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Subdirect sums of integers and reals

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1. Introduction and statement of the main theorems.

The concept of a subdirect sum of integers is important in the study of abelian latticed-ordered groups ("\( \ell \)-groups") since WEINBERG [12] has shown that a free abelian \( \ell \)-group is a subdirect sum of integers and hence each abelian \( \ell \)-group is a homomorphic image of a subdirect sum of integers. In this paper, those \( \ell \)-groups which are subdirect sums of integers are characterized. We also characterize those \( \ell \)-groups which are subdirect sums of subgroups of the naturally ordered additive group \( R \) of real numbers. TOPPING [10] has shown that each vector lattice is a homomorphic image of such an \( \ell \)-group.

PAPPERT [9] has determined a necessary and sufficient condition for a vector lattice to be a subdirect sum of reals, and BERNAU [2] has shown that with a slight modification her theory applies to an arbitrary \( \ell \)-group. Both of these authors use the fact that an archimedean \( \ell \)-group can be represented by almost finite functions on a Stone space to obtain their results. Our condition is simpler, and the proof is elementary.

In [3], BERNAU characterizes those subdirect sums of integers which contain the small sum, and those which contain a dense subset of bounded elements. We can also characterize these classes of \( \ell \)-groups. These and other special cases and corollaries of our two main theorems are contained in Section 3.

For each \( \lambda \in \Lambda \), let \( G_\lambda \) be a totally ordered group ("\( o \)-group") that is \( o \)-isomorphic to a subgroup of \( R \). Thus, each \( G_\lambda \) is an archimedean \( o \)-group, or equivalently an \( o \)-group without proper convex subgroups. \( \prod G_\lambda \) will denote the large or unrestricted direct sum of the \( G_\lambda \) ordered pointwise, the large cardinal sum of the \( G_\lambda \), and \( \sum G_\lambda \) will denote the \( s \)-all cardinal sum of the \( G_\lambda \). In particular, \( \prod G_\lambda \) is an \( \ell \)-group, and \( \sum G_\lambda \) is an \( \ell \)-ideal of \( \prod G_\lambda \). If there exists an \( \ell \)-isomorphism of an \( \ell \)-group \( G \) onto a subdirect sum of \( \prod G_\lambda \), then we say that \( G \) is a subdirect sum of reals. If, in addition, each \( G_\lambda \) is cyclic, then we say that \( G \) is a subdirect sum of integers.

Let \( G \) be an \( \ell \)-group, \( G^+ = \{ g \in G \mid g > 0 \} \), and let \( Z^+ \) be the set of all strictly positive integers. An element \( x \in G^+ \) will be called real, if there exists a map \( y \rightarrow \overline{y} \) of \( G^+ \) into \( Z^+ \) such that:
(I) \((\bar{y}x - y) \wedge (\bar{z}x - z) \not\leq 0\) for all \(y, z \in G^+\).

If, in addition, for all \(y \in G^+\) and all \(n \in \mathbb{Z}^+\):

(II) \(\bar{y} = 1\) implies \(\bar{ny} = 1\),

(III) \(x \geq 2y\) implies \(\bar{y} = 1\),

then \(x\) will be called an integral element of \(G\).

**THEOREM 1.** - An \(\lambda\)-group \(G\) is a subdirect sum of reals if, and only if, each \(y \in G^+\) exceeds a real element.

**THEOREM 2.** - An \(\lambda\)-group \(G\) is a subdirect sum of integers if, and only if, each \(y \in G^+\) exceeds an integral element.

2. **Proofs of theorems 1 and 2.**

In all that follows, let \(G \neq 0\) be an \(\lambda\)-group. A convex \(\lambda\)-subgroup \(M\) of \(G\) is a subgroup that satisfies

\[ |x| \leq |a| \quad \text{for} \quad x \in G \quad \text{and} \quad a \in M \quad \text{implies} \quad x \in M , \]

or equivalently \(M\) is a sublattice and a convex subset of \(G\). In particular, the set of all right cosets of a convex \(\lambda\)-subgroup \(M\) is a distributive lattice such that, for all \(a, b \in G\),

\[ M + a \lor M + b = M + a \lor b , \]

and dually, where, by definition, \(M + a \geq M + b\) if \(x + a \geq b\) for some \(x \in M\).

A prime subgroup of \(G\) is a convex \(\lambda\)-subgroup for which the lattice of right cosets is totally ordered. For a convex \(\lambda\)-subgroup \(M\) of \(G\), the following properties are equivalent:

(a) \(M\) is prime;

(b) The set of convex \(\lambda\)-subgroups that contain \(M\) is a chain with respect to inclusion;

(c) If \(a, b \in G^+ \setminus M\), then \(a \wedge b \in G^+ \setminus M\).

Let \(\mathbb{M}\) be the set of all maximal prime subgroups of \(G\). If \(M \in \mathbb{M}\) and \(M \triangleleft G\), then \(G/M\) is \(o\)-isomorphic to a subgroup of \(R\) (notation \(G/M < R\)). For proofs of the above, see [6].

We shall consider the following properties of \(x \in G^+\):

(1) There exists \(M \in \mathbb{M}\) such that \(M + x\) covers \(M\) and, for each \(y \in G^+\), \(M + nx > M + y\) for some \(n \in \mathbb{Z}^+\);

(2) \(x\) is an integral element of \(G\);
LEMMA. - (1) \implies (2) \implies (3) \iff (4), and if each \( M \in \mathbb{R} \) is normal in \( G \), then (2) \implies (1).

Proof. - It follows from the definition of real and integral elements that (2) \implies (3).

(4) \implies (3): For each \( y \in G^{+} \), let \( \overline{y} \) be the least element in \( Z^{+} \) such that
\[ M + \overline{y}x > M + y. \]
Then, for all \( y, z \in G^{+} \),
\[ M + (\overline{y}x - y) \land (\overline{z}x - z) = M + (\overline{y}x - y) \land M + (\overline{z}x - z) > M. \]
Thus \((\overline{y}x - y) \land (\overline{z}x - z) \not\in 0\), and so \( x \) is real.

(1) \implies (2): Define \( \overline{y} \) as above. Since \( M + x \) covers \( M \), for \( y \in G^{+} \) and \( n \in Z^{+} \), the following are equivalent:
\[ \overline{y} = 1, \quad y \in M, \quad ny \equiv M \quad \text{and} \quad \overline{ny} = 1. \]
If \( y \in G^{+} \) and \( x \not> 2y \), then \( y \in M \), and so \( \overline{y} = 1 \). For if \( y \not\in M \), then \( M + x \not> M + 2y > M + y > M \), but this contradicts the fact that \( M + x \) covers \( M \). Therefore \( x \) is an integral element in \( G \).

(3) \implies (4): For \( y, z \in G^{+} \),
\[ [(\overline{y}x - y) \lor O] \land [(\overline{z}x - z) \lor O] = [(\overline{y}x - y) \land (\overline{z}x - z)] \lor O \in G^{+}. \]
Thus, \( Q_{x} = \{(\overline{y}x - y) \lor O \mid y \in G^{+}\} \) is contained in an ultrafilter \( K \) of \( G^{+} \). That is, \( 0 < a \land b \in K \) for all \( a, b \in K \), and \( K \) is maximal with respect to this property. It follows that
\[ N = \bigcup_{k \in K} k', \]
is a minimal prime subgroup of \( G \), and \( K = G^{+} \setminus N \), where
\[ k' = \{g \in G \mid |g| \land k = 0 \} \]
is the polar of \( k \). This is theorem 5.1 in [7], and this result is also implicit in [1] and [8].

(A) \[ N + \overline{y}x > N + y, \quad \text{for each} \quad y \in G^{+}. \]
For \((\overline{y}x - y) \lor O \in K = G^{+} \setminus N \), and hence \( N + (\overline{y}x - y) \lor O > N \), and so \[ N + \overline{y}x - y > N. \]
Since the convex \( \lambda \)-subgroups of \( G \) that contain \( N \) form a chain, there is a unique convex \( \lambda \)-subgroup \( M \supset N \) that is maximal, with respect to \( x \not\in M \).

\[ M \in \mathbb{P} \]

For if \( y \in G^+ \), then \( N + \bar{y}x > N + y \), and hence \( a + \bar{y}x > y > 0 \) for some \( a \in N \). But clearly, \( a + \bar{y}x \) is contained in any convex \( \lambda \)-subgroup that properly contains \( M \). Therefore, \( G \) covers \( M \), and hence \( M \in \mathbb{P} \). It follows from (A) that

\[ M + (\bar{y} + 1)x > M + \bar{y}x > M + y \]

Therefore (4) is satisfied.

To complete the proof, we need to show that (2) \( \implies \) (1), provided that each \( M \in \mathbb{P} \) is normal in \( G \). Let \( x \) be an integral element, and let \( M \) and \( N \) be as above. Suppose (by way of contradiction) that \( M + x > M + y > M \) for some \( y \in G \).

Then, since

\[ M + y \vee 0 = M + y \vee M = M + y \quad \text{and} \quad M + x \wedge y = M + x \wedge M + y = M + y \]

we may assume that \( x > y > 0 \). Now, \( x = x - y + y \), and since \( x - y , y \in G^+ \setminus M \), and \( M \) is prime, \( d = (x - y) \wedge y \in G^+ \setminus M \). Clearly, \( x \geq 2d \), and hence \( \overline{d} = 1 \) and \( \overline{nd} = 1 \) for all \( n \in \mathbb{Z}^+ \). Thus,

\[ M + x = M + \overline{nd}x > M + nd \geq M + d > M , \quad \text{for all} \ n \in \mathbb{Z}^+ , \]

but this is impossible, because \( G/M < R \).

**COROLLARY.** - Suppose that each \( M \in \mathbb{P} \) is normal in \( G \), and consider \( x \in G^+ \).

(a) \( x \) is a real element of \( G \) if, and only if, \( x \in G \setminus M \) for some \( M \in \mathbb{P} \).

(b) \( x \) is an integral element of \( G \) if, and only if, \( M + x \) covers \( M \) for some \( M \in \mathbb{P} \).

**Proof.** - This is an immediate consequence of the lemma and the fact that \( G/M < R \) is an archimedean \( \alpha \)-group for each \( M \in \mathbb{P} \).

**BYRD [4]** has shown that \( G \) is a subdirect sum of \( \alpha \)-groups if, and only if, for each prime subgroup \( M \) and each \( g \in G \), \(-g + M + g \leq M \) or \(-g + M + g \supset M \).

Thus, for this class of \( \lambda \)-groups, each \( M \in \mathbb{P} \) is normal.

**Proof of theorem 1.** - Suppose that \( G \) is a sublattice and a subdirect sum of \( \coprod R_\lambda \ (\lambda \in \Lambda) \), where each \( R_\lambda \in R \). If \( x \in G^+ \), then \( x_\lambda > 0 \) for some \( \lambda \in \Lambda \). Let \( M = \{ g \in G \mid g_\lambda = 0 \} \). Then \( M \in \mathbb{P} \) and \( x \in G \setminus M \). Thus, by the corollary, \( x \) is real, and so each \( x \in G^+ \) is real.
Conversely, suppose that each element in $G^+$ exceeds a real element, and consider $y, z \in G^+$. There exists a real element $x \leq z$. Thus $\overline{y}x \leq \overline{y}y$, and hence $\overline{yz} \leq \overline{y}$. Therefore $G$ is archimedean, and hence abelian. By the corollary, $x \in G \setminus M$ for some $M \in \mathbb{M}$, and hence $z \in G \setminus M$. Therefore, $0 = \cap \{M \mid M \in \mathbb{M}\}$, and so $G$ is a subdirect sum of reals.

Proof of theorem 2. - Suppose that $G$ is a sublattice and a subdirect sum of $\Pi Z_{\lambda}$ (where each $Z_{\lambda} = \mathbb{Z}$). If $g \in G^+$, then $g \geq x > 0$ for some $x \in G$, where $x_{\lambda} = 1$ for some $\lambda \in \Lambda$. Let $M = \{g \in G \mid g_{\lambda} = 0\}$. Then $M \in \mathbb{M}$, and $M + x$ covers $M$, and hence, by the corollary, $x$ is integral. Therefore each element in $G^+$ exceeds an integral element.

Conversely, suppose that each element in $G^+$ exceeds an integral element. Then, as in the proof of theorem 1, $G$ is abelian. Let $\mathcal{E} = \{M \in \mathbb{M} \mid G/M$ is cyclic\}. Then, by the corollary, $\cap \{M \mid M \in \mathcal{E}\}$ must be zero, since it contains no integral element. Therefore $G$ is a subdirect sum of integers.

3. Special cases of theorems 1 and 2.

An element $s \in G^+$ is called basic, if $\{g \in G \mid 0 \leq g \leq s\}$ is totally ordered.

PROPOSITION A. - For an $\ell$-group $G$, the following properties are equivalent:

1. $G$ is a subdirect sum of reals that contains the small sum;
2. Each element in $G^+$ exceeds a real element that is also basic;
3. $G$ is archimedean, and each element in $G^+$ exceeds a basic element.

Proof. - It is shown in [5] that (1) $\iff$ (3). If each element in $G^+$ exceeds a real element, then $G$ is archimedean, and hence (2) $\implies$ (3). If (1) holds, then each element in $G^+$ is real, and hence (1) and (3) imply (2).

There are many other equivalent conditions proven in the literature (see for example [11]).

An element $a \in G^+$ is an atom, if it covers 0. It is shown in [5] that $x$ is a basic element in an archimedean $\ell$-group $G$ if, and only if, $x'' < R$, and $G$ is the cardinal sum of $x''$ and $x'$. Thus a basic element $x$ is integral if, and only if, $x''$ is cyclic, and hence if, and only if, $x$ is an atom.

PROPOSITION B. - For an $\ell$-group $G$, the following properties are equivalent:

1. $G$ is a subdirect sum of integers that contains the small sum;
2. Each element in $G^+$ exceeds an integral element that is also basic;
(3) \( G \) is archimedean, and each element in \( G^+ \) exceeds an atom.

**Proof.** - Clearly (1) \( \Rightarrow \) (2) \( \Rightarrow \) (3).

(3) \( \Rightarrow \) (1): Since each atom is a basic element, it follows from proposition A that \( G \) is a subdirect sum of reals that contains the small sum. Thus, without loss of generality,

\[ \sum R_\lambda \subseteq G \subseteq \prod R_\lambda, \]

where \( R_\lambda \subseteq \mathbb{R} \) for each \( \lambda \in \Lambda \). If \( R_\lambda \) is not cyclic, then there exists an element in \( R_\lambda^+ \subseteq G^+ \) that does not exceed an atom, a contradiction. Therefore (1) holds.

An element \( s \in G^+ \) is called singular, if \( a \land (s - a) = 0 \) for each \( 0 \leq a \leq s \).

**PROPOSITION C.** - For an \( \lambda \)-group \( G \), the following properties are equivalent:

(1) \( G \) is a subdirect sum of integers, and each element in \( G^+ \) exceeds a bounded element;

(2) Each element in \( G^+ \) exceeds an integral element that is also singular;

(3) \( G \) is a subdirect sum of reals, and each element in \( G^+ \) exceeds a singular element.

**Proof.** - In [7], it is shown that (1) \( \iff \) (3), and clearly (2) \( \Rightarrow \) (3).

Suppose that (1) and (3) hold. Then, without loss of generality, \( G \subseteq \bigcap Z_\lambda \), where for each \( \lambda \in \Lambda \), \( Z_\lambda = Z \), and in [7], it is shown that if \( s \in G \) is singular, then \( s_\lambda = 1 \) or \( 0 \). Thus, it follows that \( s \) is integral, and hence we have (2).

BERNAU [3] has established (1) \( \iff \) (3) in proposition B, and has derived a condition that is equivalent to (1) in proposition C.

Suppose that \( x \in G^+ \) is real, and let \( A_x \) be the set of all maps \( \pi : G^+ \rightarrow \mathbb{Z}^+ \), such that for all \( y, z \in G^+ \),

\[ ((\nu y)x - y) \land ((\nu z)x - z) \leq 0. \]

For \( \alpha, \beta \in A_x \), define \( \alpha \preceq \beta \) if \( y\alpha \preceq y\beta \) for all \( y \in G^+ \). Then \( (A, \preceq) \) is a po-set, and each element in \( A_x \) exceeds a minimal element in \( A_x \). For if \[ \{a_\lambda \mid \lambda \in \Lambda \} \]

is a chain in \( A_x \), then for each \( y \in G^+ \), define

\[ y\pi = \min\{y\alpha_\lambda \mid \lambda \in \Lambda \}. \]

If \( y, z \in G^+ \), then there exists \( \lambda \in \Lambda \) such that \( y\alpha_\lambda \) and \( z\alpha_\lambda \) are minimal,
and so
\[(yn - y) \wedge (zn - z) = ((yw_\lambda)x - y) \wedge ((zw_\lambda)x - z) \leq 0 .\]
Therefore \( n \in A_x \), and hence, by Zorn's lemma, each map in \( A_x \) exceeds a minimal map.

Definition. - A real element \( x \in G^+ \) for which there exists a minimal map \( y \rightarrow \bar{y} \) in \( A_x \) that also satisfies (II), will be called a \(*\)-element.

PROPOSITION D. - For an \( \lambda \)-group, the following properties are equivalent:

1. Each element in \( G^+ \) exceeds a \(*\)-element;
2. \( G \) is \( \lambda \)-isomorphic to a subdirect sum \( \prod Z_\lambda \), where for each \( \lambda \in \Lambda \), \( Z_\lambda = Z \), and \( G_\lambda = \{ g \in G \mid g_\lambda = 0 \} \) is both a maximal and a minimal prime subgroup of \( G \).

Proof.

(1) \( \Rightarrow \) (2): Since each \(*\)-element is real, it follows from theorem 1 that \( G \) is abelian. Let \( x \) be a \(*\)-element in \( G \), and let \( y \rightarrow \bar{y} \) be a minimal map in \( A_x \) that also satisfies (II). Construct \( M \) and \( N \) as in the proof of (3) \( \Rightarrow \) (4) in the lemma. Since \( N + \bar{y}x > N + y \) for all \( y \in G^+ \), and the map \( y \rightarrow \bar{y} \) is minimal, it follows that \( \bar{y} \) is the least element in \( Z^+ \) for which \( N + \bar{y}x > N + y \). Suppose (by way of contradiction) that \( M \not\geq N \), and pick \( 0 < z \in M \setminus N \), and let \( y = -(x \wedge z) + x \). Then,

\[ M + x = M + y \quad \text{and} \quad N + x > N + y . \]
Therefore \( \bar{y} = 1 \), and hence \( \bar{2y} = 1 \), but clearly \( N + 2yx = N + x < N + 2y \), that is a contradiction. Thus, \( N = M \) is both maximal and minimal. If \( M + x > M + y \), then \( \bar{y} = 1 \), and hence \( M + x = M + M + ny \geq M + ny \) for all \( n \in Z^+ \). Thus, since \( G/M < R \), it follows that \( y \in M \), and so \( G/M \) is cyclic.

(2) \( \Rightarrow \) (1): We may assume that \( G \subseteq \prod Z_\lambda \). If \( z \in G^+ \), then \( z \geq x \in G^+ \), where \( x_\lambda = 1 \) for some \( \lambda \in \Lambda \). For \( y \in G^+ \), define \( \bar{y} \) to be the least element in \( Z^+ \) such that \( \bar{y}x_\lambda > y_\lambda \). Then, the map \( y \rightarrow \bar{y} \) satisfies (I), (II) and (III). It remains to be shown that this map is minimal in \( A_x \). Suppose that \( y \rightarrow \tilde{y} \) is a map in \( A_x \), and \( \bar{y} \leq \tilde{y} \) for all \( y \in G^+ \). Construct \( M \) and \( N \) as above, using the map \( y \rightarrow \tilde{y} \). In particular, \( N + \tilde{y}x > N + y \) and \( M + \tilde{y} \geq M + y \) for all \( y \in G^+ \).

If \( M \neq C_\lambda \), then there exists \( y \in G^+ \) such that \( y_\lambda = 0 \) and \( y \notin M \). Since \( y_\lambda = 0 \), \( \bar{y} = 1 \), and so \( n\bar{y} = n\bar{y} = 1 \) for all \( n \in Z^+ \), but this means that
M + x > M + ny for all n ∈ Z⁺, and this contradicts the fact that G/H < R.

If M = G₀, then, since G₀ is a minimal prime, M = N, and so M + yx > M + y for all y ∈ G⁺, and it follows that y = y for all y ∈ G⁺. Therefore x is a *-element, and hence (1) is satisfied.

REFERENCES