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On lifting of automorphic forms

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ON LIFTING OF AUTOMORPHIC FORMS

by Hiroshi SAI TO

Q. - Let F be a totally real algebraic number field with the degree \([F: \mathbb{Q}] = \lambda\), and \(\mathcal{O}\) its maximal order. For the sake of simplicity, we assume that the class number of F is one, and \(\mathcal{O}\) has a unit with arbitrary signature distribution. For an even positive integer \(k\) and for the subgroup \(\Gamma = \text{GL}_2(\mathcal{O})^+\) of \(\text{GL}_2(\mathcal{O})\) consisting of all elements with totally positive determinants, we denote by \(S_k(\Gamma)\) the space of Hilbert cusp forms of weight \(k\) with respect to \(\Gamma\), namely the set of all holomorphic functions \(f\) on the \(\lambda\)-fold product of the complex upper half plane \(\mathbb{H}\), which satisfy

\[
1^o \quad f(\gamma z_1, \gamma z_2, \ldots, \gamma z_\lambda) = \prod_1 (c(i) z_1 + d(i))k f(z_1, \ldots, z_\lambda)
\]

for \(\gamma \in \Gamma\),

\[
2^o \quad f\text{ vanishes at every cusp},
\]

where \(\gamma(i) = \begin{pmatrix} a(i) & b(i) \\ c(i) & d(i) \end{pmatrix}\) are all distinct conjugates of \(\gamma\) over \(\mathbb{Q}\). It is known that an element \(f\) of \(S_k(\Gamma)\) has a Fourier expansion of the form

\[
f(z_1, \ldots, z_\lambda) = \sum_{\mathfrak{u}} C(\mathfrak{u}) \sum_{(\nu) = \mathfrak{u}/\mathfrak{D}, \nu > 0} \exp 2\pi i (\nu(1) z_1 + \ldots + \nu(\lambda) z_\lambda),
\]

where \(\mathfrak{u}\) runs through all integral ideals of \(F\), and \(\mathfrak{D}\) is the different of the extension \(F/\mathbb{Q}\). We denote by \(\Phi_f\) the associated Dirichlet series of \(f\), that is,

\[
\Phi_f(s) = \sum_{\mathfrak{u}} C(\mathfrak{u}) \mathfrak{N}^{s}.\]

For a place (archimedean or non-archimedean) of \(F\), we denote by \(F_v\) the completion of \(F\) at \(v\), and for a non-archimedean prime \(v = p\), we denote by \(\mathcal{O}_p\) the ring of all \(p\)-adic integers of \(F_p\). Let \(F_A\) be the adele ring of \(F\), and \(U_F\) be the open subgroup of \(\text{GL}_2(F_A)\) given by

\[
\prod_{\mathfrak{p}: \text{non-archimedean}} \text{GL}_2(\mathcal{O}_\mathfrak{p}) \times \prod_{v: \text{archimedean}} \text{GL}_2(F_v).
\]

Then we can consider the Hecke ring \(\mathcal{R}(U_F, \text{GL}_2(F_A))\) with respect to \(\text{GL}_2(F_A)\) and \(U_F\), and its action \(T\) on \(S_k(\Gamma)\) as in S. SHIMURA [9]. It is known that \(S_k(\Gamma)\) has a basis consisting of common eigenfunctions for all Hecke operators and that if \(f\) is a common eigenfunction for all Hecke operators with \(C(\mathfrak{O}) = 1\), then the associated Dirichlet series \(\Phi_f\) has an Euler product of the form

\[
\Phi_f(s) = \prod_p (1 - C(p) \mathfrak{N}_p^{-s} + \mathfrak{N}_p^{k-1-s})^{-1},
\]

where \(p\) runs through all prime ideals of \(F\).
1. On the following, we assume that $F$ is a totally real algebraic number field which satisfies

1° $F$ is a cyclic extension of $\mathbb{Q}$ with a prime degree $\lambda$,

2° $F$ is a tamely ramified extension of $\mathbb{Q}$,

3° The class number of $F$ is one,

4° The index $[E:E_+]$ is $2^\lambda$,

where $E$ is the group of all units of $O$ and $E_+$ is its subgroup consisting of all totally positive units. It follows from these conditions that the conductor of the extension $F/\mathbb{Q}$ is a prime $q$ with $q \equiv 1 \mod \lambda$.

We fix an embedding of $F$ into the real number field $\mathbb{R}$, and consider $F$ as a subfield of $\mathbb{R}$. We fix a generator $\sigma$ of the Galois group $\text{Gal}(F/\mathbb{Q})$. With this $\sigma$, we consider $\text{GL}_2(F)$ as a subgroup of $\text{GL}_2(\mathbb{R})$ by

$$\gamma \mapsto (\gamma, \sigma \gamma, \ldots, \sigma^{\lambda-1} \gamma) \text{ for } \gamma \in \text{GL}_2(F).$$

For this fixed generator $\sigma$, we define a linear operator $T_\sigma$ on $S_k(\Gamma)$ by

$$T_\sigma f(Z_1, Z_2, \ldots, Z_\lambda) = f(Z_2, \ldots, Z_\lambda, Z_1).$$

Using this $T_\sigma$ and Hecke operators, we define a subspace $SS_k(\Gamma)$ of $S_k(\Gamma)$ as follows

$$SS_k(\Gamma) = \left\{ f \in S_k(\Gamma) \mid T_\sigma T(e) f = T(e) T_\sigma f \text{ for any } e \in \mathbb{R}(U_F, \text{GL}_2(F)) \right\}.$$

It is easy to see that this subspace is stable under the action of Hecke operators, and that if $f$ is a common eigen function for all Hecke operators, then

$$f \in SS_k(\Gamma) \iff g(\mathfrak{U}) = g(\mathfrak{U}^\sigma) = \ldots = g(\mathfrak{U}^{\sigma^{\lambda-1}}) \text{ for any integral ideal } \mathfrak{U}.$$

Our purpose is to show that this subspace $SS_k(\Gamma)$ is closely related with spaces of cusp forms of one variable, in fact, this subspace can be lifted from spaces of cusp forms of one variable.

Let $S_k(\text{SL}_2(\mathbb{Z}))$ be the space of cusp forms of weight $k$ with respect to $\text{SL}_2(\mathbb{Z})$. Let us introduce other spaces of cusp forms of one variable. From the condition on $F$, it follows that there exist $\lambda - 1$ characters mod $q$ of order $\lambda$ corresponding to the extension $F/\mathbb{Q}$ in the sense of class field theory. We denote them by $\chi_i$, $1 \leq i \leq \lambda - 1$. For each character $\chi_i$, we denote by $S_k(\Gamma_0(q), \chi_i)$ the space of cusp forms $g$ which satisfy

$$g(\gamma Z) = (cz + d)^k \chi_i(d) g(Z) \text{ for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(q),$$

where

$$\Gamma_0(q) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) \mid c \equiv 0 \mod q \right\}.$$

The Hecke ring $\mathbb{R}(U_F, \text{GL}_2(\mathfrak{M}))$ acts on these spaces of cusp forms. On $S_k(\text{SL}_2(\mathbb{Z}))$, it acts in the usual manner. On the other spaces, we make it act in the following way. For a prime $p$, let $T(p)$ and $T(p, p)$ be the elements of $\mathbb{R}(U_F, \text{GL}_2(\mathfrak{M}))$
given in the next section. For \( p \neq q \), \( T(p) \) and \( T(p, p) \) acts in the usual manner. For \( p = q \), we define the action of \( T(q) \) and \( T(q, q) \) on \( S_k(\Gamma_0(q), \chi_1) \) by

\[
T(q) \, g = g \left[ \Gamma_0(q) \begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix} \Gamma_0(q) \right]_{k, \chi_1} + g \left[ \Gamma_0(q) \begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix} \Gamma_0(q) \right]^{*}_{k, \chi_1},
\]

where \( \left[ \Gamma_0(q) \begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix} \Gamma_0(q) \right]_{k, \chi_1} \) is the action of the double coset \( \Gamma_0(q)(1 \ 0 \ 0 \ q)03930(q) \) defined in G. SHIMURA [10], and \( [ \ ]_{k, \chi_1}^{*} \) means the adjoint operator of \( [ \ ]_{k, \chi_1} \) with respect to the Petersson inner product. To compare the above two kinds of representations of Hecke rings, namely the representation of \( R(\mathbb{U}_F, \text{GL}_2(F_\Lambda)) \) on \( SS_k(\Gamma) \) and those of \( R(\mathbb{U}_\mathbb{Q}, \text{GL}_2(\mathbb{Q}_\Lambda)) \) on \( S_k(\text{SL}_2(\mathbb{Z})) \) and \( S_k(\Gamma_0(q), \chi_1) \), we define a natural homomorphism \( \lambda \) from \( R(\mathbb{U}_F, \text{GL}_2(F_\Lambda)) \) to \( R(\mathbb{U}_\mathbb{Q}, \text{GL}_2(\mathbb{Q}_\Lambda)) \) in the next section. First assuming this \( \lambda \), we will state our theorem. By means of \( \lambda \), the spaces \( S_k(\mathbb{Q}_\Lambda) \) and \( S_k(\Gamma_0(q), \chi_1), 1 \leq 1 \leq k - 1 \), can be regarded as \( R(\mathbb{U}_F, \text{GL}_2(F_\Lambda)) \)-modules. On these notations, we can prove [7], the following theorem.

**Theorem.** - If \( k > 4 \), there exists a subspace \( S \) of \( \bigoplus_{i=1}^{k-1} S_k(\Gamma_0(q), \chi_1) \) such that

\[
SS_k(\Gamma) = S_k(\text{SL}_2(\mathbb{Z})) \oplus S,
\]

and \( \bigoplus_{i=1}^{k-1} S_k(\Gamma_0(q), \chi_1) \) is isomorphic to \( R(\mathbb{U}_\mathbb{Q}, \text{GL}_2(\mathbb{Q}_\Lambda)) \)-modules.

Let \( g \in S_k(\text{SL}_2(\mathbb{Z})) \) be a common eigen function for all Hecke operators and let \( f \in SS_k(\Gamma) \) be a common eigen function for all Hecke operators which corresponds to \( g \) in the above isomorphism, then it holds the following relation between the associated Dirichlet series \( \varphi_g \) of \( g \) and \( \varphi_f \) of \( f \), namely,

\[
\varphi_f(s) = \varphi_g(s) \prod_{i=1}^{k-1} \varphi_g(s, \chi_i),
\]

where

\[
\varphi_g(s, \chi_i) = \sum_{n=1}^{\infty} a_n \chi_i(n) n^{-s} \quad \text{for} \quad \varphi_g(s) = \sum_{n=1}^{\infty} a_n n^{-s}.
\]

This theorem can be considered an analogue for automorphic forms of the decomposition theorem of Dedekind zeta-functions.

The above theorem can be derived easily from the following theorem on trace of Hecke operators.

**Theorem.** - If \( k > 4 \),

\[
\text{tr} \, T(e)/SS_k(\Gamma) = \text{tr} \, T(\lambda(e))/S_k(\text{SL}_2(\mathbb{Z})) + \frac{1}{2} \sum_{i=1}^{k-1} \text{tr} \, T(\lambda(e))/S_k(\Gamma_0(q), \chi_i)
\]

for any \( e \in R(\mathbb{U}_F, \text{GL}_2(F_\Lambda)) \), where \( \text{tr} \, T(e)^*/\text{tr} \, T(e) \) is the trace of \( T(e^*) \) on the space \( * \).

**Remark.** - The above theorem is a generalization and a refinement of the result of
K. DOI and H. NAGANUMA ([2], [6]), which treated the lifting for quadratic extensions. H. JACQUET [3] studied the lifting for quadratic extensions from the viewpoint of representation theory. Alternative proofs for K. DOI and H. NAGANUMA's result are given by D. ZAGIER [12] and S. T. ASAI [1] treated the lifting in the case of imaginary quadratic extensions over \( \mathbb{Q} \).

2. In this section, we give the definition of \( \lambda \). Since it is known that
\[
\text{R}(u_F, \text{GL}_2(F_A)) = \bigotimes_p \text{R}(\text{GL}_2(\mathcal{O}_p), \text{GL}_2(F_p)),
\]
\[
\text{R}(u_Q, \text{GL}_2(\mathbb{Q}_A)) = \bigotimes_p \text{R}(\text{GL}_2(\mathbb{Q}_p), \text{GL}_2(\mathbb{Q}_p)),
\]
it is enough to define a homomorphism \( \lambda_p \) from \( R_p = \text{R}(\text{GL}_2(\mathcal{O}_p), \text{GL}_2(F_p)) \) to \( R_p = \text{R}(\text{GL}_2(\mathbb{Q}_p), \text{GL}_2(\mathbb{Q}_p)) \) for each prime ideal \( p \) of \( F \), where \( p \) is a prime such as \( p \mid \mathfrak{p} \). Let \( T(p) \) and \( T(p, p) \) (resp. \( T(p) \) and \( T(p, p) \)) be the elements of \( R_p \) (resp. \( R_p \)) given by the double cosets
\[
\text{GL}_2(\mathcal{O}_p) \left< \begin{pmatrix} 1 & 0 \\ 0 & \pi \end{pmatrix} \right> \text{GL}_2(\mathcal{O}_p) \quad \text{and} \quad \text{GL}_2(\mathcal{O}_p) \left< \begin{pmatrix} 1 & 0 \\ 0 & \pi \end{pmatrix} \right> \text{GL}_2(\mathcal{O}_p)
\]
(resp. \( \text{GL}_2(\mathbb{Q}_p) \left< \begin{pmatrix} 1 & 0 \\ 0 & \pi \end{pmatrix} \right> \text{GL}_2(\mathbb{Q}_p) \) and \( \text{GL}_2(\mathbb{Q}_p) \left< \begin{pmatrix} 1 & 0 \\ 0 & \pi \end{pmatrix} \right> \text{GL}_2(\mathbb{Q}_p) \)) respectively, where \( \pi \) is a prime element of \( \mathcal{O}_p \). We denote by \( R^I_p \) (resp. \( R^I_p \)) the subring of \( R_p \) (resp. \( R_p \)) generated by \( T(p) \) and \( T(p, p) \) (resp. \( T(p) \) and \( T(p, p) \)). If we put
\[
T(p) = X + Y \quad (\text{resp.} \quad T(p) = X + Y),
\]
\[
T(p, p) = N_pXY \quad (\text{resp.} \quad T(p, p) = pxy)
\]
we can embed \( R_p \) (resp. \( R_p \)) into the polynomial ring \( \mathbb{Q}[X, Y] \) (resp. \( \mathbb{Q}[x, y] \)) of two variables over \( \mathbb{Q} \). Now, consider the mapping from \( \mathbb{Q}[X, Y] \) to \( \mathbb{Q}[x, y] \) given by
\[
X \mapsto x^f, \quad Y \mapsto y^f,
\]
where \( f \) is an integer such that \( N_p = p^f \). Then we see easily that this mapping can be extended to a homomorphism from \( R_p \) to \( R_p \).

2. On this section, we give a numerical example of our theorem. We take as \( F \) the maximal real subfield of 7-th root of unity, then \( [F: \mathbb{Q}] = 3 \), and \( F \) satisfies the condition in § 1. Let \( \chi \) be the character mod 7 of order 3 given by \( \chi(3) = \omega, \omega = (-1 + \sqrt{-1})/2 \). For \( k = 4 \), we have \( \dim S_4(\Gamma) = 1 \) and \( \dim S_4(\Gamma_0(7), \chi) = 1 \). In this case, the subspace \( \mathbb{S}_4(\Gamma) \) coincides with \( S_4(\Gamma) \), hence \( S_4(\Gamma) \) is isomorphic to \( S_4(\Gamma_0(7), \chi) \) as \( R(u_F, \text{GL}_2(F_A)) \)-modules. Let \( f \) (resp. \( g \)) be an element of \( S_4(\Gamma) \) (resp. \( S_4(\Gamma_0(7), \chi) \)) with the associated Dirichlet series
\[
\varphi_f(s) = \sum \text{C}(\mathfrak{a}) n^{r_\mathfrak{a}} \quad (\text{resp.} \quad \varphi_g(s) = \sum_{n=1}^{\infty} a_n n^{-s}).
\]

We may assume \( \text{C}(\mathfrak{O}) = 1 \) and \( a_1 = 1 \). Then our theorem asserts that it holds the
following relation between \( C(p) \) and \( a_p \), namely

\[
C(p) = \begin{cases} 
\frac{a_p}{p} & (p) = pp'p'' \\
\frac{a_p^3 - 3\chi(p)p^3}{a_p} & (p) = p \\
a_p^{-1}a_p & (p) = p^3 
\end{cases}
\]

where \( p, p', p'' \) are the distinct prime divisors of \( (p) \). This relation can be checked for several \( p \) and \( p' \). The coefficients \( a_p \) can be calculated by Eichler-Selberg's trace formula using the class numbers of imaginary quadratic fields. On the other hand, \( C(p) \) can be obtained by Shimizu's trace formula \([8]\) using the class numbers of totally imaginary quadratic extensions of \( F \). For example, to calculate \( C((2)) \), we need the following class numbers.

\[
h(F(\sqrt{-8})) = 1, \quad h(F(\sqrt{-7})) = 1, \quad h(F(\sqrt{a^2 - 8})) = 1, \quad h(F(\sqrt{a^2 + 2a - 7})) = 1.
\]

Here \( h(K) \) is the class number of \( K \) and \( a \) is a root of the equation

\[
x^3 + x^2 - 2x - 1 = 0.
\]

On this way, we have the following table.

<table>
<thead>
<tr>
<th>( p )</th>
<th>( \chi(p) )</th>
<th>( a_p )</th>
<th>( p )</th>
<th>( C(p) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>( \omega^2 )</td>
<td>2( \omega )</td>
<td>(2)</td>
<td>-40</td>
</tr>
<tr>
<td>3</td>
<td>( \omega )</td>
<td>7( \omega^2 )</td>
<td>(3)</td>
<td>-224</td>
</tr>
<tr>
<td>7</td>
<td>0</td>
<td>7 - 14( \omega )</td>
<td>(2 - ( a ))</td>
<td>28</td>
</tr>
<tr>
<td>13</td>
<td>1</td>
<td>-14</td>
<td>( (a^2 + 1) )</td>
<td>-14</td>
</tr>
<tr>
<td>29</td>
<td>1</td>
<td>58</td>
<td>( (3 - \alpha) )</td>
<td>58</td>
</tr>
</tbody>
</table>

4.1. - Let \( F \) be as in § 1, and \( \mathfrak{a} \) an integral ideal of \( F \) such as \( \mathfrak{a} = \mathfrak{a} \), then we can define a subspace \( S_k(\Gamma_0(\mathfrak{a})) \) of \( S_k(\Gamma_0(\mathfrak{a})) \) in the same way as in § 1, where \( \Gamma_0(\mathfrak{a}) = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma, \quad c \equiv 0 \mod \mathfrak{a} \} \), and we can prove a similar but more complicated result on this \( S_k(\Gamma_0(\mathfrak{a})) \). Also, in the case of definite quaternion algebras, we can consider a similar problem, and the case where \( \mathfrak{a} \geq 3 \) has been treated by H. Hijikata. In the case of quadratic extensions, we can prove the following. Let \( F \) be \( \mathbb{Q}(\sqrt{q}) \) with a prime \( q \), \( q \equiv 1 \mod 4 \), and \( B \) a definite quaternion algebra over \( \mathbb{Q} \) which ramifies at \( q \) and at the archimedean prime. Let \( R \) be a maximal order of \( B \otimes \mathbb{F} \) which satisfies \( \sigma R = R \), where \( \sigma \) is the generator of \( \text{Gal}(\mathbb{Q}/q) \). For a non-negative even integer \( k \), let \( \mathcal{M}(\text{id}, [k, k]) \) be the space of continuous functions on \( (B \otimes F)_A^x \) defined in H. Shimizu \([8]\) with respect to the open subgroup \( \prod \mathbb{R}^x \times \prod \mathbb{R}^x \) of \( (B \otimes F)_v^x \). We can define the action of \( T_\sigma \) on \( \mathcal{M}(\text{id}, [k, k]) \) by means of the action of \( \sigma \) on \( (B \otimes F)_A^x \), and in these notations we can prove the following theorem.

**Theorem.** - For any \( c \in \bigotimes_{p \neq \mathfrak{a}} R^x_p, (B \otimes F)_A^x \), it holds
where $q$ is the prime ideal such as $q^2 = (q)$ and $(q)$ is the quadratic residue symbol mod $q$.

4. 2. - The theorem in § 1 has been generalized from the viewpoint of representation theory by T. SHINTANI [11] and R. P. LANGLANDS [5] by a similar method as ours. Especially, R. P. LANGLANDS found an important application of his generalization of our theorem to the conjecture of Artin on the poles of Artin's $L$-functions.

REFERENCES


(Texte reçu le 21 janvier 1977)