Richard Becker

Some consequences of a kind of Hahn-Banach’s theorem


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SOME CONSEQUENCES OF A KIND OF HAHN-BANACH'S THEOREM

by Richard BECKER

Abstract. - The aim of this work is to give some consequences of a theorem of H. DINGES used by M. F. SAINTE-BEUVE.

Preliminaries

1. THEOREM. - Let $X$ be an ordered vector space, and $p$ an extended sub-linear functional on $X$, such that $p(x) \in \mathbb{R} \cup \{+\infty\}$ for each $x \in X$, and $p(x) \leq 0$ for each $x \leq 0$. Let $Y$ a linear subspace of $X$, and $f$ a linear form on $Y$ majorized by $p$. There exists a linear form on $Z = \{x ; \exists x_1, x_2 \in Y \text{ with } x_1 < x \leq x_2\}$ which extends $f$ and is majorized by $p$ on $Z$ ([5], [11]).

What is needed concerning conical measures can be found in [3] (§ 30, 38, 40). Notation not included in [3]. In this paper, $C$ will be the class of weakly complete convex cones, not necessarily proper.

2. Summary. - Part I is devoted to conical measures. We generalize specially (proposition 12) the theorem of Cartier-Fell-Meyer ([10] p. 112) concerning dilations of measures on a metrizable convex compact set. Positive measures on a metrizable convex compact set can be considered as conical measures on a proper convex closed cone of $\mathbb{R}^N$. Here, we will consider arbitrary conical measures on $\mathbb{R}^N$.

Part II (A) extends a result of STRASSSEN ([8], p. 300-301), from which the theorem of Cartier-Fell-Meyer can be derived. We weaken, here, a condition of compactness (proposition 21). Part II (B) extends some results about "theory of balayage" ([8], p. 294, 297). This theory studies cones of continuous functions on a compact set containing a strictly positive function. We weaken this condition.

Part I : The case of conical measures.

I (A). Conical measures on an arbitrary weak space.

Recall the following proposition which enlightens the definition of the order $<$.  

3. PROPOSITION. - Let $E$ a complete weak space, and $\Gamma \subseteq E$ a convex cone of $C$. For each $f \in h(E)$, such that $f|\Gamma$ is sub-linear, there exist $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{R}$, such that we have on $\Gamma$, $f = \text{lub}(\lambda_1, \lambda_2, \ldots, \lambda_n)$.

Proof. - We can suppose $E$ of finite dimension.
There exist \( u_1, \ldots, u_p \in E' \) such that, for each \( x \in E \), \( f(x) \) is equal to one of the \( u_p(x) \). Hence, for each pair \( x, y \in \Gamma \), there exist \( p_{x,y} \), an integer \( \leq p \), such that

\[
f(x) = u_{p_{x,y}}(x), \quad \text{and} \quad f(y) \geq u_{p_{x,y}}(y).
\]

For each \( x \in \Gamma \), let \( v_x = g_{\lambda \in \mathbb{R}} u_{p_{x,y}} \). The family \( (v_x)_{x \in \Gamma} \) is finite, and we have on \( \Gamma f = \inf_{x \in \Gamma} (-v_x) \); as \( -v_x \in S(E) \), we can conclude with the help of the elementary form of the theorem of Hahn-Banach because dimension of \( E \leq \infty \).

4. PROPOSITION. - If \( E \) is a complete weak space, and \( \mu \in \mathbb{M}^+(E) \), then, for each \( \lambda \in \mathcal{E}' \) with \( \lambda \neq 0 \), the two following properties are equivalent.

1° \( \forall f \in \mathcal{H}^+(E) \), we have \( \mu(f) = \lim (\mu(f \wedge n\lambda^+)) \) when \( n \to \infty \).

2° \( \exists m, \sigma \)-additive and positive functional on the tribe on \( e = \mathcal{A}^1(1) \) generated by \( h(E)|_e \), such that \( \mu(f) = m(f|_e) \), for each \( f \in h(E) \).

If \( \mu \) satisfies to 1° and 2°, then each \( \lambda \in \mathbb{M}^+(E) \), with \( \lambda < \mu \), satisfies also to 1° and 2°.

Proof. 1° and 2° are equivalent on account of ([3], 38.13).

Proof that \( \lambda \) satisfies to 1°. Note that \( h(E) = S^+(E) - S^+(E) \). Let \( f \in S^+(E) \), we have

\[
0 \leq \lambda(f - f \wedge n\lambda^+) \leq \lambda((f - n\lambda)^+) \leq \mu((f - n\lambda)^+) \to 0 \quad \text{when} \quad n \to \infty,
\]

hence

\[
\lambda(f) = \lim(\lambda(f \wedge n\lambda^+)) \quad \text{when} \quad n \to \infty.
\]

5. PROPOSITION. - Suppose \( E \) is a weak space, and \( \lambda, \mu \in \mathbb{M}^+(E) \). If \( \lambda < \mu \), then, for each sequence \( \lambda_1, \lambda_2, \ldots, \lambda_n \) of \( \mathbb{M}^+(E) \) such that \( \lambda = \sum_1^n \lambda_i \), there exists a sequence \( \mu_1, \mu_2, \ldots, \mu_n \) of \( \mathbb{M}^+(E) \) such that \( \mu = \sum_1^n \mu_i \), and \( \lambda_i < \mu_i \) for \( i = 1, 2, \ldots, n \).

Proof. - For each \( f \in h(E) \), let \( \hat{f} \) such that :

1° \( \hat{f} = g_{\lambda \in \mathbb{R}} (\lambda \wedge f) \) if \( f \) is majorized by an element of \( E' \).

(In fact, on account of [1] (chap. II, § 7, exercice 24), we have \( -\hat{f} \in S(E) \).)

2° Otherwise, \( \hat{f} = + \infty \) on \( E \).

For each \( \nu \in \mathbb{M}^+(E) \), let \( p_\nu \) such that :

1° If \( \hat{f} \neq + \infty \), \( p_\nu(f) = g_{\lambda \in \mathbb{R}} (\nu(g) ; -g \in S(E), g \geq f) \). We have \( p_\nu(f) \in \mathbb{R} \).

2° If \( \hat{f} = + \infty \), \( p_\nu(\hat{f}) = + \infty \).

For \( i = 1, 2, \ldots, n \), let \( p_i = p_\nu_i \).

On the space \( (h(E))^n \), let us consider the functional \( p \), such that

\[
(f_i)_{1 \leq i \leq n} \mapsto p((f_i)) = \sum_1^n p_i(f_i).
\]
p is sub-linear with values in $\mathbb{R} \cup \{+\infty\}$, and

\[(f_i \leq 0, \text{ for } i = 1, 2, \ldots, n) \implies (p(f_i)) \leq 0.\]

Let $\mu$ the linear form on the diagonal of $(h(E))^n$, such that

$$
\mu((f, f, \ldots, f)) = \mu(f).
$$

$\mu$ is majorized by $p$. As each element of $(h(E))^n$ is majorized by an element of the diagonal, we can apply the version of the theorem of Hahn-Banach recalled in 1. $\mu$ has an extension $\tilde{\mu} \in (h(E))^n_+$ with $\tilde{\mu} \leq p$. We can write $\tilde{\mu} = (\mu_i)_{1 \leq i \leq n}$ with $\mu_i \in h(E)^n_+$, for $i = 1, 2, \ldots, n$.

The $\mu_i$ are convenient.

6. PROPOSITION. - Suppose $E$ is a complete weak space, and $\lambda, \mu \in M^+(E)$. The two following properties are equivalent.

1° $\lambda < \mu$.

2° There exists a conical measure $\pi \in M^+(M^+(E) \times M^+(E))$ carried by the cone $B = \{(\epsilon_x, \nu); x \in E \text{ and } \epsilon_x < \nu\}$, such that $r(\pi) = (\lambda, \mu)$.

Proof. - For simplification, we will write sometimes $M$ instead of $M(E)$ and $M^+$ instead of $M^+(E)$.

1° $\implies$ 2°: For each sequence $\lambda_1, \lambda_2, \ldots, \lambda_n$ satisfying the hypothesis of proposition 5, let us choose a sequence $\mu_1, \mu_2, \ldots, \mu_n$ satisfying the conclusion of 5.

We say that a sequence $s' = \lambda'_1, \lambda'_2, \ldots, \lambda'_m$ is finer than a sequence $s = \lambda_1, \lambda_2, \ldots, \lambda_n$ if, and only if, there exists a partition of $\{1, 2, \ldots, m\}$ into $n$ subsets $p_1, p_2, \ldots, p_n$, such that $\lambda_i = \sum_{j \in p_i} \lambda'_j$ for $i = 1, 2, \ldots, n$.

Let $U_\lambda$ be the set consisting of all the (finite) sequences finer than $s$. The family of sets $U_\lambda$ is a filter basis over $U(\lambda)$ where $(\lambda)$ means the sequence $\lambda$.

Let $\varphi$ be the application

$$
\varphi(s) = \pi_s = \sum_{i=1}^n \epsilon_i(\epsilon_x(\lambda_i), \mu_i),
$$

we have $\pi_s \in M^+(M^+ \times M^+)$. The family of sets $\varphi(U_\lambda)$ is a filter basis over $M^+(M^+ \times M^+)$. We have $r(\pi_s) = \sum_{i=1}^n (\epsilon_i(\epsilon_x(\lambda_i), \mu_i))$.

Each element of $h(E)^+$ is majorized by an element of $S(E)^+$, and for each $f \in S(E)^+$, we have

$$
\sum_{i=1}^n (\epsilon_i(\epsilon_x(\lambda_i), \mu_i))(f) = \sum_{i=1}^n f(\lambda_i) \leq \sum_{i=1}^n \lambda_i(f) = \lambda(f).
$$

Hence the filter basis $\varphi(U_\lambda)$ has at least a cluster point, let $\pi$. The element $\pi$ answers the question, since each $\pi_s$ is carried by $B$, and we have

$$
r(\pi) = \lim(r(\pi_s)) = (\lambda, \mu).
$$

2° $\implies$ 1°: If $\pi \in M^+(M^+ \times M^+)$ with $r(\pi) = (\lambda, \mu)$, and if $\pi$ is carried by $B$, then for each $f \in S(E)$, we have $\pi((-f, f)) \geq 0$, since the element $(-f, f)$
of \( h(E) \times h(E) \) is \( \geq 0 \) on \( B \). Hence we have \( \lambda < \mu \).

7. **Remark.** — We can prove 6 with the method of [10] (p. 108) (and without the theorem of § 1) by looking at the convex closure of the set
\[
\{(\varepsilon_x, \nu) ; x \in E \text{ and } r(\nu) = x\}
\]
in \( M^+ \times M^+ \). Then § 5 can be obtained for \( \mathbb{R}^n \) as in [10] (p. 112) and in the general case by a projective limit argument.

8. **Definition** (of a pure pair and a pure measure). — Suppose \( \lambda, \mu \in M^+(E) \). We say the pair \( (\lambda, \mu) \) is pure if, and only if,
\[
(\mu' \in M^+(E) \text{ and } \mu' \leq \mu, \lambda < \mu') \text{ involves } (\mu' = \mu).
\]
Suppose \( \lambda \in M^+(E) \). We say that \( \lambda \) is pure, when the two following equivalent condition are fulfilled.

1° \( (\varepsilon_{x}(\lambda), \lambda) \) is a pure pair.
2° \( K_{\lambda} \) admits 0 as an extremal point.

**Proof.**
1° \( \implies \) 2°: Suppose 2° is false. Let \( \lambda_1 \leq \lambda \), and \( \lambda_2 \leq \lambda \) with \( r(\lambda_1) = -r(\lambda_2) \neq 0 \). If \( \lambda_0 = \lambda - (\lambda_1 + \lambda_2)/2 \), we have \( 0 \leq \lambda_0 \leq \lambda \), \( \lambda_0 \neq \lambda \), and \( r(\lambda_0) = r(\lambda) \), then \( (\varepsilon_{x}(\lambda), \lambda) \) is not a pure pair.

2° \( \implies \) 1°: Suppose \( \mu \leq \lambda \) with \( \mu \geq 0 \) and \( r(\mu) = 0 \). Let \( \mu = \mu_1 + \mu_2 + \cdots + \mu_n \) be any decomposition of \( \mu \) where \( \mu_i \geq 0 \). We have \( \mu_i \in K_{\lambda} \), and \( \sum_{1 \leq i \leq n} r(\mu_i) = 0 \), hence \( r(\mu_1) = 0 \) for \( i = 1, 2, \ldots, n \). Then \( \mu = 0 \).

9. **Example.** — In the cartesian product \( \mathbb{R}^2 \), let \( a, b, c, d \) be the consecutive vertices of a square of center 0. If
\[
\lambda = \varepsilon_a + \varepsilon_b + \varepsilon_c + \varepsilon_d + \varepsilon_{-(c+d)},
\]
\[
\lambda_1 = \varepsilon_a + \varepsilon_b + \varepsilon_c + \varepsilon_d,
\]
\[
\lambda_2 = \varepsilon_c + \varepsilon_d + \varepsilon_{-(c+d)},
\]
we have \( r(\lambda_1) = r(\lambda_2) = 0 \), and \( (\lambda - \lambda_1), (\lambda - \lambda_2) \) are pure.

10. **Proposition.** — Suppose \( E \) is a complete weak space, and \( \lambda, \mu \in M^+(E) \) with \( \lambda < \mu \). Then, the three following properties are equivalent.

1° The pair \( (\lambda, \mu) \) is pure.
2° For each \( \pi \in M^+(M^+ \times M^+) \), representing \( (\lambda, \mu) \) according to § 6 and carried by the cone \( B \), then the restriction \( \pi_0 \) of \( \pi \) to the cone \( A = \{(0, \nu) ; \nu \in M^+(E) \text{ and } r(\nu) = 0\} \) is equal to zero.

3° Each \( \pi \in M^+(M^+ \times M^+) \) representing \( (\lambda, \mu) \), and carried by the cone \( B \), is carried by the cone \( B_\lambda = \{(\varepsilon_x, \nu) ; x \in E, r(\nu) = x, \nu \text{ is pure}\} \).

**Proof.**
1° $\Rightarrow$ 2°: Suppose $2°$ is false. If $\pi$ represents $(\lambda, \mu)$ with $\pi_0 \neq 0$ (for the definition of $\pi_0$, see [3], 30.8), we have $r(\pi_0) = (0, \nu)$ with $\nu \neq 0$ and $r(\nu) = 0$.

For each $f \in S(E)$, we have $(-f, f) \geq 0$ on $B$. Hence

$$v(f) = \pi_0((-f, f)) \leq \pi((-f, f)) = -\lambda(f) + \mu(f)$$

Therefore $\lambda < \mu - \nu$, and $(\lambda, \mu)$ is not pure.

2° $\Rightarrow$ 3°: We can write $\pi = \lim_{\mathcal{U}} \sum \varepsilon(\varepsilon_x, \nu)$ with $(\varepsilon_x, \nu) \in B$ where $\mathcal{U}$ is an ultrafilter.

For each $\nu \in \mathbb{N}^+(E)$, let us choose $p_\nu \in \mathbb{N}^+(E)$ such that:

(a) $p_\nu$ is pure,
(b) $p_\nu \leq \nu$,
(c) $k.p_\nu = p_{k\nu}$ for any $k \geq 0$.

We will prove that $\pi = \lim_{\mathcal{U}} \sum \varepsilon(\varepsilon_x, p_\nu)$.

We have

$$(\lambda, \mu) = (\lambda, \lim_{\mathcal{U}} \sum p_\nu) + (0, \lim (\nu - p_\nu)).$$

On account of the hypothesis, we have $\lim_{\mathcal{U}} \sum (\nu - p_\nu) = 0$, hence

$\lim_{\mathcal{U}} \sum \varepsilon(\varepsilon_0, \nu - p_\nu) = 0$. For each $f \in S(\mathbb{R} \times \mathbb{R})$, we have

$$f(\varepsilon_x, p_\nu) - f(0, p_\nu - \nu) \leq f(\varepsilon_x, \nu) \leq f(0, \nu - p_\nu) + f(\varepsilon_x, p_\nu).$$

As we have $\lim_{\mathcal{U}} \sum \varepsilon(\varepsilon_0, \nu - p_\nu) = 0$, then $\pi(f) = \lim_{\mathcal{U}} \sum f(\varepsilon_x, p_\nu)$. Therefore $\pi$ is carried by $B_p$.

3° $\Rightarrow$ 1°: Suppose $1°$ is false. We have $(\lambda, \pi) = r(\varepsilon(\lambda, \omega) + \varepsilon(0, \beta))$ with $(\lambda, \omega)$ pure, and $(0, \beta) \in A$ with $\beta \neq 0$. Therefore $\varepsilon(0, \beta)$ is not carried by $B_p$.

11. Example (G. Choquet). - In $\mathbb{R}^2$ suppose $C_1$ and $C_\rho$ are the circles (for the classical distance) of center 0 with radius 1 and $\rho > 1$. For each $x \in C_1$, let $x_1, x_2 \in C_\rho$ so that $(x_1, x_2)$ is tangent to $C_1$ at $x$. Let $dx$ be the Haar measure on $C_1$. We have

$$\int_{C_1} (\varepsilon_{x_1} + \varepsilon_{x_2}) \, dx = \rho' \int_{C_1} \varepsilon_x \, dx \quad \text{(with } \rho' > 1)$$

as conical measures.

The pair $(\varepsilon_x, \varepsilon_x + \varepsilon_{x_2})$ is pure for each $x \in C_1$, but the resultant of

$$\int_{C_1} \varepsilon(\varepsilon_x, \varepsilon_x + \varepsilon_{x_2}) \, dx$$

is the pair $(\int_{C_1} \varepsilon_x \, dx, \rho' \int_{C_1} \varepsilon_x \, dx)$ which is not pure since $\rho' > 1$.

12. Proposition. - Suppose $\lambda, \mu \in M^+(\mathbb{R}^n)$ with $\lambda < \mu$, and the pair $(\lambda, \mu)$ is pure. Then, there exist:
1° a $K_0$ of $(\mathbb{R}^N \setminus \emptyset)$, let $X$ such that each half-line issued from 0 intersects $X$ into at most one point.

2° a Radon measure $\Lambda$ on $X$.

3° a Borel application $x \mapsto \mu_x$ defined on $X$ where $\mu_x$ is a Radon measure on $X$ such that $r(\mu_x) = x$.

And we have:

(a) $\Lambda$ is a localization of $\lambda$ (Note that $\Lambda$ is unique when $X$ is given).

(b) $\mu = \int_X \mu_x \, d\Lambda(x)$.

Proof (with the notations of the proof of § 6). - We had

$$\pi_s = \frac{1}{1} \varepsilon(\varepsilon_x(\lambda_{s}), \mu_1).$$

For each $n \in \mathbb{N}$, let $x_n$ be the function $n$-th coordinate on $\mathbb{R}^N$. We have

$$\pi_s(|x_p|, |x_p|) \leq \lambda(|x_p| + \mu(|x_p|) \leq 2\mu(|x_p|).$$

Let $\lambda$ be the affine l. s. c. function defined on $\mathbb{M}^+ \times \mathbb{M}^+$ by

$$\lambda(\alpha, \beta) = \sum_p (\alpha \beta| |x_p|) / \varepsilon p 1 \mu(|x_p|).$$

We have

$$\lambda(\pi_s) = \sum_p \pi_s(|x_p|, |x_p|) / \varepsilon p 1 \mu(|x_p|) \leq 1 / 2 \varepsilon \leq 1.$$

$\pi$ has a localization by a Radon measure $m$ on a cap $K$ of $\mathbb{M}^+ \times \mathbb{M}^+$, with $K = \{(\alpha, \beta) ; \alpha, \beta \in \mathbb{M}^+, \lambda(\alpha, \beta) \leq 1\}$. Moreover $m$ can be assumed to be carried by the cone $B$.

Let $\mathbb{A}$ be the l. s. c. function defined on $\mathbb{R}^N$ by $\mathbb{A}(x) = \lambda(\varepsilon_x, \varepsilon_x)$. For each $n \in \mathbb{N}$, let $K_n = \{(\varepsilon_x, \alpha) ; (\varepsilon_x, \alpha) \in K, 1/(n+1) < \mathbb{A}(x) \leq 1/n\}$. Let $m_n$ be the restriction of $m$ to $K_n$. We have $m = \sum m_n$ on account of § 10, since $(\lambda, \mu)$ is a pure pair. Let $\pi_n$ be the conical measure on $\mathbb{M}^+ \times \mathbb{M}^+$ localized by $m_n$.

Let $m'_n$ be the Radon measure on $(n + 1)K$ such that, for each continuous function $f$ on $(n + 1)K$, we have

$$m'_n(f) = \int_K \mathbb{A}(x) f(\varepsilon_x / \mathbb{A}(x), \alpha / \mathbb{A}(x)) \, dm_n(\varepsilon_x, \alpha),$$

then $m'_n$ localizes $\pi_n$.

Suppose $p$ is the projection on the first factor of the product $\mathbb{M}^+ \times \mathbb{M}^+$, then $p(m'_n)$ is carried by $K = \{\varepsilon_x ; x \in \mathbb{R}^N \text{ with } \mathbb{A}(x) = 1\}$. $K$ is a Borel set because $\mathbb{A}$ is l. s. c., moreover $K$ intersects each half-line issued from 0 in at most one point.

Suppose $x \mapsto m'_n$ is a disintegration of $m'_n$ with respect to $p$ ([2], p. 58).

Then each $m'_n$ has a resultant which is a conical measure $\nu'_n$ on $\mathbb{R}^N$, and we have $\mu(\nu'_n) = x$.

Now $\Lambda = \sum_n p(m'_n)$ can be seen as a Radon measure on a $K_0$ subset $X_\Lambda$ of $K$. We
can write, for each $n \in \mathbb{N}$, $p(m_n) = u_n \lambda$ where $u_n$ is a Borel function on $X_\lambda$. We have $\sum_n u_n = 1$, $\lambda$-a.e.

Recall that $\lambda$ represents $\lambda$, and that $\mu = \sum_n \int \nu_x d(p(m_n))$ (equality of conical measures), then we have $\mu = \int (\sum_n u_n \nu_x) d\lambda$. Therefore $\mu_x = \sum_n u_n \nu_x$ exists as a conical measure $\lambda$-a.e., and we have $r(\mu_x) = x$.

On account of [3] (38.8), there exists a compact subset $H$ of $\mathbb{R}^N$ with $H = \bigcap_{k=1}^\infty (-k, k)$ where $k_n > 0$, such that $\mu$ is localizable on $H$ by a Radon measure.

For simplification we shall use the same notation for $\mu_x$, and its unique ([7], prop. 2.13) localization on the set $E(H) = \{x ; x \in H, \forall k > 1, kx \notin H\}$. As $\mu_x$ is a Daniell integral on $\mathbb{R}(\mathbb{N})$ ([3] 38.13), and since

$$E(H) = \{x ; \text{lub}(|x|/k_n) = 1\},$$

then ([9] prop. II.7.1) $\mu_x$ can be extended to a $\sigma$-additive measure, called also $\mu$ for simplification, on the tribe $\mathcal{C}$ of $E(H)$ generated by the closed half-spaces containing $0$. Recall we know that, for each $f \in h(\mathbb{R}^N)$, the map $x \mapsto \mu_x(f)$ is Borel-measurable. Then, for each $e \in \mathcal{C}$, we have $\mu(e) = \int \mu_x(e) d\lambda$.

Let $X_{\mu}$ be a $K_\sigma$ subset of $E(H)$ which bears $\mu$. In order to show that $\mu_x$ lives on $X_{\mu}$ for $\lambda$-a.e.$x$, it is sufficient to prove the following lemma.

13. **Lemma.** Each compact subset $A$ of $E(H)$ is a member of $\mathcal{C}$.

**Proof.** - Let us suppose the sequence $(\omega_n)_{n\in\mathbb{N}}$ is a basis of open subsets of $\mathbb{R}^N$. Let $\Sigma$ be the subset of $N$ such that $n \in \Sigma$ if, and only if, there exists $h \in h^+(E)$, with $h = 0$ on $A$, and $h > 0$ on $\omega_n$. For each $n \in \Sigma$, we choose $h_n \in h^+(E)$, with $h_n = 0$ on $A$, and $h_n > 0$ on $\omega_n$.

Let us show that, for each $x \notin R^+ A$, we have $h_n(x) > 0$ for at least one $n \in \Sigma$. For each $y \in A$, there exists $h_y \in E'$ with $h_y(x) > 0$, and $h_y(y) < 0$. By compactness, there exists $h_x \in h^+(E)$, with $h_x(x) > 0$, and $h_x = 0$ on $A$. As the set $\{z ; h_x(z) > 0\}$ is open, then there exists $n \in N$ such that $x \in \omega_n$ and $h_n(x) > 0$. Therefore, we have $n \in \Sigma$ and $h_n(x) > 0$.

Now, if we let $h = \text{lub}_{n\in\Sigma}(h_n)$, then we have $h = 0$ on $A$, and $h(z) > 0$ for each $z \notin R^+ A$. Hence $A \in \mathcal{C}$.

Now, it is easy to complete the proof of § 12 by a mixture of $X_{\lambda}$ and $X_{\mu}$.

14. **Remark** (N. F. SAINTE BEUVE [11], theorem 3). - In the case of $\mathbb{R}^N$, we can take the unit sphere of $\mathbb{R}^n$ (for the usual distance) for $X$.

15. **Example** (Answer to a question of G. CHOQUET). - Let $\mathbb{M}$ be the set of Radon measures on $(0, 1)$, and $\mathbb{M}_1^+$ the subset of probability measures.

Let $E$ the vector subspace of $\mathbb{M}$ generated by the Dirac probabilities, $E$ is equipped with the weak$^\ast$-topology.

Suppose $\mu$ is the maximal measure on $\mathbb{M}_1^+$ which represents the element $dx \in \mathbb{M}_1^+$. 


The measure $\mu$ and $dx$ induce, in a canonical way, elements of $\mathbb{M}^+(E)$, $\tilde{\mu}$ and $\varepsilon_d\mu$, since $E \cap \mathbb{M}^+_1$ is dense in $\mathbb{M}^+_1$.

Let $\varphi$ be the canonical injection from $(0, 1)$ into $\mathbb{M}$, and $X = \varphi((0, 1))$. We have $\varepsilon_d\mu < \tilde{\mu}$ (in fact, $\varepsilon_d\mu = \varepsilon_d(\tilde{\mu})$ in the weak completion of $E$), however $\tilde{\mu}$ has a localization on the compact subset $X$ of $E$, while $\varepsilon_d\mu$ does not have such a localization.

Part II: Extension of a result of Strassen and "theory of balayage".

II (A). Extension of a result of Strassen.

16. Notations and definitions. - Suppose $X$ and $Y$ are two compacta (Hausdorff) spaces and $x \mapsto M_x$ is a mapping of $X$ in the set of closed convex subsets of $\mathbb{M}(Y)$ (positive Radon measures on $Y$).

For each $f \in C(Y)$ (continuous real functions on $Y$), we let

$$\forall x \in X, \quad \hat{f}(x) = \text{lub}_{\nu \in M_x} (\nu(f)),$$

we have $\hat{f}(x) \in \mathbb{R}$, and $(\hat{f}(x) = -\infty) \iff (M_x = \emptyset)$.

The map $f \mapsto \hat{f}$ has been previously considered by P.-A. Meyer ([8], p. 301).

Suppose $\lambda \in \mathbb{M}^+(X)$. For each function $\varphi$ on $X$ with values in $\mathbb{R}$, we let

$$\lambda^\varphi = g \& b(\lambda(u)); \; u \geq \varphi, \; u \text{ l.s.c. on } X, \text{ with values in } \mathbb{R} \cup (+\infty).$$

We have $\lambda^\varphi \in \mathbb{R}$.

If $\lambda \in \mathbb{M}^+(X)$ and $\mu \in \mathbb{M}^+(Y)$, we write $\lambda < \mu$ if, and only if, for each $f \in C(Y)$, we have $\mu(f) \leq \lambda^\varphi(\hat{f})$. We let $p_\lambda(f) = \lambda^\varphi(\hat{f})$.

17. Proposition. - Suppose $\lambda \in \mathbb{M}^+(X)$ and $\mu \in \mathbb{M}^+(Y)$ with $\lambda < \mu$. For each sequence $\lambda_1, \ldots, \lambda_n$, such that $\lambda = \lambda_1 + \cdots + \lambda_n$ with $\lambda_i \geq 0$, there exists a sequence $\mu_1, \ldots, \mu_n$ with $\mu_i \geq 0$, such that $\mu = \mu_1 + \cdots + \mu_n$, and $\lambda_i < \mu_i$ for $i = 1, 2, \ldots, n$.

Proof. - Let $1$ be the constant function equal to $1$ on $Y$.

We have $\lambda^\varphi(-1) \geq \mu(-1) > -\infty$. Hence, for each $f \in C(Y)$, we have

$$\lambda_i(\hat{f}) \in \mathbb{R} \cup (+\infty) \text{ for } i = 1, 2, \ldots, n,$$

then we can use the same proof than in proposition 5.

18. Proposition. - We let $H = \{(x, \nu); \; x \in X, \; \nu \in M_x\}$. If $\lambda \in \mathbb{M}^+(X)$ and $\mu \in \mathbb{M}^+(Y)$, the two following properties are equivalent

1° $(\lambda, \mu) \in \text{conv}(R^+H)$ in $\mathbb{M}^+(X) \times \mathbb{M}^+(Y)$ equipped with the weak*-topology.

2° For each $f \in C(Y)$, we have

$$\mu(f) \leq g \& b(\lambda(g)); \; g \in C(X) \text{ and } \hat{f} \leq g.$$
Proof. - We apply the theorem of Hahn-Banach.
Suppose \( g \in C(X) \) and \( f \in C(Y) \). Then \((g, -f)\) is in the polar of \( H \) if, and only if, \( \hat{f} \leq g \).

1° \( \Rightarrow \) 2°: If \( f \in C(Y) \), we have \( \lambda(g) \geq \mu(f) \) for each \( g \in C(X) \) with \( \hat{f} \leq g \), hence 2° is fulfilled.

2° \( \Rightarrow \) 1°: For each \( g \in C(X) \) and each \( f \in C(Y) \) with \( \hat{f} \leq g \), we have, on account of 2°, \( \mu(f) \leq \lambda(g) \). Hence 1° is fulfilled on account of the bipolar theorem.

19. Definition of the relation \( \ll \). - If \( \lambda \in \mathfrak{M}^+(X) \), proposition 18 invites us to let, for each \( f \in C(Y) \)

\[ q_\lambda(f) = g \& b(\lambda(g)) \quad (g \in C(X) \text{ and } g \geq \hat{f}) \]

Note we have \( p_\lambda \leq q_\lambda \). Moreover, if \( H \) is a closed subset of \( \mathfrak{M}^+(X) \times \mathfrak{M}^+(Y) \), then we have \( p_\lambda(-1(y)) = q_\lambda(-1(y)) \) because \( -\hat{f}(y) \) is negative, and u. s. c.

If \( \mu \in \mathfrak{M}^+(Y) \), we write \( \lambda \ll \mu \) if, and only if, \( \mu < q_\lambda \). We have

\[ (\lambda < \mu) \Rightarrow (\lambda \ll \mu) \]

Of course, we can prove the analogous of proposition 17 for the relation \( \ll \). Note, in the case, study by P.-A. MEYER ([8] p. 302) (i.e. \( H \) is compact), \( \hat{f} \) is u. s. c. so that \( \hat{f} = g \& b(g) \quad (g \in C(X), g \geq \hat{f}) \). Hence \( p_\lambda = q_\lambda \).

20. PROPOSITION. - Suppose \( \mathcal{R} \) is the closure, in \( \mathfrak{M}^+(X) \times \mathfrak{M}^+(Y) \), equipped with the weak*-topology, of the set

\[ \mathcal{K} = \{ (\varepsilon_x / (1 + \nu_x(1)), \nu_x / (1 + \nu_x(1))) \mid x \in X, \nu_x \in \mathcal{M}_x \} \]

If \( \lambda \in \mathfrak{M}^+(X) \) and \( \mu \in \mathfrak{M}^+(Y) \), the two following properties are equivalent:

1° \( \lambda \ll \mu \)

2° There exists a positive Radon measure \( \pi \) on the compact set \( \mathcal{K} \) such that \( r(\pi) = (\lambda, \mu) \).

Proof.

1° \( \Rightarrow \) 2°: Each element \( u \) of \( \text{conv}(\mathcal{R}^+ H) \) can be written \( u = \sum_{x \in X} \epsilon_{x}^{u}(\varepsilon_{x}, \nu_{x}^{u}) \)
where the \( \epsilon_{x}^{u} \) are unique, positive, and equal to 0 except for a finite number of \( x \in X \). We have \( \nu_{x}^{u} \in \mathcal{M}_x \).

On account of §18, there exists an ultrafilter \( \mathcal{U} \) on \( \text{conv}(\mathcal{R}^+ H) \) such that \( \lim_{\mathcal{U}}(u) = (\lambda, \mu) \).

\( u \) is the resultant of the following conical measure \( \pi_u \) on \( \mathfrak{M}^+(X) \times \mathfrak{M}^+(Y) \) with \( \pi_u = \sum_{x \in X} a_x^u \varepsilon((b_x^u, c_x^u)) \) where

\[ a_x^u = (1 + \nu_x^u(1))k_x^u, \quad b_x^u = \varepsilon_x^u / (1 + \nu_x^u(1)) \]

and \( c_x^u = \nu_x^u(1) \).
\( \pi_u \) can be also seen as a positive Radon measure on \( \bar{K} \).

We have \( \lim \nu_u(1) = \lambda(1) + \mu(1) \). Hence \( \lim \nu_u \) exists as a positive Radon measure \( \pi \) on \( \bar{K} \) and \( r(\pi) = (\lambda, \mu) \).

Each discrete positive Radon measure on \( K \) can be written

\[ m = \sum_{p \in K} a_p \epsilon((b_p, c_p)) \text{ where } (b_p, c_p) \in K, \ a_p > 0 \text{ and } a_p = 0, \]

except for a finite number of \( p \in K \).

There exists an ultrafilter \( \mathfrak{U} \) on the discrete positive measures on \( K \) such that \( \lim_{\mathfrak{U}}(m) = \pi \).

If \( g \in C(X) \) and \( f \in C(Y) \) with \( g \geq f \), we have

\[ \lambda(g) = \lim_{\mathfrak{U}}(\sum_{p \in K} a_p b_p(1) g(b_p/b_p(1))) \]

and

\[ \mu(f) = \lim_{\mathfrak{U}}(\sum_{p \in K} a_p c_p(f)). \]

As \( g \geq f \), we have \( b_p(1) g(b_p/b_p(1)) \geq c_p(f) \), hence \( \lambda(g) > \mu(f) \).

21. PROPOSITION (Extension of a result of STRASSEN [8], p. 302). - Suppose moreover that \( X \) and \( Y \) are metrizable, and that \( H \) is a closed subset of \( \mathfrak{M}(X) \times \mathfrak{M}(Y) \) equipped with the weak*-topology. If \( \lambda \in \mathfrak{M}(X) \) and \( \mu \in \mathfrak{M}(Y) \)

with \( \lambda \ll \mu \), then, there exists a Borel mapping \( x \mapsto \nu_x \) defined on \( X \) such that \( \nu_x \in \mathfrak{M}_x \lambda-a.e., \) and \( \mu \gg \int \nu_x d\lambda(x) \), and \( \mu \ll \nu - \int \nu_x d\lambda(x) \).

Proof. - Note that \( \{x ; M_x = \emptyset\} \) is a \( G_\delta \lambda \)-null subset of \( X \) since \( \lambda(1(y)) \geq \mu(-1(y)) > -\infty \). We shall use the notations of the proof of §20.

Suppose \( \nu \) is the projection of \( \mathfrak{M}(X) \times \mathfrak{M}(Y) \) on \( \mathfrak{M}(X) \). We have \( \nu(\pi) = \lambda \) as conical measures on \( \mathfrak{M}(X) \). Let \( A_0 = \{(0, \beta) ; \beta \in \mathfrak{M}(Y) \} \), and \( \pi_0 \) the part of \( \pi \) carried by \( A_0 \).

Let \( \pi' = \pi - \pi_0 \).

Suppose \( \pi' = \pi_1' + \ldots + \pi_n' + \ldots \) is a decomposition of \( \pi' \) such that, for each \( n \), \( \pi_n' \) lives on \( A_n = \{(\alpha, \beta) ; \alpha \in \mathfrak{M}(X), \beta \in \mathfrak{M}(Y) \text{ and } \alpha(1) > 1/n\} \). Let \( \pi_n^\mu \) the Radon measure on \( (1, n)[X] \) such that, for each \( f \in C((1, n)[X]) \), we have

\[ \pi_n^\mu(f) = \int \alpha(1) f(\alpha/\alpha(1), \beta/\alpha(1)) \, d\nu_n^\mu(\alpha, \beta). \]

\( \pi_n^\mu \) and \( \pi_n^\mu \) induce the same conical measure on \( \mathfrak{M}(X) \times \mathfrak{M}(Y) \). Then \( \nu(\pi_n^\mu) \) is carried by \( \{\varepsilon_x, x \in X\} \); the image of \( \lambda \) by the map \( x \mapsto \varepsilon_x \) of \( X \) into \( \mathfrak{M}(X) \) is \( \sum \nu(\pi_n^\mu) \). If \( \varepsilon_x \mapsto \nu_x^\pi \) is a disintegration ([2], p. 58) of \( \pi_n^\mu \) with respect to \( \nu \), then, for each \( n \), we have \( \nu(\pi_n^\mu) \)-a.e., that \( \nu_x^\pi \) lives on the set \( \{(\varepsilon_x, v) ; v \in M_x \text{ and } \nu(1) \leq n\} \), let \( \nu_x^\pi \in M_x \) such that

\[ r(\varepsilon_x \otimes \nu_x^\pi) = (\varepsilon_x, \nu_x^\pi). \]

For each \( x \in X \), we identify \( x \) and \( \varepsilon_x \). Let \( \mu_0 \in \mathfrak{M}(Y) \) be such that \( (0, \mu_0) = r(\pi_0) \). We have \( \lambda = \sum \nu(\pi_n^\mu) \) and \( \mu - \mu_0 = \sum \nu_x^\pi \, d\nu(\pi_n^\mu) \).
If we let \( \psi = \frac{1}{n} \sum_{n=1}^{\infty} \psi_{n}(\theta_{n})/\lambda \), then, we have \( \psi \in H_{\lambda} \), \( \lambda \)-a.e. and \( \mu - \mu' = \int \psi_{x} \, d\lambda(x) \). As \( \pi_{0} \) is carried by \( A_{0} \), we have \( 0 \ll \mu' \).

22. **Remark.** - Strictly speaking, in [8] (chap. 11), Strassen theorem is T51 which admits T52 as a consequence, but T51 can be also derived from T52. We sketch a proof, with the notations of [8]. Suppose \( E_{1}^{+} \) is the unit ball of \( E' \) equipped with the weak*-topology. For each \( \omega \in \Omega \), let \( P_{\omega} \) be the set
\[
\{ y \, ; \, y \in E' \land \, y \leq p_{\omega} \}.
\]
We suppose \( P_{\omega} \subset E_{1}^{+} \). Let \( M_{\omega} = \{ \nu \, ; \, \nu \in \pi_{1}^{+}(E) \land \, r(\nu) \in P_{\omega} \} \).

Now, suppose \( (x_{n}) \) is a sequence of \( E \) everywhere norm-dense in the unit ball \( E_{1} \) of \( E \). Let \( \Phi \) be the map \( \Omega \rightarrow \{-1, 1\}^{N} \) such that \( (\Phi(\omega))_{n} = p_{n}(x_{n}) \). We let \( X = \Phi(\Omega) \) and \( \Lambda = \Phi(\lambda) \), which is a regular Borel measure on \( X \) ([9] prop. II.7.2).

For each \( t = (t_{n}) \) in \( X \), because of [8] (p. 300 footnote), there exists a sublinear form \( p_{t} \) on \( E \) such that \( p_{t}(x_{n}) = t_{n} \), for each \( n \), and \( p_{t}(E_{1}) \in (-1, 1) \).

Then the definition of \( P_{t} \) and \( M_{t} \) are meaningful, and the set \( \{(t, \nu) \, ; \, t \in X \land \, \nu \in M_{t} \} \) is a compact subset of \( X \times \pi_{1}^{+}(E_{1}) \).

Now it is sufficient to apply T52 to \( X \) and measure \( \Lambda \), with \( Y = E_{1}^{+} \) using the map \( X \rightarrow \Phi(\pi_{1}^{+}(Y)) \) defined by \( t \mapsto M_{t} \), and taking for \( \mu \) an extension of \( x' \) to \( C(Y) \) such that, for each \( f \in C(Y) \), \( \mu(f) \leq \Lambda(f) \), then T51 follows since in \( X \), \( \Phi(\Omega) \) is of \( \Lambda \)-outer measure equal to \( \Lambda(1) \).

**II (B). Theory of balayage.**

23. **Notations.** - Suppose \( X \) is a compact (HAUSDORFF) space, \( \Gamma \) a convex subcone of \( C(X) \) which is an inf-lattice (i.e. if \( f, g \in \Gamma \), then \( g \& b(f, g) \in \Gamma \)), and \( \Gamma^{0} \) is the polar of \( \Gamma \) in \( \pi(X) \).

Using the previous notations, we take \( Y = X \),
\[
M_{X} = (\varepsilon - \Gamma^{0}) \cap \pi^{+}(X) = \{ \mu \, ; \, \mu \in \pi^{+}(X) \land \mu|_{\Gamma} \leq \varepsilon|_{\Gamma} \}.
\]
Note that we do not suppose as in [8] (p. 294-297) that \( \Gamma \) contains a strictly positive function.

24. **Definition (of \( f_{\Gamma} \) and \( r_{\lambda} \)).** - For each \( f \in C(X) \), we let
\[
f_{\Gamma} = g \& b(g \land g \geq f)
\]
and for each \( \lambda \in \pi^{+}(X) \), we let \( r_{\lambda}(f) = \lambda(f_{\Gamma}) \). \( r_{\lambda} \) is a sublinear functional on \( C(X) \), with values in \( R \cup (+\infty) \), and we have \( p_{\lambda} \leq q_{\lambda} \leq r_{\lambda} \).

25. **Proposition (Extension of a balayage formula of MOKOBODZKI [8] chap. 11 T45).**

For each \( f \in C(X) \) with \( f < 0 \), we have \( f_{\Gamma} = \overline{f} \).

Moreover the following properties are equivalent:

1° There is no element \( > 0 \) in \( \Gamma \),
2° $\hat{1} = + \infty$ everywhere on $X$.

3° $\hat{1}$ is equal to infinity in at least one point of $X$.

4° $\hat{1}$ is unbounded on $X$.

Proof. - Let us prove that $f_{\Gamma} = \hat{1}$ for each $f < 0$ of $C(X)$.

If $\lambda \in \mathbb{R}^+(X)$, because of the theorem of Hahn-Banach recalled in § 1, for each $k \in \mathbb{R}^+$, there exists $\mu_k \in \mathbb{R}^+(X)$ with $\mu_k(f) = k$ and $\mu_k \leq r_\lambda$.

It suffices now to take $\lambda = \varepsilon_x$ and $k = f_\Gamma(x) = r_\lambda(f)$. Now $1° \implies 2°$ can be proved in the same way, and we see that $4° \implies 1°$.

26. PROPOSITION.

(a) Suppose $f$ is an u. s. c. function $< 0$ on $X$. We have $f_{\Gamma} = \hat{f}$ (the definition of $\hat{f}$ is as in § 16 and that of $f_{\Gamma}$ as in § 24).

(b) If $(f_n)$ is a family of u. s. c. functions $< 0$ on $X$, directed downward, having a limit $f$, we have $(f_n) \to f_{\Gamma}$.

Proof.

(a) can be proved as in [7] (prop. 5.6) because it is enough to work, for each $x \in X$, on a compact subset of $M_x$.

(b) can be proved as in [7] (prop. 5.6).

Proposition 25 enables us to give a balayage proof of the following result of Choquet-Deny [4].

27. PROPOSITION. - Suppose $\Gamma$ is a closed convex subcone of $C^-(X)$ which is an inf-lattice and contains $-1$. If we let

$$\hat{\Gamma} = \{f ; f \in C^-(X) \text{ with } m(f) \leq f(x), \forall x \in X, \forall m \in \mathbb{R}^+(X) \text{ with } m|\Gamma \leq \varepsilon_x|\Gamma\},$$

then we have $\Gamma = \hat{\Gamma}$.

Proof. - $\hat{\Gamma}$ is a closed convex subcone of $C^-(X)$ which is an inf-lattice and $\Gamma \subseteq \hat{\Gamma}$. For each $f \in C(X)$ such that $f < 0$, we have, because of 25,

$$f_{\Gamma}(x) = \operatorname{lub}_{x \in M_x}(v(f)),$$

and we see that $f_{\hat{\Gamma}}(x) = \operatorname{lub}_{x \in M_x}(v(f))$, hence $f_{\Gamma} = f_{\hat{\Gamma}}$. Therefore, by Dini lemma, we have $f = f_{\Gamma}$ if, and only if, $f \in \Gamma$ and $f = f_{\hat{\Gamma}}$ if, and only if, $f \in \hat{\Gamma}$, hence $\Gamma \cap \{f < 0\} = \hat{\Gamma} \cap \{f < 0\}$. Then $\Gamma = \hat{\Gamma}$, since $\Gamma$ and $\hat{\Gamma}$ are the closure of $\Gamma \cap \{f < 0\}$ and $\hat{\Gamma} \cap \{f < 0\}$.

28. Remark. - Suppose $\Gamma$ is separating. Then we can apply to $\Gamma$ the theorem 48 of [8] (chap. 11) about the Silov compacts. It is enough to apply [8] (chap. 11, th. 48) to the cone $\Gamma_1 = \{f ; f = g + a \text{ with } g \in \Gamma \text{ and } a \geq 0\}$ which is an inf-lattice.
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Richard BECKER
Equipe d'Analyse, Tour 46
Université Pierre et Marie Curie
4 place Jussieu
75230 PARIS CEDEX 05