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ASYMPTOTIC MORSE INEQUALITIES FOR ANALYTIC SHEAF COHOMOLOGY
[according to J.-P. Demailly]

by Yum-Tong SIU

Recently J.-P. Demailly [6,7] obtained some Morse inequalities estimating asymptotically the dimensions (and also their partial alternating sums) of cohomology groups of the tensor powers of a Hermitian holomorphic line bundle over a compact complex manifold in terms of some integrals involving the curvature of the line bundle as the power tends to infinity. When the complete alternating sum is used, one recovers the asymptotic case of the theorem of Riemann-Roch. These Morse inequalities give sufficient conditions for a compact complex manifold to be Moishezon. We will discuss these Morse inequalities, their background, Demailly's proof, a later probabilistic proof by Bismut, and the applications.

1. STATEMENT OF THE INEQUALITIES

Let X be a compact complex manifold of complex dimension n, F be a holomorphic vector bundle of rank r over X, and E be a Hermitian holomorphic line bundle over X. We denote by $c(E)$ the curvature form of E, which equals $\partial\bar{\partial}\varphi$ when the Hermitian metric of E is locally given by $e^{-\varphi}$, and denote by $X(q)$ the open subset of X consisting of all points x of X where $c(E)(x)$ has exactly q negative eigenvalues and $(n-q)$ positive eigenvalues. Let $X(\leq q) = X(0)\cup X(1)\cup\cdots\cup X(q)$. Demailly's result is the following.

THEOREM 1.1.– (*a*) (*Morse inequality*)

$$\dim H^q(X,E^k\otimes F) \leq r\,\frac{k^n}{n!}\int_{X(q)}(-1)^q\left(\frac{i}{2\pi}c(E)\right)^n + o(k^n)$$

(*b*) (*Strong Morse inequality*)

$$\Sigma_{j=0}^{q}(-1)^{q-j}\dim H^j(X,E^k\otimes F) \leq r\,\frac{k^n}{n!}\int_{X(\leq q)}(-1)^q\left(\frac{i}{2\pi}c(E)\right)^n + o(k^n).$$

(*c*) (*Asymptotic formula of Riemann-Roch*)

$$\Sigma_{j=0}^{n}(-1)^{j}\dim H^j(X,E^k\otimes F) = r\,\frac{k^n}{n!}\int_{X}\left(\frac{i}{2\pi}c(E)\right)^n + o(k^n).$$

Here $o(k^n)$ *is the Landau symbol denoting a term of order less than that of* k^n.

Before we present Demailly's proof we discuss first the background of his result.

2. BACKGROUND

The question of asymptotically estimating the dimensions of cohomology groups with coefficients in the tensor power of a fixed line bundle was motivated by a conjecture of Grauert and Riemenschneider [8]. Kodaira proved that a compact complex manifold is projective algebraic if and only if it admits a positive holomorphic line bundle. The conjecture of Grauert and Riemenschneider was an attempt to generalize Kodaira's theorem to Moishezon manifolds. A *Moishezon manifold* is a compact complex manifold with the property that the transcendence degree of its meromorphic function field equals its complex dimension. Moishezon showed that such manifolds are precisely those which can be transformed into a projective algebraic manifold by proper modifications. One similarly defines the concept of a Moishezon space. The conjecture of Grauert-Riemenschneider asserts that a compact complex space is Moishezon if there exists on it a torsion-free coherent analytic sheaf of rank one with a Hermitan metric whose curvature form is positive definite on an open dense subset. Here a Hermitian metric for a sheaf is defined by going to the linear space associated to the sheaf and the curvature form is defined only on the set of points where the sheaf is locally free and the space is regular. The conjecture is easily reduced to the following special case.

2.1. *A Special Case of the Conjecture of Grauert-Riemenschneider.* Let X be a compact complex manifold of complex dimension n. If X admits a Hermitian holomorphic line bundle E whose curvature form is positive definite on an open dense subset of X. Then X is Moishezon.

This conjecture was proved in [14,15]. The proof relies on the asymptotic dimension estimate that for any positive number ϵ one has $\dim H^q(X,E^k) \leq \epsilon k^n$ for $q \geq 1$ and for k sufficiently large. From this estimate and the theorem of Hirzebruch-Riemann-Roch one gets a positive constant c such that $\dim \Gamma(X,E^k) \geq ck^n$ for k sufficiently large. By taking quotients of elements of $\Gamma(X,E^k)$, one gets enough meromorphic functions to make X Moishezon. In [14,15] the asymptotic dimension estimate was obtained by imitating a familiar technique in analytic number theory of using the Schwarz lemma to prove the identical vanishing of a function by estimating its order and making it vanish to high order at a sufficient number of points. Such a technique applied to the holomorphic sections of a holomorphic line bundle was used by Serre [12] and also later by Siegel [13] to obtain an alternative proof of Thimm's theorem [17] that the transcendence degree of the meromorphic function field of a compact complex manifold cannot exceed its complex dimension. To get the asymptotic dimension estimate for sheaf cohomology groups one shows that the cocycle constructed in the natural way by solving $\bar\partial$ equations from a harmonic form must be identically zero if it vanishes on a lattice of points in the manifold which are, roughly speaking, spaced $(\lambda k)^{-1/2}$ distances apart along directions in which the curvature form of E has eigenvalue λ. In the oral presentation of [15] it was conjectured that there should be an asymptotic estimate of $\dim H^q(X,E^k)$ given by some integral expression of the curvature of E. Demailly's result gives such an asymptotic estimate and as a consequence gives better sufficient conditions than the conjecture of Grauert and Riemenschneider for a compact complex manifold to be Moishezon.

3. MAIN STEPS OF DEMAILLY'S PROOF

3.1. *Reduction to a problem of asymptotic eigenvalue distribution.* Let $\mathscr{A}^{0,q}(X,E^k\otimes F)$ be the group of all global $E^k\otimes F$-valued $(0,q)$-forms on X whose coefficients have locally L^2 first-order derivatives. Fix $\mu > 0$. Let $\mathscr{A}_\mu^{0,q}(X,E^k\otimes F)$ be the subgroup of $\mathscr{A}^{0,q}(X,E^k\otimes F)$ spanned by all eigenfunctions of the Laplace operator $\bar\partial\bar\partial^* + \bar\partial^*\bar\partial$ whose eigenvalues are $\le \mu$. On $\mathscr{A}^{0,q}(X,E^k\otimes F)$ let H be the projection operator onto the harmonic forms and G be the Green operator for $\bar\partial\bar\partial^* + \bar\partial^*\bar\partial$. Since G maps $\mathscr{A}_\mu^{0,q}(X,E^k\otimes F)$ to itself, it follows from $I - H = (\bar\partial^* G)\bar\partial + \bar\partial(\bar\partial^* G)$ that the q^{th} cohomology group of the complex $\left[\mathscr{A}_\mu^{0,\bullet}(X,E^k\otimes F),\ \bar\partial\right]$ is isomorphic to the group of all $E^k\otimes F$-valued harmonic $(0,q)$-forms on X and is therefore isomorphic to $H^q(X,E^k\otimes F)$. Let $\tilde N_k^q(\mu)$ be the complex dimension of $\mathscr{A}_\mu^{0,q}(X,E^k\otimes F)$, which is equal to the number of eigenvalues of $\bar\partial\bar\partial^* + \bar\partial^*\bar\partial$ on $\mathscr{A}^{0,q}(X,E^k\otimes F)$ that are $\le \mu$. By standard linear algebra arguments we have

$$\dim H^q(X,E^k\otimes F) \le \tilde{N}_k^q(\mu),$$
$$\Sigma_{\nu=0}^q(-1)^{\nu-q}\dim H^\nu(X,E^k\otimes F) \le \Sigma_{\nu=0}^q(-1)^{\nu-q}\,\tilde{N}_k^\nu(\mu).$$

The proof of Demailly's theorem is therefore reduced to estimating $\frac{1}{k^n}\tilde{N}_k^q(\mu)$ asymptotically as $k \to \infty$ and $\mu \to 0+$. One estimates $\tilde{N}_k^q(\mu)$ by a localization procedure. The use of the localization procedure requires that the two limits $k \to \infty$ and $\mu \to 0+$ be taken in the following special way. Let $N_k^q(\lambda) = \tilde{N}_k^q(k\lambda)$. One fixes λ and estimates $\frac{1}{k^n}N_k^q(\lambda)$ asymptotically as $k \to \infty$ and then lets $\lambda \to 0$.

The localization procedure originated with the work of H. Weyl. Weyl [18] introduced the localization procedure and the minimax principle to get the asymptotic estimate of the distribution of eigenvalues of linear partial differential equations. The intuition is that for the Dirichlet problem with zero boundary value, as the eigenvalue increases the nodal hypersurfaces (the zero-sets of the eigenfunctions) divide the domain into smaller and smaller subdomains and the Dirichlet problem looks more and more like the union of Dirichlet problems for all the subdomains.

Though Demailly attributed part of the motivation of his method to Witten's papers [19,20], his localization procedure is quite different from Witten's. On the other hand it bears a close resemblance to Weyl's original localization procedure.

We state first the result on asymptotic eigenvalue distribution obtained by the localization procedure and use it to finish the proof of Demailly's result. We will later prove the result of asymptotic eigenvalue distribution.

3.2. *Statement of result on asymptotic eigenvalue distibution.* Let M be a smooth compact Riemannian manifold of real dimension m, L be a Hermitian smooth line bundle over M with a Hermitian connection D, G be a Hermitian smooth vector bundle of rank t over M with a Hermitian connection ∇. Let ∇_k be the Hermitian connection on $L^k\otimes G$ induced by D and ∇. Let S be a smooth section of $\Lambda^1T^*M\otimes_{\mathbb{R}}\mathrm{Hom}_{\mathbb{C}}(G,G)$ and V be a global Hermitian endomorphism of G.

For a global section u of $L^k \otimes G$ consider the quadratic form

$$Q_k(u) = \int_M \left[\frac{1}{k}|\nabla_k u + Su|^2 - \langle Vu, u\rangle\right].$$

Let $N_{M,k}(\lambda)$ be the number of eigenvalues of the quadratic form $Q_k(\cdot)$ which are $\leq \lambda$. For $a \in M$ Let $V_1(a) \leq V_2(a) \leq \cdots \leq V_t(a)$ be the eigenvalues of $V(a)$. Let B be the curvature form of the connection D. For $a \in X$ write $B(a) = \Sigma_{j=1}^{s} B_j(a)dx_j \wedge dx_{j+s}$ where $(x_1, \cdots, x_m)$ is a normal coordinate system at a and $B_1(a) \geq B_2(a) \geq \cdots \geq B_s(a) > 0$. Let

$$\nu_B(\lambda) = \frac{2^{s-m}\pi^{-m/2}}{\Gamma(\frac{m}{2}-s+1)} B_1 \cdots B_s \sum_{(p_1,\cdots,p_s)\in\mathbb{N}^s} (\lambda - \Sigma_{j=1}^{s}(2p_j+1)B_j)_+^{\frac{m}{2}-s}.$$

Here λ_+ means the maximum of λ and 0 except when we are raising λ_+ to the zeroth power, in which case λ_+^0 means 0 if $\lambda \leq 0$ and means 1 if $\lambda > 0$. Let $\bar{\nu}_B(\lambda) = \lim_{0<\epsilon\to 0} \nu_B(\lambda+\epsilon)$. Then one has the following result on asymptotic eigenvalue distribtuion by the localization procedure.

THEOREM 3.2.1.- *(a)* $\limsup_{k\to\infty} k^{-m/2} N_{M,k}(\lambda) \leq \Sigma_{j=1}^{t} \int_M \bar{\nu}_B(V_j+\lambda)$.

(b) $\liminf_{k\to\infty} k^{-m/2} N_{M,k}(\lambda) \geq \Sigma_{j=1}^{t} \int_M \nu_B(V_j+\lambda)$.

We will prove this theorem later. We now apply this estimate to the case where $M = X$, $L = E$, G is the tensor product of F and the bundle of $(0,q)$-forms $\overline{\Lambda^q T^* X}$ of X, and $Q_k(u) = \frac{2}{k}\int_X \langle(\bar\partial\bar\partial^* + \bar\partial^*\bar\partial)u, u\rangle$. We have to relate the operator $\bar\partial\bar\partial^* + \bar\partial^*\bar\partial$ to the curvature of E and F and this is done through the formula of Weitzenbock, Bochner, and Kodaira.

3.3. *Formulae of Weitzenbock, Bochner, and Kodaira.* Let $\Omega = \Omega_{i\bar{j}} dz^i \wedge \overline{dz^j}$ be the curvature form of a holomorphic vector bundle W over X which is a $\mathrm{Hom}(W,W)$-valued $(1,1)$-form, where $(z^1, \cdots, z^n)$ is a local coordinate system and the summation convention is used. Consider first the case when X admits a

Kähler metric $g_{i\bar{j}}$. Let $R_{i\bar{j}}$ denote the Ricci curvature of the Kähler metric. Then for any W-valued (0,q)-form φ we have the following formulae of Weitzenbock, Bochner, and Kodaira:

$$(\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial})\varphi = -g^{i\bar{j}}\nabla_i\nabla_{\bar{j}}\varphi + \Omega\varphi + R\varphi,$$

$$(\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial})\varphi = -g^{i\bar{j}}\nabla_{\bar{j}}\nabla_i\varphi + \Omega\varphi - (\mathrm{tr}\Omega)\varphi.$$

Here ∇ denotes the covariant differential operator and

$$(\Omega\varphi)_{\bar{j}_1\cdots\bar{j}_q} = \Sigma_{\nu=1}^{q}\,\Omega^{\bar{\ell}}{}_{\bar{j}_\nu}\varphi_{\bar{j}_1\cdots(\bar{\ell})_\nu\cdots\bar{j}_q},$$

$$(R\varphi)_{\bar{j}_1\cdots\bar{j}_q} = \Sigma_{\nu=1}^{q}\,R^{\bar{\ell}}{}_{\bar{j}_\nu}\varphi_{\bar{j}_1\cdots(\bar{\ell})_\nu\cdots\bar{j}_q}.$$

$$\mathrm{tr}\Omega = g^{i\bar{j}}\Omega_{i\bar{j}}.$$

(The usual rule of raising indices is used. The notation $(\bar{\ell})_\nu$ means that the index in the ν^{th} position is replaced by $\bar{\ell}$, and $\varphi_{\bar{j}_1\cdots\bar{j}_q}$ denotes the components of φ with respect to the local coordinates $z^1,\cdots,z^n$.)

For the case $W = E^k\otimes F$ one chooses normal local coordinates at the point under consideration so that the curvature form of E is diagonalized and its eigenvalues are α_j $(1 \le j \le n)$. For $I = \{i_1,\cdots,i_\ell\}$ let $\alpha_I = \Sigma_{\nu=1}^{\ell}\,\alpha_{i_\nu}$ and let CI denote the complement of the set I in $\{1,\cdots,n\}$. Then we have

$$\frac{2}{k}\int_X\langle(\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial})\varphi,\varphi\rangle = \frac{1}{k}\int_X|\nabla\varphi|^2 + \int_X\Sigma_J(\alpha_J - \alpha_{CJ})\,|\varphi_{\bar{J}}|^2 + O\left[\frac{1}{k}\int_X|\varphi|^2\right].$$

Here $|\nabla\varphi|^2$ is the pointwise L^2 norm of the (covariant) first-order derivatives of φ in *both the (1,0) and the (0,1) directions* and the notation $O\left[\frac{1}{k}\int_X|\varphi|^2\right]$ means a term whose absolute value is dominated by $\frac{C}{k}\int_X|\varphi|^2$ with a constant C independent of k and φ. The summation Σ_J is over all $J = (j_1,\cdots,j_q)$ with $1 \le j_1 <\cdots< j_q \le n$. For the case of a non-Kähler Hermitain metric we have

$$\frac{2}{k}\int_X \langle(\bar\partial\bar\partial^* + \bar\partial^*\bar\partial)\varphi,\varphi\rangle = \frac{1}{k}\int_X |\nabla\varphi + S\varphi|^2 + \int_X \Sigma(\alpha_J - \alpha_{CJ})|\varphi_J|^2 + O\left(\frac{1}{k}\int_X |\varphi|^2\right).$$

Here S is a section of $\Lambda^1 T^* X \otimes_{\mathbb{R}} \mathrm{Hom}_{\mathbb{C}}(F,F)$ coming from the torsion of the Hermitan metric and is independent of k. The other error contributions from the torsion of the Hermitian metric are absorbed in the term $O\left(\frac{1}{k}\int_X |\varphi|^2\right)$. Now we apply Theorem 3.2.1 to the quadratic form $\frac{2}{k}\int_X \langle(\bar\partial\bar\partial^* + \bar\partial^*\bar\partial)\varphi,\varphi\rangle$ (with the obvious modifications due to the term $O\left(\frac{1}{k}\int_X |\varphi|^2\right)$).

The rank t of $F\otimes(\overline{\Lambda^q T^* X})$ equals r times the binormial coefficient $\binom{n}{q}$. The set $\{V_i\}_{i=1}^t$ is equal to the set $\{\alpha_{CJ} - \alpha_J\}$ repeated r times, where J runs through the set of all $(j_1,\cdots,j_q)$ with $1 \le j_1 < \cdots < j_q \le n$. We have $\{B_j\} = \{\text{nonzero } |\alpha_j|\}$. In the following formula for asymptotic eigenvalue distribution

$$\nu_B(\lambda) = \frac{2^{s-2n}\pi^{-n}}{\Gamma(n-s+1)} B_1\cdots B_s \sum_{(p_1,\cdots,p_s)\in\mathbb{N}^s} (\lambda - \Sigma_{j=1}^s (2p_j+1)B_j)_+^{n-s},$$

at any given point when we compute $\nu_B(V_i+\lambda)$ for a sufficiently small positive number λ, the only nonzero term we can possibly pick up from the sum is the term with all the p_j's equal to 0. Even then the only possible nonzero contributions must come from the case where

$$\alpha_{CJ} - \alpha_J = \Sigma_{j=1}^n |\alpha_j|,$$

which means that α_j is nonpositive for $j \in J$ and α_j is nonnegative for $j \in CJ$. . We have a nonzero factor $B_1\cdots B_s$ only when $s = n$ which means that all the α_j's must be nonzero to give a nonzero contribution. In that case $B_1\cdots B_s = \Pi_{i=1}^n |\alpha_i|$ and the factor $\frac{2^{s-2n}\pi^{-n}}{\Gamma(n-s+1)}$ becomes $\frac{1}{(2\pi)^n}$. At a given point of X where all the α_j's are nonzero, out of the t numbers $\nu_B(V_i+\lambda)$ there are only r nonozero ones (when the positive number λ is sufficiently small) and they are all identical, because at that point there is only one J with the property that $\alpha_j < 0$ for $j \in J$ and $\alpha_j > 0$ for $j \in CJ$. Thus

$$\limsup_{\lambda \to 0+} \Sigma_{j=1}^{t} \int_X \bar{\nu}_B(V_j+\lambda)d\sigma = \frac{r}{n!} \int_{X(q)} (-1)^q (\tfrac{i}{2\pi} c(E))^n,$$

$$\liminf_{\lambda \to 0+} \Sigma_{j=1}^{t} \int_X \nu_B(V_j+\lambda)d\sigma = \frac{r}{n!} \int_{X(q)} (-1)^q (\tfrac{i}{2\pi} c(E))^n.$$

Demailly's result now follows from these two limits and Theorem 3.2.1.

4. PROOF OF THE ASYMPTOTIC EIGENVALUE DISTRIBUTION

4.1. *The minimax principle*. The p^{th} eigenvalue λ_p for a quadratic form $Q(\cdot)$ is the minimum, over all p-dimensional vector subspaces F, of the maximum of $Q(f)$ over all elements f of unit length in F. Moreover, λ_p is also equal to the maximum, over all $(p-1)$-codimensional vector subspaces G, of the minimum of $Q(g)$ over all elements g of unit length in G.

4.2. *The case of a cube and constant curvature and potential*. On the cube

$$P(R) = \{(x_1, \cdots, x_m) \in \mathbb{R}^m \mid |x_j| < \tfrac{R}{2}, \ 1 \le j \le m\}$$

consider the quadratic form

$$Q_{P(R)} = \int_{P(R)} \left[\Sigma_{1 \le j \le s} \left(\left|\frac{\partial u}{\partial x_j}\right|^2 + \left|\frac{\partial u}{\partial x_{j+s}} + iB_j x_j u\right|^2 \right) + \Sigma_{j > 2s} \left|\frac{\partial u}{\partial x_j}\right|^2\right].$$

Let $N_{Q_{P(R)}}(\lambda)$ be the number of eigenvalues of $Q_{P(R)}$ that are $\le \lambda$ for the Dirichlet problem with zero boundary value. Let

$$\nu_B(\lambda) = \frac{2^{s-m}\pi^{-m/2}}{\Gamma(\frac{m}{2}-s+1)} B_1 \cdots B_s \sum_{(p_1, \cdots, p_s) \in \mathbb{N}^s} (\lambda - \Sigma_{j=1}^{s}(2p_j+1)B_j)_+^{\frac{m}{2}-s}.$$

LEMMA 4.2.1.- $\quad \lim_{R \to \infty} R^{-n} N_{Q_{P(R)}}(\lambda) = \nu_B(\lambda).$

A change of scale and a translation of the eigenvalues gives us the following more general case with a constant potential V. Let V be a real number and k be a positive integer. Let

$$Q_{P(R),k} = \frac{1}{k}\int_{P(R)}\left[\Sigma_{1\le j\le s}\left(\left|\frac{\partial u}{\partial x_j}\right|^2+\left|\frac{\partial u}{\partial x_{j+s}}+ikB_jx_ju\right|^2\right) + \Sigma_{j>2s}\left|\frac{\partial u}{\partial x_j}\right|^2 + V|u|^2\right].$$

Then

$$\lim_{k\to\infty} k^{-n/2}N_{Q_{P(R),k}}(\lambda) = \nu_B(V+\lambda),$$

where $N_{Q_{P(R),k}}(\lambda)$ is the number of eigenvalues of $Q_{P(R),k}$ that are $\le \lambda$ for the Dirichlet problem with zero boundary value. To prove Lemma 4.2.1 one uses the method of separation of variables by Fourier series and uses comparison with the solution to the classical problem of the harmonic oscillator. For $1 \le j \le s$ and $p_j \in \mathbb{N}$ let $\psi_{p_j,\ell_j}(x_j)$ be the p_j^{th} eigenfunction of the quadratic form

$$q(f) = \int_{\mathbb{R}}\left[\left|\frac{\partial f}{\partial x_j}\right|^2 + \left(\frac{2\pi}{R}\ell_j + B_jx_j\right)^2|f|^2\right]dx_j$$

for f with compact support in $(-\frac{R}{2}, \frac{R}{2})$ and let λ_{p_j,ℓ_j} be its eigenvalue. We compare this with the following two eigenvalue problems of the harmonic oscillator with explicit known solutions: (i) the same quadratic form $q(f)$ but with no support condition on the function f on $\mathbb{R}$ (see e.g. [2]); (ii) the quadratic form

$$\int_{|x_j|\le R/2}\left[\left|\frac{\partial f}{\partial x_j}\right|^2 + \left(\frac{2\pi}{R}|\ell_j| - \frac{R}{2}B_j\right)_+^2|f|^2\right]dx_j$$

with zero boundary value for f. We conclude from this comparison that the eigenvalue λ_{p_j,ℓ_j} is strictly bounded from below by the maximum of the eigenvalues $(2p_j+1)B_j$ and $\frac{4\pi^2}{R^2}\left[\left[\frac{p_j+1}{2}\right]^2 + \left(|\ell_j| - \frac{B_jR^2}{4\pi}\right)_+^2\right]$ of the above two eigenvalue problems of the harmonic oscillator. Moreover, by using a cut-off function on the interval $(-\frac{R}{2},\frac{R}{2})$ and comparison with the eigenvalue problem (i) of the harmonic oscillator, we conclude that for every $p \in \mathbb{N}^s$ there exists a nonnegative constant C depending on p and B such that $\lambda_{p,\ell} \leq (1+\frac{C}{R})\Sigma_{j=1}^s(2p_j+1)B_j$ when $|\ell_j| \leq \frac{B_jR^2}{4\pi}(1-R^{-1/2})$, $1 \leq j \leq s$. Now write

$$u(x) = R^{-(m-s)/2} \sum_{(p,\ell)\in\mathbb{N}^s\times\mathbb{Z}^{m-s}} u_{p,\ell}\, \psi_{p,\ell}(x')\exp(\tfrac{2\pi i}{R}\ell\cdot x'')$$

with $u_{p,\ell} \in \mathbb{C}$ and $\psi_{p,\ell} = \Pi_{1\leq j\leq s}\, \psi_{p_j,\ell_j}(x_j)$, where $x' = (x_1,\cdots,x_s)$, $x'' = (x_{s+1},\cdots,x_m)$, and $\ell\cdot x'' = \ell_1x_{s+1} +\cdots+ \ell_{m-s}x_m$. The condition that u satisfies the zero boundary value condition is equivalent to the equations

$$(*) \qquad \sum_{\ell_j\in\mathbb{Z}} (-1)^{\ell_j} u_{p,\ell} = 0,$$

for all $1 \leq j \leq s$ and all $p,\ell_1,\cdots,\ell_{j-1},\ell_{j+1},\cdots,\ell_s$. The L^2 norm of u over $P(R)$ is $\Sigma\, |u_{p,\ell}|^2$ and

$$Q_{P(R)}(u) = \Sigma\, (\lambda_{p,\ell} + \frac{4\pi^2}{R^2}\,|\ell''|^2)\, |u_{p,\ell}|^2,$$

where $\lambda_{p,\ell} = \Sigma_{1\leq j\leq s}\, \lambda_{p_j,\ell_j}$ and $\ell'' = (\ell_1,\cdots,\ell_{m-s})$. By the minimax principle $N_{Q_{P(R)}}(\lambda)$ is dominated by the number of (p,ℓ) in $\mathbb{N}^s\times\mathbb{Z}^{m-s}$ with $\lambda_{p,\ell} + \frac{4\pi^2}{R^2}\cdot |\ell''|^2 \leq \lambda$. From the lower bound of λ_{p_j,ℓ_j} obtained above by comparison with the two eigenvalue problems of the harmonic oscillator and from a simple estimate of the number of integral points inside a ball we obtain $\limsup_{R\to\infty} R^{-m}N_{Q_{P(R)}}(\lambda) \leq \nu_B(\lambda)$. Using the upper bound of $\lambda_{p,\ell}$ obtained above by comparison with the eigenvalue problem (i) of the harmonic oscillator and using the fact that the number of equations in $(*)$ is of an order no higher than that of R^{m-1}, we conclude that $\liminf_{R\to\infty} R^{-m}N_{Q_{P(R)}}(\lambda) \geq \nu_B(\lambda)$.

4.3. *The general manifold case.* To get the asymptotic eigenvalue distribution for the general manifold case, the main tool is the following two observations. Let Ω be an open subset of M. In a way analogus to the definition of $N_{M,k}(\lambda)$, we define $N_{\Omega,k}(\lambda)$ to be the number of eigenvalues $\leq \lambda$ for the Dirichlet problem on Ω with zero boundary value. If $\Omega_1,\cdots,\Omega_N$ are mutually disjoint open subsets of Ω, then $N_{\Omega,k}(\lambda) \geq \Sigma_{j=1}^N N_{\Omega_j,k}(\lambda)$. If $(\Omega_j')_{1\leq j\leq N}$ covers $\bar\Omega$ and ψ_j is a smooth function with compact support in Ω_j and if $\Sigma_{j=1}^N \psi_j^2 = 1$, then $N_{\Omega,k}(\lambda) \leq \Sigma_{j=1}^N N_{\Omega_j',k}(\lambda+\frac{C}{k})$, where C is $\sup_\Omega \Sigma_{j=1}^N |d\psi_j|^2$.

First one applies these observations to reduce the general case to the case of a bounded smooth subdomain Ω of $\mathbb{R}^m$ with the bundles differentially trivial over Ω. Then on each Ω we consider two kinds of cubes centered at $k^{-1/3}\alpha$ with $\alpha \in \mathbb{Z}^m$: a smaller kind whose side has length $k^{-1/3}$ and a larger kind whose side has length $k^{-1/3} + k^{-11/24}$. The smaller kind of cubes that are inside Ω are used as the mutually disjoint open subsets. The larger kind of cubes that intersect $\bar\Omega$ are used as a cover of $\bar\Omega$. For the asymptotic eigenvalue distribution of each cube we compare the original problem with the one obtained as follows. The original Hermian metric is replace by a constant Hermitian metric which is equal to the original Hermitian metric at the center. The original connection is replaced by a connection whose curvature is constant and equal to the original curvature at the center. The original potential V is replaced by the constant potential which is equal to the original potential at the center. By applying the above observations to the cubes and using the comparison to the local case with constant curvature and potential, we get the asymptotic eigenvalue distribution for a general manifold.

5. BISMUT'S HEAT EQUATION PROOF.

Bismut [5] later gave a probabilistic proof of Demailly's result using the heat equation. We very briefly discuss his proof. We use the same notations as in Demailly's proof. Let $C_q(E^k\otimes F)$ be the set of smooth global sections of $\Lambda^q T^* X\otimes E^k\otimes F$ over X. Let $\square^k$ be the operator $\bar\partial\bar\partial^* + \bar\partial^*\bar\partial$ on the complex

$C_\bullet(E^k \otimes F)$. Set $t = \frac{1}{k}$. For $s > 0$ the operator $e^{-st\Box^k}$ on $C_\bullet(E^k \otimes F)$ has a smooth kernel P_s^k. Let $\mathrm{Tr}_q e^{-st\Box^k}$ be the trace of $e^{-st\Box^k}$ on $C_q(E^k \otimes F)$. Then $\mathrm{Tr}_q e^{-st\Box^k}$ equals $\int_X \mathrm{Tr}_q(P_s^k(x,x))dx$ and

$$\Sigma_{j=0}^q (-1)^{j-q} \dim H^j(X, E^k \otimes F) \le \Sigma_{j=0}^q (-1)^{j-q}\, \mathrm{Tr}_j e^{-st\Box^k}.$$

This step corresponds to the step in 3.1. The curvature form of E can be naturally regarded as an endomorphism of the bundle of $(0,q)$-forms and we denote this endomorphism by θ. The next step is to prove that

$$\lim_{k\to\infty} \left(\frac{2\pi}{k}\right)^n \mathrm{Tr}_q(P_s^k(x,x)) = r\left[\frac{\det(\theta)\mathrm{Tr}_q(e^{s\theta})}{\det(I-e^{-s\theta})}\right](x).$$

This is the most difficult step. It corresponds to the asymptotic eigenvalue distribution of Section 4. For this step the method of Brownian motion, Ito's formula and Bismut's earlier work [3,4] on the asymptotic representations of $P_s^k(x,x)$ are used. The final step is

$$\lim_{s\to\infty} \frac{\det \theta}{\det(I-e^{-s\theta})} (\mathrm{Tr}_q e^{s\theta})(x) = 1_{X(q)}\ (-1)^q(\det \theta),$$

where $1_{X(q)}$ is the characteristic function of $X(q)$. This step corresponds to the step of letting $\lambda \to 0$ in Demailly's proof. Demailly's result now follows from integration over X. Bismut's method gives also asymptotic bounds for the dimension of the kernel of Dirac operators in both the even and odd dimension cases.

6. APPLICATIONS.

6.1. Demailly's result gives the following result on Moishezon manifolds that is better than the conjecture of Grauert and Riemenschneider proved in [14,15], where only condition (c) is given.

THEOREM 6.1.1.- *Let* X *be a compact complex manifold of complex dimension* n. *For* X *to be Moishezon, it suffices to assume that* X *admits a Hermitian* *holomorphic line bundle satisfying one of the following conditions:*

(a) $\int_{X(\leq 1)} (ic(E))^n > 0$.

(b) $c_1(E)^n > 0$ *and the curvature form* $c(E)$ *does not have precisely a positive even number of negative eigenvalues at any point of* X.

(c) $c(E)$ *is semipositive everywhere and positive at one point of* X.

6.2. Another application of Demailly's result is the following integral inequalities involving the Monge-Ampère operator. Let Ω be a relatively compact smooth subdomain of a Stein manifold M of complex dimension n such that the complex Hessian of a defining function for the boundary of Ω has at least $n - r + 1$ nonnegative eigenvalues at every boundary point. Let φ be a smooth real-valued function on the closure of Ω such that the complex Hessian H_φ of φ has at least $n - p + 1$ nonnegative eigenvalues near the boundary of Ω. Then for $q \geq p + r - 2$ the integral of $(-1)^q \det H_\varphi$ over Ω is dominated by its integral over the set of points where H_φ is nonsingular and has no more than q negative eigenvalues, where det denotes the determinant. There is also a real analog in which M is replaced by $\mathbb{R}^n$ and the complex Hessian is replaced by the real Hessian.

For the proof one applies Demailly's result to a compact n-dimensional complex manifold with a Hermitian line bundle such that the curvature form θ of the line bundle has at least $n - q$ positive eigenvalues outside a subdomain D and $\theta|D$ equals $\partial\bar{\partial}\psi$ for some smooth real-valued function ψ on D. One constructs a sequence of such pairs (D, ψ) approaching the given pair (Ω, φ). The real analog is obtained via the correspondence between the complex and real Hessians of functions on a Reinhardt domain and its associated real domain.

Though the formulation of these inequalities are so elementary, yet except in the case $q = 0$ so far there is no way to prove these inequalities by the method of integration by parts. For the case $q = 0$ A. Taylor has the following proof. One solves the Monge-Ampère equation for a plurisubharmonic function f with boundary value equal to that of φ so that $(i\partial\bar{\partial}f)^n$ equals $(i\partial\bar{\partial}\varphi)^n$ times the characteristic function of the set of points where H_φ is positive definite. One then uses the inequality $f \geq \varphi$ proved in [11] and apply Stokes' theorem.

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