1$. Therefore, if $\rho_{o,1} = 2 \cdot (6\pi)^{\frac{1}{2}n(n-1)} \cdot d(M)$ does not have the desired property, one tries $\rho_{o,2} = 4 \cdot \rho_{o,1}$; after at most N such 4-fold increases one must have found a suitable ρ_o , since it cannot be true at each step that one finds a rotational part for a loop of length between $2\rho_o$ and $4\rho_o$ which does not η_1-almost occur among the loops $\le \rho_o$.

3.2. The almost translational set of loops. Consider the set Γ_{ρ_1} of loops with lengths $\le \rho_1 = e^{3n^2}\rho_o$ and rotational parts $\le \frac{1}{3}$. (The large ratio $\frac{\rho_1}{\rho_2}$ is needed in 4.1 to have sufficiently many products available to determine the Malcev polynomials.) Under the curvature assumption $3\Lambda\rho_1 \le \frac{1}{3}$ we have 2.14 for Γ_{ρ_1} , i.e. a short basis of length $\le d = 1.76^{n(n+1)}$ so that all d-fold commutators vanish. Let $\bar{\Gamma}_{\rho_1}$ be the set of all Gromov products of elements in Γ_{ρ_1} such that the products are inductively defined and have lengths $\le \rho_1$. We claim :

3.2.1. All rotational parts in $\bar{\Gamma}_{\rho_1}$ are in fact $\le 2^{-d}$; in particular $\bar{\Gamma}_{\rho_1} = \Gamma_{\rho_1}$.

<u>Proof</u>. Let $\delta \in \bar{\Gamma}_{\rho_1}$ be a loop with $\|r(\delta)\| = \theta > 2^{-d}$. Because of the inductive definition of $\bar{\Gamma}_{\rho_1}$ it is sufficient to assume $\theta \le \frac{1}{3}$. We choose a vector $v \in T_pM$ with $|\measuredangle(r(\delta)\cdot v, v)| = \theta$ and with 3.1 find a loop α such that $\|r(\alpha)\| \le \eta_1$ and $|t(\alpha) - v| \le \rho_o$. Consider the d-fold commutator $[\ldots[\alpha,\delta],\ldots,\delta]$; 2.14 shows that it is trivial ; on the other hand we can estimate its translational part directly and after some computation find it $\neq 0$ if the following is true :

3.2.2. $$\eta_1 \cdot \left(3.2d + (2\sin\frac{\theta}{2})^{-1} \cdot \frac{\rho_1}{\rho_o} \cdot (0.5 + 2\cdot 10^{-4}d) \cdot \left(\frac{2.4\,\theta}{2\sin\frac{\theta}{2}}\right)^d\right) < \frac{1}{2} .$$

From $\eta_1 \le (2.6\pi)^{-d}$ follows that 3.2.2 is true for $\theta \in [2^{-d}, \frac{1}{3}]$, therefore these θ cannot occur in $\bar{\Gamma}_{\rho_1}$.

In the proof of 3.2.2 one has to use the estimates of homotopy errors from chapter 2·; in particular one needs $3\Lambda|t(\alpha)| \le \|r(\alpha)\|$ or $6\Lambda 4^N \cdot (6\pi)^{\frac{1}{2}n(n-1)} \cdot d(M) \le \eta_1 = (2.6\pi)^{-d}$ (compare 3.1).

More explicitly,

3.2.3. $\Lambda^2 \cdot d(M)^2 \cdot \exp(\exp(\exp 2n^2)) \le 1$ is a sufficient curvature assumption.

We repeat : this assumption is so strong that homotopy errors at all <u>other</u> parts of the proof do not significantly change the estimates.

An immediate consequence of 3.2.1 is (since at least $\rho_1 \cdot |t(\alpha)|^{-1}$ iterations of α are possible in Γ_{ρ_1}) :

3.2.4 If $\alpha \in \Gamma_{\rho_1}$ then $\|r(\alpha)\| \le 2^{-d} \cdot \frac{|t(\alpha)|}{\rho_1}$.

Therefore we have the following <u>almost translational behaviour</u> ($\varepsilon \ll 1$ contains the homotopy error).

3.2.5. If $\alpha,\beta \in \Gamma_{\rho_1}$ then $|t(\alpha * \beta) - t(\alpha) - t(\beta)| \le 2^{-d}\,\frac{|t(\alpha)| \cdot |t(\beta)|}{\rho_1} \cdot (1+\varepsilon)$.

Moreover we have from 2.12 (as a consequence of 2.11) already at this point a commutator estimate which Gromov derives only later.

3.2.6. If $\alpha, \beta \in \Gamma_{\rho_1}$ then $|t([\alpha,\beta])| \le 2 \cdot 2^{-d} \cdot \frac{|t(\alpha)| \cdot |t(\beta)|}{\rho_1} \cdot (1+\varepsilon)$.

3.3. The index estimate. We estimate the index of the group Γ generated by Γ_{ρ_1} in $\pi_1(M,p)$ as follows :

(i) The finitely many loops at p of length $\le 2 \cdot d(M)$ generate $\pi_1(M,p)$.

(ii) If all the words of wordlength $= \ell + 1$ in these generators occur already in equivalence classes mod Γ_{ρ_1} of the words of wordlength $\le \ell$, then there are no further equivalence classes in longer words.

(iii) Two short loops ($\le \rho_1$) are in the same equivalence class mod Γ_{ρ_1} if their rotational parts have a distance $\le \frac{1}{3}$ in $O(n)$ (homotopy errors neglected). Therefore there are at most $W = \operatorname{vol} O(n) \big/ \operatorname{vol} B_{\frac{1}{3}} \le 2 \cdot (6\pi)^{\dim SO(n)}$ different equivalence classes among the short loops.

(iv) Words of wordlength $\le W$ are still short as loops $(2W \cdot d(M) \le 2\rho_o)$. Therefore (ii) must occur among the words of wordlength $\le W$, so that there are not more than W equivalence classes mod Γ in $\pi_1(M)$.

3.4. The lattice identification. The almost translational behaviour 3.2.5 allows to pick generators in Γ_{ρ_1} in the same way as in a discrete translational subgroup of $\mathbb{R}^n$. Let δ_1 be the shortest loop in Γ_{ρ_1} ; δ_1 commutes with all other loops because of 3.2.6. For each $\delta \in \Gamma_{\rho_1}$ consider the orbit $\{\delta_1^i * \delta\} \subset \Gamma_{\rho_1}$. Scalar products $\langle t(\delta_1), t(\delta_1^i * \delta)\rangle$ and lengths $|t(\delta_1^i * \delta)|$ along the orbit can be controlled with 3.2.5 to find a unique representative $\tilde{\delta}$ in the orbit determined by $\langle t(\delta_1), t(\tilde{\delta})\rangle > 0$, $\langle t(\delta_1), t(\delta_1^{-1} * \tilde{\delta})\rangle \le 0$. Starting from δ one needs at most $\left(1 + (1-2^{-d})^{-1} \circ \frac{|t(\delta)|}{|t(\delta_1)|}\right)$ multiplications by $\delta_1^{\pm 1}$ to reach $\tilde{\delta}$.

Let Γ' be the set of orthogonal projections of representatives $\tilde{\delta}$ onto the orthogonal complement of $t(\delta_1)$ in $\mathbb{R}^n = T_pM$ and define for $\alpha', \beta' \in \Gamma'$ the product $\alpha' * \beta'$ to be the projection of the representative of $\tilde{\alpha} * \tilde{\beta}$. Starting from $|\delta'| \le \tilde{\delta} \le 1{,}5 \cdot |\delta'|$ one proves that the inequalities 3.2.5 and 3.2.6 hold in Γ' with 2^{-d} replaced by $8 \cdot 2^{-d}$; note that even $8^n \cdot 2^{-d}$ is still much smaller than needed for the present arguments. To define a product $\alpha' * \beta'$ one needs the product $\tilde{\alpha} * \tilde{\beta}$ of somewhat longer elements in Γ, but for $\alpha', \beta' \in \Gamma'$ with $|\alpha'|, |\beta'| \le \frac{1}{3} \cdot \rho_1$ the product is clearly defined. Therefore one is ready for an induction which for dimension reasons terminates after at most n steps : If inductively the basis $\delta'_2, \dots, \delta'_n$, for Γ' is already selected then choose $\delta_1, \tilde{\delta}_2, \dots, \tilde{\delta}_n$, as basis for Γ_{ρ_1}. Since the loops from Γ_{ρ_1} are ρ_o-dense in the $4\rho_o$-ball in $\mathbb{R}^n$ (see 3.1), and since we do not lose significantly from this relative denseness through n inductive steps (recall $d = 1.76^{n(n+1)}$), we will obtain <u>exactly</u> n generators $\delta_1, \dots, \delta_n$ for Γ_{ρ_1}, which is Gromov's "normal basis". 3.2.6 shows at each inductive step that the shortest element is in the center ; therefore all loops $\delta \in \Gamma_{\rho_1}$ of length $\le 3^{-n} \cdot \rho_1$ have a unique representation as a normal word $\delta_1^{k_1} * \dots * \delta_n^{k_n}$. (The factor 3^{-n} stems from $|\tilde{\delta}| \le 1.5|\delta'|$; it could be almost removed since for $|\tilde{\delta}| >> |\delta_1|$ a much sharper inequality is true.) Clearly we can identify the loop $\delta_1^{k_1} * \dots * \delta_n^{k_n}$ with the n-tuple $(k_1, \dots, k_n)$ or even with the lattice vector $\sum_{i=1}^{n} k_i \cdot \delta_i$ in T_pM. This identification is much better than one might expect since the inductive choice of the normal basis gives

3.4.1. $$|\det(\delta_1, \dots, \delta_n)| \ge 0.8^{n(n-1)} \cdot |\delta_1| \cdot \ldots \cdot |\delta_n| .$$

From 3.2.5 and 3.4.1 we prove that the lattice-identification is very close to the translational part, namely (if $|t(\delta_1^{k_1} * \dots * \delta_n^{k_n})| \le 3^{-n} \cdot \rho_1$) :

3.4.2. $$\Big|t(\delta_1^{k_1} * \dots * \delta_n^{k_n}) - \sum_{i=1}^{n} k_i \cdot \delta_i\Big|_{T_pM} \le 2^{-d} \cdot 2^{n^2} \cdot \frac{1}{\rho_1} \cdot \Big|\sum_{i=1}^{n} k_i \cdot \delta_i\Big|^2 .$$

We interpret now Gromov's product of loops as a product between the lattice points $\sum k_i \cdot \delta_i$ of T_pM and since lattice length and loop length almost coincide by 3.4.2 we have :

3.4.3. Inequalities 3.2.5 and 3.2.6 hold for lattice vectors of length $\le 3^{-n} \cdot \rho_1$ if loop length $|t(\delta_1^{k_1} * \dots * \delta_n^{k_n})|$ is replaced by lattice length $|\sum k_i \cdot \delta_i|$ and ε is increased slightly.

Finally we note that at each inductive step the shortest vector is $\le 2\rho_o$, therefore we have for the normal basis

3.4.4. $|\delta_i| < 2\rho_o \cdot (1.5)^{i-1}$ $(i = 1,\dots,n)$.

4. The nilpotent Lie group

4.1. The Malcev polynomials. 3.4.3 shows that commutators $[\delta_i , \delta_j]$ are generated by $\delta_1,\dots,\delta_{\min(i,j)-1}$. Therefore the product of two words $\delta_1^{k_1} * \dots * \delta_n^{k_n} * \delta_1^{\ell_1} * \dots * \delta_n^{\ell_n}$ is a new word $\delta_1^{p_1} * \dots * \delta_n^{p_n}$ where the p_i are polynomials of degree $\le n+1-i$ in the exponents $k_1,\dots,k_n$, $\ell_1,\dots,\ell_n$ [11]. (Commutators are so much shorter than their factors that the rearranging of the product into its normal form does not change its length very much ; therefore the rearranging can be considered an algebraic procedure as in [11].) We want to use these so called "Malcev polynomials" to extend the product from a ball in the lattice $\sum k_i.\delta_i$ to all of $\mathbb{R}^n$ and thus obtain the desired n-dimensional nilpotent Lie group G . If one knows associativity, inverses and the nilpotency relations on sufficiently many lattice points then the polynomials expressing these relations are satisfied on all of $\mathbb{R}^n$ and therefore define the nilpotent Lie group structure on $\mathbb{R}^n$.

The inverse is given by a polynomial of degree $\le n$, associativity is expressed by a polynomial of degree n^3 and the vanishing of the various n-fold commutators is expressed by polynomials of degree $\le n^{3n}$. Since commutators are shorter than their factors one stays in the domain where products are defined. Together with $\max\{ |\sum k_i.\delta_i| ; |k_i| \le N\} \le n \cdot N \cdot 2\rho_o \cdot 1.5^{n-1}$ it follows that it is sufficient to have products defined for all loops of length $\le 2n \cdot n^{3n} \cdot 1.5^{n-1} \cdot \rho_o$. This leads to $\rho_1 = e^{3n^2} \cdot \rho_o$, the assumption made in 3.2. Therefore the Malcev polynomials are uniquely determined by the Gromov products of loops in Γ_{ρ_1} and they satisfy all relations to define a nilpotent Lie group structure on $\mathbb{R}^n$! The set Γ_{ρ_1} of loops $\le \rho_1$ with rotational part $\le \frac{1}{3}$ is identified in a product preserving way with a subset of this Lie group G , and the group Γ (which is abstractly generated from Γ_{ρ_1} with the short relations (2.5) between its elements) is identified as an uniform discrete subgroup of G via the integer lattice points $\sum k_i.\delta_i$ in $\mathbb{R}^n$.

4.2. Injectivity. Obviously Γ has a natural homomorphic image in $\pi_1(M)$; we need this to be an isomorphic one. Therefore one has to exclude the possibility that the other short loops, i.e. those with rotational parts $> \frac{1}{3}$, generate (in $\pi_1(M)$) additional relations between the elements of Γ . To achieve this we identify (in 4.2.1) all loops $\leq 3^{-n} \cdot \rho_1$ bijectively and product preserving with transformations of some set S . Clearly, the group generated from the loops is isomorphic to the group generated from the transformations ; therefore there are no further identifications in the generated group. Recall, that all relations in $\pi_1(M)$ are generated from the short relations between loops of length $\leq 5d(M)$ - which is $\leq 3^{-n} \cdot \rho_1$; this proves that the natural image of Γ in $\pi_1(M)$ is an isomorphic one. -

4.2.1. The definition of the set S . Consider two loops $\leq 3^{-n} \cdot \rho_1$ equivalent if they differ by a loop in Γ_{ρ_1} , then take A as a set of shortest representatives from these equivalence classes and put $S = A \times \Gamma$. To define the action of any loop b ($\leq 3^{-n} \cdot \rho_1$) on $(a,\delta) \in A \times \Gamma$ write $b * a = a' * \delta'$ ($a' \in A$, $\delta' \in \Gamma$) and put $b \cdot (a,\delta) = (a' , \delta' * \delta)$. To check that this identifies the loops $\leq 3^{-n} \cdot \rho_1$ injectively and product preserving with transformations on S , one uses that Γ_{ρ_1} is fairly dense among all loops $\leq 3^{-n}\rho_1$ (see 3.4, in particular 3.4.2) and that Γ_{ρ_1} can be identified with its left actions on Γ (see end of 4.1).

4.3. The left invariant metric on G . We lift the "normal basis" $\delta_1,\dots,\delta_n \in G$ with the exponential map Exp of G to a basis of the tangent space T_eG and use this basis for an isometric identification of T_eG with T_pM ; then we left translate this metric to all of G . Next, the curvature tensor of this metric - or equivalently the norm of the Lie bracket - has to be estimated. We do not understand Gromov's "interpolation argument" , but we estimate the third order remainder term of the Campbell-Hausdorff power series inductively over the subgroups spanned by $\delta_1,\dots,\delta_i$:

4.3.1. If $H(X,Y)$ is defined by $\mathrm{Exp}\, X \cdot \mathrm{Exp}\, Y = \mathrm{Exp}\, H(X,Y)$, then we have

$$|H(X,Y) - X - Y - \tfrac{1}{2}[X,Y]| \leq |[X,Y]| \cdot \varepsilon \cdot (|X| + |Y|) ,$$

where $X \in T_eG$ is arbitrary, $Y \in T_e\mathrm{span}(\delta_1,\dots,\delta_i)$ and ε depends on the norm of the Lie bracket on $T_e\mathrm{span}(\delta_1,\dots,\delta_{i-1})$.

Consequently we have (side conditions as in 4.3.1) :

4.3.2. $$|H(e^{\mathrm{Ad}X} \cdot Y , -Y) - [X,Y]| \leq |[X,Y]| \cdot \varepsilon' \cdot (|X| + |Y|) .$$

Because of 4.3.2 and $\mathrm{Exp}\, X \cdot \mathrm{Exp}\, Y \cdot \mathrm{Exp}(-X) \cdot \mathrm{Exp}(-Y) = \mathrm{Exp}\, H(e^{\mathrm{Ad}X} \cdot Y , -Y)$ we can use the commutator estimates 3.4.3 to get, inductively over the subgroups span$(\delta_1,\ldots,\delta_i)$, estimates for the Lie bracket which are about as good as 3.2.6. (In other words : the elements $\delta_1,\ldots,\delta_n$ are indeed so close to the identity in G that the higher than second order terms in the Campbell-Hausdorff series can be neglected for the computation of commutators.) In particular, the curvature of G is very small. (We do not give any more numbers, since the curvature assumption we were forced to make in 3.2.3 makes all estimates ridiculously small compared to what the present arguments would need.)

4.4. The Γ-equivariant diffeomorphism. Γ acts isometrically by left translations on G and - as the deck group of a finite covering of M - Γ also acts isometrically on the universal covering $\tilde{M}$. From the "normal basis" $\delta_1,\ldots,\delta_n$ in Γ and the exponential maps of G and $\tilde{M}$ we obtain natural basis for T_eG and $T_p\tilde{M}$; therefore, after left translation by Γ , we have corresponding natural basis in the tangent spaces of all "lattice points" in G and $\tilde{M}$ which identify these tangent spaces almost isometrically. Then, with the exponential maps of G and $\tilde{M}$ we obtain maps from large balls around the lattice points in G onto corresponding balls in $\tilde{M}$. These local maps are compatible with the action of Γ and they are very close to isometries since the curvatures of G and $\tilde{M}$ are so small (see 2.1). Moreover, their differentials can be described by Jacobi fields, hence, again because of the small curvatures, these differentials are close to the identity (if we identify different tangent spaces by Levi-Civita parallel translation). Therefore a center-of-mass-average [9] of these local maps will produce a Γ-equivariant map of maximal rank from G to $\tilde{M}$, i.e. a Γ-equivariant diffeomorphism.