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AFFINE LIE ALGEBRAS AND MODULAR FORMS

I.G. MACDONALD

Introduction

This lecture is in some sense a sequel to that of Demazure [4], although its point of view will be somewhat different.

Affine Lie algebras are particular examples of Lie algebras defined by Cartan matrices, or Kac-Moody Lie algebras. These are infinite-dimensional complex Lie algebras defined by generators and relations, for which there exists a satisfactory structure theory and representation theory which mirrors precisely (and includes) the classical theory of finite-dimensional complex semisimple Lie algebras, and culminates in an analogue of Weyl's character formula and denominator formula. In the case of affine Lie algebras, these formulas can be made quite explicit, at any rate for certain modules, and lead to formal identities for theta-functions and modular forms. The simplest example is that of the trivial representation , which leads to the so-called denominator formula; this is an identity between formal power series in several variables, and can be specialised to give a large number of identities for Dedekind's η-function.

Apart from these connections with arithmetic and modular forms, which form the subject of this lecture, it has become apparent in the last few years that affine Lie algebras have connections with many other areas of mathematics: combinatorics (partitions, Rogers-Ramanujan identities) [5, 21]; topology (loop spaces and loop groups) [8,9,20]; linear algebra (representations of quivers) [14]; singularities [26]; completely integrable systems [1,2] and the structures of mechanics and particle physics [6,7]. There appear also to be tantalising but as yet little understood connections with the "Monster" simple group [3,15]. The range of these applications, all of which are in a stage of active development, continues to increase at an alarming rate.

1. Finite-dimensional simple Lie algebras

In order to set the scene, we shall briefly review some of the salient facts about a finite-dimensional complex simple Lie algebra $\underline{g}$. Let $\underline{h}$ be a Cartan subalgebra of $\underline{g}$ (i.e., a maximal abelian diagonalizable subalgebra); let $\underline{h}^*$ be the dual of $\underline{h}$, and ℓ the dimension of $\underline{h}$. There exists a non-degenerate symmetric bilinear form (x, y) on $\underline{g}$ which is invariant, i.e., $([x, z], y) = (x, [z, y])$ for all $x, y, z \in \underline{g}$, for example the Killing form $\mathrm{tr}(\mathrm{ad}(x)\mathrm{ad}(y))$. The restriction of this form to $\underline{h}$ is non-degenerate and hence determines a symmetric bilinear form (λ, μ) on $\underline{h}^*$.

Root system. For each $\alpha \in h^*$ let $\underline{g}_\alpha$ denote the set of $x \in \underline{g}$ such that $[h, x] = \alpha(h)x$ for all $h \in \underline{h}$. Then $\underline{g}_0 = \underline{h}$, and the non-zero $\alpha \in \underline{h}^*$ such that $\underline{g}_\alpha \neq 0$ are the roots of $\underline{g}$ relative to $\underline{h}$. They form a finite subset R of $\underline{h}^*$, called the root system of $(\underline{g}, \underline{h})$. We have

$$\underline{g} = \underline{h} + \sum_{\alpha \in R} \underline{g}_\alpha \tag{1.1}$$

and each $\underline{g}_\alpha$ is 1-dimensional. For each $\alpha \in R$, the only roots proportional to α are $\pm\alpha$. The bilinear form on $\underline{g}$ may be chosen so that $|\alpha|^2 = (\alpha, \alpha)$ is real and positive for each $\alpha \in R$.

It is possible to choose roots $\alpha_1, \ldots, \alpha_\ell \in R$ such that each root $\alpha \in R$ is of the form $\alpha = \Sigma n_i \alpha_i$ with integer coefficients n_i, which are either all ≥ 0 (positive roots) or all ≤ 0 (negative roots). The α_i are called a set of simple roots or a basis of R, and we shall assume that they have been chosen once and for all. There is then a unique highest root, for which Σn_i is a maximum.

Weyl group. For each $\alpha \in R$, let w_α denote the reflection in the hyperplane orthogonal to α in $\underline{h}^*$. We have

$$w_\alpha(\lambda) = \lambda - (\alpha^\vee, \lambda)\alpha \qquad (\lambda \in \underline{h})$$

where $\alpha^\vee = 2\alpha/(\alpha, \alpha)$ is the coroot of α. The reflections w_{α_i} corresponding to the simple roots generate a finite group of isometries of $\underline{h}^*$ called the Weyl group W of $\underline{g}$ relative to $\underline{h}$. Each reflection w_α belongs to W; the root system R is stable under W, and each root is of the form $w\alpha_i$ for some $w \in W$ and some simple root α_i.

Cartan matrix. The numbers $a_{ij} = (\alpha_i^\vee, \alpha_j)$ are integers, and the $\ell \times \ell$ matrix $A = (a_{ij})$ is called the Cartan matrix of $\underline{g}$. It satisfies the following conditions:

(C1) $a_{ii} = 2$ for all i; $a_{ij} \leq 0$ if $i \neq j$; $a_{ij} = 0$ whenever $a_{ji} = 0$.

(C2) All the principal minors of A are > 0.

Generators and relations. The Cartan matrix A determines $\underline{g}$ up to isomorphism. Choose generators e_i of $\underline{g}_{\alpha_i}$, f_i of $\underline{g}_{-\alpha_i}$ $(1 \leq i \leq \ell)$ such that $(e_i, f_i) = 1$, and elements $h_i \in \underline{h}$ such that $(h_i, h) = \alpha_i^\vee(h)$, so that $\alpha_j(h_i) = (\alpha_i^\vee, \alpha_j) = a_{ij}$. Then the 3ℓ elements e_i, f_i, h_i generate $\underline{g}$ subject to the following relations:

$$\begin{aligned} &[e_i, f_j] = \delta_{ij} h_i, \qquad [h_i, h_j] = 0, \\ (1.2) \qquad &[h_i, e_j] = a_{ij} e_j, \qquad [h_i, f_j] = -a_{ij} f_j, \\ &(\mathrm{ad}\, e_i)^{1-a_{ij}} e_j = (\mathrm{ad}\, f_i)^{1-a_{ij}} f_j = 0 \qquad (i \neq j). \end{aligned}$$

Modules with highest weight. Let Q (resp. Q_+) denote the set of all $\Sigma n_i \alpha_i$ with $n_i \in \mathbb{Z}$ (resp. $\mathbb{N}$), and let P (resp. P_+) denote the set of all $\lambda \in \underline{h}^*$ such that $\lambda(h_i) \in \mathbb{Z}$ (resp. $\mathbb{N}$) for $1 \leq i \leq \ell$. We have $Q \subset P$ (but $Q_+ \not\subset P_+$).

If V is a $\underline{g}$-module and $\lambda \in \underline{h}^*$, let V_λ denote the set of $v \in V$ such that $h.v = \lambda(h)v$ for all $h \in \underline{h}$. If $V_\lambda \neq 0$, λ is said to be a weight of V with multiplicity $\dim(V_\lambda)$. If V is finite-dimensional, then V is the direct sum of its weight spaces, and all the weights of V lie in the lattice P.

For each $\lambda \in P_+$ there exists a unique finite-dimensional simple $\underline{g}$-module $V(\lambda)$ generated by an element $v_\lambda \in V(\lambda)_\lambda$ such that $e_i.v_\lambda = 0$ $(1 \leq i \leq \ell)$. The set of weights of $V(\lambda)$ is stable under W, and is contained in $\lambda - Q_+$, and λ (the highest weight) has multiplicity 1.

In particular, $V(0)$ is the trivial 1-dimensional $\underline{g}$-module.

Character formula. In the group ring $\mathbb{Z}[P]$ of the free abelian group P, let e^λ denote the element corresponding to λ, so that $e^\lambda.e^\mu = e^{\lambda+\mu}$. The character of a finite-dimensional $\underline{g}$-module V is defined to be

$$\mathrm{ch}(V) = \sum_\mu \dim(V_\mu).e^\mu$$

summed over the weights of V.

Let $\rho \in h^*$ be defined by $\rho(h_i) = 1$ $(1 \leq i \leq \ell)$. Then for $\lambda \in P_+$ we have the character formula (of H Weyl)

$$\text{(1.3)} \qquad \mathrm{ch}\, V(\lambda) = \Big(\sum_{w\in W} \det(w) e^{w(\lambda+\rho)}\Big)/e^\rho \prod_{\alpha>0} (1-e^{-\alpha}).$$

When $\lambda = 0$, $\mathrm{ch}\, V(\lambda) = 1$ and hence (1.3) becomes the denominator formula

$$\text{(1.4)} \qquad \prod_{\alpha>0} (1-e^{-\alpha}) = \sum_{w\in W} \det(w)\, e^{w\rho-\rho}.$$

Using (1.4) we can rewrite (1.3) as

$$\text{(1.5)} \qquad \mathrm{ch}\, V(\lambda) = \Big(\sum_{w\in W} \det(w) e^{w(\lambda+\rho)}\Big)/\Big(\sum_{w\in W} \det(w) e^{w\rho}\Big).$$

2. Kac-Moody Lie algebras [10,11,12,13,24]

Let $A = (a_{ij})_{i,j \in I}$ be any (finite) matrix of integers satisfying (C1), and let $\underline{g}' = \underline{g}'(A)$ denote the complex Lie algebra generated by elements e_i, f_i, h_i $(i \in I)$ subject to the relations (1.2). Unless A is of finite type (i.e., satisfies (C2)), the algebra $\underline{g}'(A)$ will be infinite-dimensional. The h_i

are linearly independent in $\underline{g}'$, and generate a Cartan subalgebra $\underline{h}'$ of $\underline{g}'$. Following the pattern of the finite-dimensional case, we should define simple roots $\alpha_j \in \underline{h}'^*$ by the relations $\alpha_j(h_i) = a_{ij}$ $(i, j \in I)$. However, it may happen that $\det(A) = 0$, in which case the α_j so defined will be linearly dependent. This inconvenience may be avoided by enlarging $\underline{g}'$ as follows. Let $\underline{d}_o$ denote the space of derivations of $\underline{g}'$ generated by d_i $(i \in I)$, where $d_i(e_j) = \delta_{ij}e_j$, $d_i(f_j) = -\delta_{ij}f_j$; define $\phi : \underline{h}' \to \underline{d}_o$ by $\phi(h_i) = \mathrm{ad}\, h_i = \Sigma a_{ij}\, d_j$, and let $\underline{d}$ be any subspace of $\underline{d}_o$ supplementary to $\phi(\underline{h}')$ (so that $\dim \underline{d} = \mathrm{corank}\ A$). Up to isomorphism, the semidirect product $\underline{g} = \underline{g}' \oplus \underline{d}$ does not depend on the choice of $\underline{d}$, and is the Kac-Moody Lie algebra defined by the Cartan matrix A. The subspace $\underline{h} = \underline{h}' \oplus \underline{d}$ is a Cartan subalgebra of $\underline{g}$, and the centre $\underline{c}$ consists of all $h = \Sigma a_i h_i \in \underline{h}'$ such that $\sum_i a_i a_{ij} = 0$ for all $j \in I$, so that $\dim \underline{c} = \mathrm{corank}\ A$.

The matrix A is said to be indecomposable if there does not exist a partition of I into non-empty disjoint subsets J, K such that $a_{jk} = 0$ for all $(j, k) \in J \times K$; and symmetrizable if there exists a nonsingular diagonal matrix D such that DA is symmetric. Symmetrizability is a necessary and sufficient condition for the existence of a non-degenerate invariant symmetric bilinear form (x, y) on $\underline{g}$. We shall assume these conditions satisfied from now on, and that the bilinear form has been chosen so that (h_i, h_i) is real and > 0 for all $i \in I$. Just as in the finite-dimensional case, the restriction of (x, y) to $\underline{h}$ is nondegenerate, and hence determines a symmetric bilinear form (λ, μ) on $\underline{h}^*$.

After these preliminaries, all the features of the finite-dimensional case pointed out in §1 have their counterparts in this more general setting.

Root system. The root system R of $\underline{g}$ (relative to $\underline{h}$) is defined exactly as in §1, but is now an infinite subset of $\underline{h}^*$ (unless A is of finite type). The decomposition (1.1) of $\underline{g}$ remains valid, and each $\underline{g}_\alpha$ is finite-dimensional, but no longer necessarily of dimension 1, so that each root $\alpha \in R$ has a multiplicity $m(\alpha) = \dim \underline{g}_\alpha \geq 1$. The simple roots α_i $(i \in I)$ are defined by $[h, e_i] = \alpha_i(h)e_i$

for all $h \in \underline{h}$, so that $\alpha_j(h_i) = a_{ij}$; they are linearly independent and have multiplicity 1, and they form a basis of R in the sense of §1. So we have the notions of positive roots and negative roots (but no highest root).

For each root α, the number (α, α) is real, and is > 0 for each simple root α_i. However, it is no longer true that $(\alpha, \alpha) > 0$ for all roots α (unless A is of finite type). The roots $\alpha \in R$ for which $(\alpha, \alpha) > 0$ are called real roots, and have multiplicity 1; those for which $(\alpha, \alpha) \leq 0$ are called imaginary roots, and may have multiplicity > 1. If α is real, the only roots proportional to α are $\pm\alpha$; if α is imaginary, then $n\alpha$ is a root for all non zero integers n.

Weyl group. Just as before, the Weyl group W is defined to be the group of isometries of $\underline{h}^*$ generated by the reflections w_{α_i} $(i \in I)$, and contains w_α for each real root α. The set of real roots, the set of positive imaginary roots and the set of negative imaginary roots are each stable under W, and a root $\alpha \in R$ is real if and only if it is of the form $w\alpha_i$ for some $w \in W$ and $i \in I$.

Modules with highest weight. Define P, P_+, Q, Q_+ as in §1. For each $\lambda \in P_+$ there exists a unique simple (in general infinite-dimensional) $\underline{g}$-module $V(\lambda)$ generated by an element $v_\lambda \in V(\lambda)_\lambda$ such that $e_i v_\lambda = 0$ $(i \in I)$. All the weight spaces $V(\lambda)_\mu$ are finite-dimensional, and $V(\lambda)$ is their direct sum; the set of weights of $V(\lambda)$ is stable under W, and is contained in $\lambda - Q_+$. The highest weight λ has multiplicity 1. The character

$$\mathrm{ch}\, V(\lambda) = \sum_\mu \dim(V_\mu).e^\mu$$

summed over the weights of $V(\lambda)$ is now an infinite sum of formal exponentials, such that $e^{-\lambda}\mathrm{ch}\, V(\lambda)$ lies in the formal power series ring generated by the $e^{-\alpha_i}$ $(i \in I)$.

Define $\rho \in \underline{h}^*$ by $\rho(h_i) = 1$ $(i \in I)$, $\rho|\underline{d} = 0$. Then the character formula (of V Kac) states that, for all $\lambda \in P_+$,

$$\text{(2.1)} \qquad \mathrm{ch}\, V(\lambda) = \sum_{w \in W} \det(w) e^{w(\lambda+\rho)} / e^\rho \prod_{\alpha > 0} (1-e^{-\alpha})^{m(\alpha)} .$$

When $\lambda = 0$, $\operatorname{ch} V(\lambda) = 1$ and we have the denominator formula

$$\prod_{\alpha>0} (1-e^{-\alpha})^{m(\alpha)} = \sum_{w\in W} \det(w)e^{w\rho-\rho} \tag{2.2}$$

and hence also

$$\operatorname{ch} V(\lambda) = \Big(\sum_{w\in W} \det(w)e^{w(\lambda+\rho)}\Big)\Big/\Big(\sum_{w\in W} \det(w)e^{w\rho}\Big). \tag{2.3}$$

3. Affine Lie algebras [7,11,13,25]

The root system of a Kac-Moody Lie algebra is in general a rather elusive object, and at present no method is known for systematically listing the roots and their multiplicities. However, there is one class of infinite-dimensional Kac-Moody algebras, the affine Lie algebras, for which the root system can be explicitly described and hence the character formula (2.1) and denominator formula (2.2) exploited to produce explicit identities.

Affine Lie algebras can be characterised in various ways: as the infinite-dimensional Kac-Moody algebras for which all roots α satisfy $(\alpha, \alpha) \geq 0$, or equivalently as those defined by affine Cartan matrices, i.e., matrices A which satisfy (C1) and

(C2') $\det(A) = 0$, and all proper principal minors of A are > 0.

Let R be an irreducible finite root system, $\alpha_1, \ldots, \alpha_\ell$ a basis of R, and let $-\alpha_0$ be the highest root relative to this basis. Let $a_{ij} = (\alpha_i^\vee, \alpha_j)$. Then the matrix $\tilde{A} = (a_{ij})_{0 \leq i, j \leq \ell}$ is an affine Cartan matrix, and therefore so also is its transpose ${}^t\tilde{A}$; and these are all the affine Cartan matrices.

For $\tilde{A}$ as above, where R is the root system of a finite-dimensional simple Lie algebra $\underline{g}$ as in §1, the affine Lie algebra $\underline{g}(\tilde{A})$ may be constructed as follows. Let $L = \mathbb{C}[t, t^{-1}]$ be the ring of Laurent polynomials in one variable, and form $L(\underline{g}) = L \otimes_{\mathbb{C}} \underline{g}$. This is an infinite-dimensional Lie algebra, which may be identified with the Lie algebra of polynomial maps $\mathbb{C}^* \to \underline{g}$ (the element

$\Sigma t^i \otimes x_i$ of $L(\underline{g})$ corresponding to the mapping $z \to \Sigma z^i x_i$). For $x = \Sigma t^i \otimes x_i$, $y = \Sigma t^j \otimes y_j$, define $(x, y)_t = \sum_{i,j} t^{i+j}(x_i, y_j) \in L$, and let (x, y) denote the constant term in $(x, y)_t$. Then (x, y) is an invariant bilinear form on $L(\underline{g})$.

This algebra $L(\underline{g})$ is in fact isomorphic to $\underline{g}'(\tilde{A})$ modulo its (1-dimensional) centre $\underline{c}$. To construct $\underline{g}'(\tilde{A})$ from $L(\underline{g})$ we must therefore construct a 1-dimensional central extension, which we do as follows. The function on $L(\underline{g})$ $\psi(x, y) = \mathrm{Res}_t(dx/dt, y)_t$ may be verified to be a 2-cocycle with values in $\mathbb{C}$, hence determines a central extension $\tilde{L}(\underline{g})$ of $L(\underline{g})$, and $\tilde{L}(\underline{g})$ is isomorphic to $\underline{g}'(A)$. Explicitly, $\tilde{L}(\underline{g}) = L(\underline{g}) \oplus \underline{c}$ where $\underline{c} = \mathbb{C}c$, and the multiplication is given by $[x+\lambda c, y+\mu c] = [x, y]+\psi(x, y)c$ $(x, y \in L(g);\ \lambda, \mu \in \mathbb{C})$.

The affine Lie algebra $\underline{G} = \underline{g}(\tilde{A})$ is then obtained by adjoining to $\tilde{L}(\underline{g})$ a derivation d which acts on $L(\underline{g})$ as td/dt and which kills c. In other words, $\underline{G} \simeq \tilde{L}(\underline{g}) \oplus \underline{d}$, where $\underline{d} = \mathbb{C}d$ and the multiplication is given by $[x+\lambda d, y+\mu d] = [x, y]+\lambda dy-\mu dx$ $(x, y \in \tilde{L}(\underline{g});\ \lambda, \mu \in \mathbb{C})$.

Not all the affine Lie algebras are constructed in this way (about half of them are). The remainder are obtained by a variant of the construction above, in which one starts with a simple Lie algebra $\underline{g}$ as above and a graph automorphism σ of $\underline{g}$, of order say k (so that $k = 1, 2$ or 3). Let ω be a primitive kth root of unity, and for each $n \in \mathbb{Z}$ let $\underline{g}_n$ be the set of $x \in \underline{g}$ such that $\sigma(x) = \underline{\omega}^n x$ (so that $\underline{g}_n$ depends only on n modulo k). In place of $L(\underline{g})$ we form $L(\underline{g}, \sigma) = \bigoplus_{n\in\mathbb{Z}} t^n \otimes \underline{g}_n$, and the rest of the construction is unaltered. If $\underline{g}$ is of type X, where X is one of the symbols A_n, B_n, ..., G_2, the affine Lie algebra so constructed is said to be of type $X^{(k)}$.

For simplicity of exposition, we shall concentrate on the affine Lie algebras $\underline{G} = \underline{g}(\tilde{A})$, for which $k = 1$ (i.e., $L(\underline{g}, \sigma) = L(\underline{g})$).

Root system. It will be convenient to normalise the bilinear form (x, y) on $\underline{g}$ so that $|\alpha_o|^2 = 2$ (where $-\alpha_o$ is the highest root). We then define the standard invariant bilinear form (X, Y) on $\underline{G} = L(\underline{g}) \oplus \underline{c} \oplus \underline{d}$ as follows: if $X = x+\lambda c+\mu d$, $Y = y+\lambda' c+\mu' d$, $(x, y \in L(\underline{g});\ \lambda, \mu, \lambda', \mu' \in \mathbb{C})$, then

$$(X, Y) = (x, y)+\lambda\mu'+\lambda'\mu.$$

We shall identify $\underline{g}$ with the subalgebra $1 \otimes \underline{g}$ of G. Then $\underline{H} = \underline{h} \oplus \underline{c} \oplus \underline{d}$ is a Cartan subalgebra of $\underline{G}$. The restriction of (X, Y) to $\underline{H}$ is nondegenerate, hence defines an isomorphism $\omega : \underline{H} \to \underline{H}^*$ and a bilinear form (λ, μ) on $\underline{H}^*$. Each $\lambda \in \underline{h}^*$ we regard as a linear form on $\underline{H}$ by setting $\lambda(c) = \lambda(d) = 0$. Let $\gamma = \omega(d)$, $\delta = \omega(c)$. Then $\gamma(c) = \delta(d) = 1$ and $\gamma(d) = \delta(c) = 0$, and $\underline{H}^* = \underline{h}^* \oplus \mathbb{C}\gamma \oplus \mathbb{C}\delta$.

We have

$$L(\underline{g}) = \underline{h} + \sum_{n,\lambda} t^n \otimes \underline{g}_\lambda$$

where $n \in \mathbb{Z}$ and $\lambda \in R \cup \{0\}$, the pair $(0, 0)$ being excluded. Since d acts on $t^n \otimes \underline{g}_\lambda$ as multiplication by n, it follows that the roots of $\underline{G}$ relative to $\underline{H}$ are $n\delta+\alpha$ $(n \in \mathbb{Z}, \alpha \in R)$ and $n\delta$ $(n \in \mathbb{Z}, n \neq 0)$. We have $(n\delta+\alpha, n\delta+\alpha) = (\alpha, \alpha) > 0$, so that the $n\delta+\alpha$ are real roots, and the $n\delta$ $(n \neq 0)$ are imaginary roots (since $(\delta, \delta) = 0$), each of multiplicity ℓ.

Let S denote the root system of $\underline{G}$ relative to $\underline{H}$. The simple roots are $a_i = \alpha_i$ $(1 \le i \le \ell)$ and $a_o = \delta+\alpha_o$. Since $|\alpha_o|^2 = 2$ we have $a_o^\vee = \delta+\alpha_o^\vee$. We have $h_i = \omega^{-1}(a_i^\vee)$ $(1 \le i \le \ell)$, and we define $h_o = \omega^{-1}(a_o^\vee) \in \underline{H}$. The positive roots $a \in S^+$ are

$$(3.1) \qquad (n-1)\delta+\alpha, \quad n\delta-\alpha, \quad n\delta \qquad (n \ge 1,\ \alpha \in R^+).$$

Weyl group. Let $\widetilde{W}$ denote the Weyl group of $\underline{G}$, generated by the reflections w_{a_i} $(0 \le i \le \ell)$ in $\underline{H}^*$. Since $(a_i, \delta) = 0$ for all i, we see that $\widetilde{W}$ fixes

δ. We can realise $\widetilde{W}$ as an "affine Weyl group" as follows. First, the action of $\widetilde{W}$ on $\underline{H}^*$ may be transported to $\underline{H}$ by means of the isomorphism ω; it fixes c and hence acts on the real vector space $V = \underline{H}_{\mathbb{R}}/\mathbb{R}c$, where $\underline{H}_{\mathbb{R}}$ is generated by $h_1, \ldots, h_\ell$, c, d. Since $\delta(c) = 0$, δ may be regarded as a real linear form on V, and it is easily verified that each affine hyperplane δ = constant in V is stable under $\widetilde{W}$. In particular, let E denote the hyperplane $\delta = 1$ in V; the roots $a \in S$ may be regarded as linear functions on V, hence as affine-linear functions on E; and $\widetilde{W}$ acts faithfully on E as an affine Weyl group, generated by the reflections in the hyperplanes $a_i(x) = 0$ $(0 \leq i \leq \ell)$ in E, as in [23].

For each real root $a = n\delta+\alpha$, $w_{a-\delta} \circ w_a$ is the composition of reflections in two parellel hyperplanes in E, hence is a <u>translation</u>, namely $x \to x+\omega^{-1}(\alpha^\vee)$. The subgroup of $\widetilde{W}$ which fixes the point $x_o \in E$ defined by $a_i(x_o) = 0$ $(1 \leq i \leq \ell)$ may be identified with the (finite) Weyl group W of $\underline{g}$, and $\widetilde{W}$ is the semidirect product of W with the translation subgroup T, isomorphic to the lattice $M = \sum_{i=1}^{\ell} \mathbb{Z}\alpha_i^\vee$. For each $\mu \in M$ let $t_\mu \in T$ denote the corresponding translation, so that $t_\mu(x) = x+\omega^{-1}(\mu)$ on E. Then the action of t_μ on $\underline{H}^*$ is given by the formula

$$t_\mu(v) = v^o+n\mu+n\gamma+ \frac{1}{2n}(|v|^2 - |v^o+n\mu|^2)\delta \tag{3.2}$$

where $n = (v, \delta) \neq 0$ and v^o is the projection of $v \in \underline{H}^*$ on $\underline{h}^*$.

<u>Modules with highest weight</u>. Let $\widetilde{P}$ (resp. $\widetilde{P}_+$) denote the set of $\lambda \in \underline{H}^*$ such that $\lambda(h_i) \in \mathbb{Z}$ (resp. $\mathbb{N}$) for $0 \leq i \leq \ell$, and for each $\lambda \in \widetilde{P}_+$ let $V(\lambda)$ denote the simple $\underline{G}$-module with highest weight λ, as in §2. The set of weights of $V(\lambda)$ is a $\widetilde{W}$-stable subset of $\widetilde{P}$, and weights congruent under $\widetilde{W}$ have the same multiplicity. If μ is a weight, so is $\mu-n\delta$ for all integers $n \geq 0$, so that the weights are distributed into "strings".

A weight μ of $V(\lambda)$ is said to be <u>maximal</u> if $\mu+\delta$ is not a weight. For example, λ is a maximal weight. The set $\mathrm{Max}(\lambda)$ of maximal weights is stable under $\widetilde{W}$ (since $\widetilde{W}$ fixes δ) and is the union of a finite number of $\widetilde{W}$-orbits. For each weight μ there exists a unique integer $n \geq 0$ such that $\mu+n\delta$ is maximal.

The central element c acts on $V(\lambda)$ as multiplication by the positive integer $m = \lambda(c) = (\lambda, \delta)$, called the level of λ. We have $\mu(c) = m$ for all weights μ of $V(\lambda)$, and $m = 0$ if and only if $\lambda = 0$.

Character formula. In §2 we regarded the e^{λ} as formal exponentials. From now on we shall regard them as functions on $\underline{H}$, defined by $e^{\lambda}(h) = \exp\text{-}2\pi i\lambda(h)$.

Let $\underline{H}_+$ denote the set of $h \in \underline{H}$ such that $\delta(h)$ lies in the upper half-plane $\mathcal{H} = \{x+iy \in \mathbb{C} : y > 0\}$. For each $\lambda \in \widetilde{P}_+$, the character of $V(\lambda)$ and the series

$$J(\lambda + \rho) = \sum_{w\in\widetilde{W}} \det(w)e^{w(\lambda+\rho)}$$

converge absolutely for all $h \in \underline{H}_+$, and define holomorphic functions on this half-space. (Define $\rho \in \underline{H}^*$ by $(\rho, a_i) = 1$ $(0 \leq i \leq \ell)$, $\rho(d) = (\rho, \gamma) = 0$.) Then the character formula (2.3) takes the form

$$\text{ch } V(\lambda) = J(\lambda+\rho)/J(\rho) \tag{3.3}$$

as an identity between holomorphic functions on $\underline{H}_+$. The denominator formula (2.2) takes the form

$$J(\rho) = e^{\rho} \prod_{n=1}^{\infty} (1-q^n)^{\ell} \prod_{\alpha>0} (1-q^{n-1}e^{-\alpha})(1-q^n e^{\alpha}) \tag{3.4}$$

where $q = e^{-\delta}$, in view of the description (3.1) of the positive roots.

4. Specialisations of the denominator formula [13,15,19,20,23]

We recall that Dedekind's η-function is

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1-q^n)$$

where $q = \exp 2\pi i\tau$ and $\tau \in \mathcal{H}$.

Let σ be an automorphism of finite order m of the simple Lie algebra $\underline{g}$. Assume that the characteristic polynomial $\det(X-\sigma)$ has integer coefficients; then it is a product of cyclotomic polynomials, hence can be expressed uniquely in the form

$$\det(X-\sigma) = \prod_{i=1}^{r} (X^{m_i}-1)^{\varepsilon_i}$$

where each ε_i is ± 1, the m_i divide m, and $\Sigma\varepsilon_i m_i = \dim \underline{g}$. Each such σ leads to two η-function identities (which may coincide): the first gives a power series expansion of $\Pi\eta(m_i\tau)^{\varepsilon_i}$ and the second a power series expansion of $\Pi\eta(m_i^{-1}\tau)^{\varepsilon_i}$.

In the particular case that σ is an inner automorphism $\mathrm{Ad}(\exp 2\pi i h)$ where $h \in \underline{h}$, the first identity is obtained by evaluating both sides of the denominator formula (3.4) at $h+\tau d$, and the second by evaluation at $\tau h+\tau d$.

The simplest case is that in which σ is the identity. Then both identities coincide and express $\eta(\tau)^{\dim \underline{g}}$ as a power series in q. When $\underline{g} = \underline{sl}(2, \mathbb{C})$, this is the formula of Gauss and Jacobi for $\prod_{n=1}^{\infty} (1-q^n)^3$.

Another example is obtained by taking h to be the element of $\underline{h}$ defined by $\alpha_i(h) = 1$ $(1 \le i \le \ell)$. Then the characteristic polynomial of $\sigma = \mathrm{Ad}(\exp 2\pi i h)$ is $(X^m-1)^\ell$ (where $m = 1 +$ Coxeter number of $\underline{g}$); again the two identities coincide and express $\eta(\tau)^\ell$ as a power series in q. When $\underline{g} = \underline{sl}(2, \mathbb{C})$, this is Euler's expansion of $\Pi(1-q^n)$ ("pentagonal number theorem").

5. Theta functions and string functions

The numerator and denominator of the character formula (3.3) can be expressed as alternating sums of theta functions, by summing first over the translation subgroup T of $\tilde{W}$ and then over the finite Weyl group W. For each $\nu \in \tilde{P}$, let $\nu^o \in P$ be the projection of ν on $\underline{h}^*$. If $\nu \in \tilde{P}$ is such that $(\nu, \delta) = n > 0$, it follows from (3.2) that

$$\sum_{t\in T} e^{t(\nu)} = \exp \frac{|\nu|^2}{2n}\delta \cdot \Theta_{\nu^{o},n} \tag{5.1}$$

where for any $\mu \in P$ and any positive integer m

$$\Theta_{\mu,\, m} = \sum_{\phi\in M+m^{-1}\mu} e^{m(\gamma+\phi-\frac{1}{2}|\phi|^2\delta)} . \tag{5.2}$$

The series (5.1) and (5.2) converge absolutely on $\underline{H}_+$, and $\Theta_{\mu,\, m}$ is a classical theta function, which depends only on μ modulo mM. Using (5.1) we calculate that for $\lambda \in \widetilde{P}_+$

$$J(\lambda+\rho) = \sum_{w\in W} \det(w) \sum_{t\in T} e^{tw(\lambda+\rho)}$$

$$= \exp \frac{|\lambda+\rho|^2\delta}{2(m+g)} \sum_{w\in W} \det(w)\, \Theta_{w(\lambda+\rho)^{o},\, m+g}$$

where $m = \lambda(c)$ is the level of λ (and is a nonnegative integer) and $g = \rho(c)$. Hence we obtain

$$e^{-r_\lambda\delta} \operatorname{ch} V(\lambda) = \frac{\sum_{w\in W} \det(w)\, \Theta_{w(\lambda+\rho)^{o},\, m+g}}{\sum_{w\in W} \det(w)\, \Theta_{w\rho^{o},\, g}} \tag{5.3}$$

where

$$r_\lambda = \frac{|\lambda+\rho|^2}{2(m+g)} - \frac{|\rho|^2}{2g} .$$

(Incidentally, $|\rho|^2/2g = 1/\rho^\vee(c)$ is equal to $\frac{1}{24}\dim \underline{g}$ by the "strange formula".) The formula (5.3) resembles Weyl's character formula (1.5), except that the exponentials are replaced by theta-functions.

We shall next derive another expression for the character $\operatorname{ch} V(\lambda)$. If $m_\lambda(\mu) = \dim V(\lambda)_\mu$ is the multiplicity of the weight μ, we have

$$\operatorname{ch} V(\lambda) = \sum_{\mu\in \operatorname{Max}(\lambda)} e^{\mu} \sum_{n\geq 0} m_\lambda(\mu-n\delta)e^{-n\delta}$$

$$= \sum_{\substack{\mu\in \operatorname{Max}(\lambda) \\ \mu \bmod T}} \sum_{t\in T} e^{t(\mu)} \sum_{n\geq 0} m_\lambda(\mu-n\delta)e^{-n\delta}$$

since the weights $\mu-n\delta$ and $t(\mu-n\delta) = t(\mu)-n\delta$ have the same multiplicity. Using (5.1) we obtain

$$e^{-r_\lambda \delta} \operatorname{ch} V(\lambda) = \sum_{\substack{\mu \in \operatorname{Max}(\lambda) \\ \mu \bmod T}} c^\lambda_{\mu^o} \Theta_{\mu^o, m} \tag{5.4}$$

where

$$c^\lambda_{\mu^o} = e^{-r_\lambda(\mu)\delta} \sum_{n \geq 0} m_\lambda(\mu - n\delta) e^{-n\delta} \tag{5.5}$$

depends only on λ and μ^o, and

$$r_\lambda(\mu) = \frac{|\lambda+\rho|^2}{2(m+g)} - \frac{|\rho|^2}{2g} - \frac{|\mu|^2}{2m} .$$

The $c^\lambda_{\mu^o}$ are called <u>string functions</u>: they are functions of $\tau = \delta(h)$, holomorphic on the upper half-plane $\mathcal{H}$. If $\nu \in P$ is not the projection on $\underline{h}^*$ of any weight of $V(\lambda)$, set $c^\lambda_\nu = 0$. Then c^λ_ν is defined for all $\nu \in P$, and $c^\lambda_\nu = c^\lambda_{w(\nu)}$ for $w \in W \ltimes mM$. From (5.3) and (5.4) it follows that

$$\frac{\sum_{w \in W} \det(w)\, \Theta_{w(\lambda+\rho)^o, m+g}}{\sum_{w \in W} \det(w)\, \Theta_{\rho^o, g}} = \sum_{\nu \in P/mM} c^\lambda_\nu \Theta_{\nu, m} . \tag{5.6}$$

The theta-functions $\Theta_{\nu, m}$ on the right-hand side of (5.6) are linearly independent, and hence the string functions c^λ_ν are uniquely determined by (5.6). The transformation law for theta functions then gives rise to a transformation law for the string functions which expresses $\tau^{\ell/2} c^\lambda_\nu(-1/\tau)$ as a linear combination of the $c^{\lambda'}_{\nu'}(\tau)$, where λ' runs through the elements of level m in $\widetilde{P}_+$, and ν' through $P \bmod mM$. From this it follows that $\eta(\tau)^{\dim \underline{g}} c^\lambda_\mu(\tau)$ is a cusp form of weight $\frac{1}{2}|R|$ for the group $\Gamma(Nm) \cap \Gamma(N(m+g))$, where N is the least positive integer such that $\frac{1}{2}N|\mu|^2 \in \mathbb{Z}$ for all $\mu \in P$. These facts make it possible in principle to compute the string functions for any highest weight module $V(\lambda)$.

6. Examples

(a) The basic representation [7,16,17,18]. Kac and Peterson in [18] give many examples in which the string functions are explicitly determined (as linear combinations of products of η-functions). Here we shall consider only the simplest case. Suppose that all roots $\alpha \in R$ have the same length (so that R is of type A, D or E). Define the fundamental weights λ_i by $\lambda_i(h_j) = \delta_{ij}$, $\lambda_i(d) = 0$ (so that $\lambda_o = \gamma$). Then all $\lambda \in \tilde{P}_+$ of level $\lambda(c) = 1$ are conjugate to γ under automorphisms of the set S^+ of positive roots of $\underline{G}$. The maximal weights of the "basic representation" $V = V(\gamma)$ form a single orbit $\tilde{W}\cdot\gamma$, hence all string functions c_ν^λ for λ of level 1 are equal. This common string function $c(\tau)$ is of the form

$$c(\tau) = q^{-\ell/24} \sum_{n\geq 0} a_n q^n$$

where $q = e^{2\pi i\tau}$ and $a_o = 1$. Hence $\eta(\tau)^\ell c(\tau)$ is $SL_2(\mathbb{Z})$-invariant, holomorphic and equal to 1 at $i\infty$; hence is identically 1, i.e., $c(\tau) = \eta(\tau)^{-\ell}$. The character of V is therefore

$$\text{ch } V = \phi(e^{-\delta})^{-\ell} \sum_{\mu\in M} e^{\mu-\frac{1}{2}|\mu|^2\delta} \tag{6.1}$$

where $\phi(X) = \prod_{n=1}^{\infty} (1-X^n)$. The simplicity of this formula suggests that there should be a simple explicit construction of the $\underline{G}$-module V; such a construction has recently been found by Frenkel and Kac [7] and displays remarkable connections with certain notions (vertex operators, dual resonance models) of particle physics.

(b) Moonshine [3,15]. As in (a), let $\underline{g}$ be of type A, D or E and let V denote the basic representation of $\underline{G}$. For each integer $n \geq 0$ let V_n denote the subspace of V on which d acts as multiplication by $-n$, so that V_n is the sum of the weight spaces V_μ for which $\mu(d) = -n$. Each V_n is a finite-dimensional

$\underline{g}$-module, so that $V = \bigoplus_{n \geq 0} V_n$ may be regarded as a graded $\underline{g}$-module. From (6.1) it follows that the Poincaré series of V as graded $\underline{g}$-module is

$$\sum_{n\geq 0} \dim(V_n)q^n = (\mathrm{ch}\ V)(\tau d) = \phi(q)^{-\ell}\Theta(M, q)$$

where $q = e^{2\pi i\tau}$ and $\Theta(M, q) = \sum_{\mu\in M} q^{|\mu|^2/2}$ is the theta-series of the lattice M (which here is the root lattice Q, since $\alpha = \alpha^\vee$ for all $\alpha \in R$). In the case where $\underline{g}$ is of type E_8 we obtain

$$\sum_{n\geq 0} \dim(V_n)q^n = (qj(q))^{1/3}$$

where $j(q) = q^{-1}+744+196884q+ \ldots$ is the modular invariant, by known properties of j.

In [3] Conway and Norton conjecture that there should exist a graded module $H = \sum_{n\geq 0} H_n$ for the Monster simple group $\mathfrak{M}$ such that

$$q^{-1} \sum_{n\geq 0} \dim(H_n)q^n = j(q)$$

and more generally such that for each element $\sigma \in \mathfrak{M}$ the "Thompson series"

$$T_\sigma = q^{-1} \sum_{n\geq 0} \mathrm{trace}(\sigma, H_n)q^n$$

is the normalised generator of a function field of genus zero arising from a certain subgroup of $SL_2(\mathbb{Z})$. They also conjecture the following relationship between the Leech lattice L and the Monster group $\mathfrak{M}$. For each automorphism σ of L, let L^σ the sublattice of L fixed by σ and let $\Theta(L^\sigma, q)$ be its theta-series. Then there should exist an element $\sigma \in \mathfrak{M}$ such that

$$T_\sigma = q^{-1}\ \Theta(L^\sigma, q)/ \prod_{n=1}^{\infty} \det(1-q^n\sigma). \tag{6.2}$$

In [15] Kac shows that an analogue of (6.2), with the Leech lattice L replaced by the root lattice M, and the Conway group by the Weyl group W, is in fact true.

Let G be the simply-connected complex Lie group with $\underline{g}$ as Lie algebra. Then the action of $\underline{g}$ on each V_n can be "integrated" to a G-action. Each element $\sigma \in W$ of order m can be lifted to an element $\tilde{\sigma} \in G$ of order 2m, and Kac shows that

$$\sum_{n\geq 0} \operatorname{trace}(\tilde{\sigma}, V_n)q^n = \Theta(M^\sigma, q)/\prod_{n=1}^{\infty} \det(1-q^n\sigma)$$

in perfect analogy with the conjecture (6.2).

(c) Rogers-Ramanujan [21]. Let $r = \omega^{-1}(\rho) \in \underline{H}$, and let $\tau \in \mathcal{H}$ (the upper half plane). For any $\lambda \in \tilde{P}_+$ we have

$$J(\lambda+\rho)(\tau r) = \sum_{w\in\tilde{W}} \det(w)e^{-2\pi i(w(\lambda+\rho),\ \rho)\tau}$$

which by the denominator formula (2.2) factorises:

$$J(\lambda+\rho)(\tau r) = q^{-(\rho,\lambda+\rho)} \prod_{a\in S^+}(1-q^{(a,\ \lambda+\rho)})^{m(a)}$$

where $q = e^{2\pi i\tau}$. Hence

$$\text{(6.3)} \qquad \operatorname{ch} V(\lambda)(\tau r) = q^{-(\lambda,\ \rho)}\prod_{a\in S^+}\left(\frac{1-q^{(a,\ \lambda+\rho)}}{1-q^{(a,\ \rho)}}\right)^{m(a)}.$$

We shall apply this formula when $\underline{g} = \underline{sl}(2, \mathbb{C})$ and λ has level 3, so that (up to automorphisms) $\lambda = 3\lambda_o$ or $2\lambda_o + \lambda_1$. Since $\rho = \lambda_o + \lambda_1$ the product in (6.2) is easily calculated; we find

$$\text{(6.4)} \qquad e^{-\lambda}\operatorname{ch} V(\lambda)(\tau r) = \frac{1}{\prod_{n=1}^{\infty}(1-q^{2n-1})(1-q^{5n-2})(1-q^{5n-3})}$$

when $\lambda = 3\lambda_o$, and

$$\text{(6.5)} \qquad e^{-\lambda}\operatorname{ch} V(\lambda)(\tau r) = \frac{1}{\prod_{n=1}^{\infty}(1-q^{2n-1})(1-q^{5n-1})(1-q^{5n-4})}$$

when $\lambda = 2\lambda_o + \lambda_1$. Now, apart from the factor $\prod_{n=1}^{\infty}(1-q^{2n-1})^{-1}$, the right-hand sides of (6.4) and (6.5) feature in the Rogers-Ramanujan identities

$$\frac{1}{\prod_{n=1}^{\infty}(1-q^{5n-2})(1-q^{5n-3})} = \sum_{m=0}^{\infty} \frac{q^{m(m+1)}}{(1-q)\ \dots\ (1-q^m)}\ , \tag{6.6}$$

$$\frac{1}{\prod_{n=1}^{\infty}(1-q^{5n-1})(1-q^{5n-4})} = \sum_{m=0} \frac{q^{m^2}}{(1-q)\ \dots\ (1-q^m)}\ . \tag{6.7}$$

The product sides of various generalisations of these identities can also be interpreted in the same way, for suitable choices of the highest weight λ. This suggests that it should be possible to prove (6.6) and (6.7) (and their generalisations) by a construction and analysis of the modules $V(\lambda)$; and such a proof has recently been found by Lepowsky and Wilson [21].