

SÉMINAIRE DUBREIL. ALGÈBRE ET THÉORIE DES NOMBRES

ROLANDO E. PEINADO

Finite semigroups admitting ring structure

Séminaire Dubreil. Algèbre et théorie des nombres, tome 23, n° 2 (1969-1970), exp. n° DG 13, p. DG1-DG3

http://www.numdam.org/item?id=SD_1969-1970__23_2_A12_0

© Séminaire Dubreil. Algèbre et théorie des nombres
(Secrétariat mathématique, Paris), 1969-1970, tous droits réservés.

L'accès aux archives de la collection « Séminaire Dubreil. Algèbre et théorie des nombres » implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
<http://www.numdam.org/>

FINITE SEMIGROUPS ADMITTING RING STRUCTURE

by Rolando E. PEINADO

A semigroup S is said to admit a ring structure if S is isomorphic to the multiplicative structure of some (associative) ring R (in case S has no zero elements, we consider S^0), where R belongs to a class of rings \mathcal{R} , we call S an \mathcal{R} -semigroup. It is clear that the class of \mathcal{R} -semigroup is not empty since the multiplicative structure of any ring is such a semigroup. By considering left (right) zero semigroup with more than one element, it is clear that not all semigroups are \mathcal{R} -semigroups. The problem of characterizing the class of all \mathcal{R} -semigroups, where \mathcal{R} is the class of all rings, was first formally stated by P. DUBREIL [1] and KOGALOVSKIJ in [2], shows that it is impossible to give an axiomatic characterization of such class.

Here we proposed to give a brief account of results obtained by several people on the specific case of finite \mathcal{R} -semigroup. For a more complete survey of the general case (includes unique addition \mathcal{R} -semigroups) and a more complete bibliography, see R. E. PEINADO [3], to appear in Semigroup Forum.

If S is a finite \mathcal{R} -semigroup without zero (S^0 admits a ring structure), and suppose R is a ring whose multiplicative structure is isomorphic to S^0 . Then R has no zero divisors, and hence it is cancellative and being finite, it is a finite integral domain, and hence R a field, moreover it is a Galois field of order p^n . This implies that S has order $p^n - 1$ and is a cyclic group. The converse is obvious.

THEOREM 1. - A finite semigroup S without zero is an \mathcal{R} -semigroup if, and only if, S is a cyclic group of order $p^n - 1$ for some prime p .

This theorem enables us to restrict our attention to finite semigroups with zero element and zero divisors. In considering finite \mathcal{R} -semigroup, the following results are important.

THEOREM 2. - Let R be a finite ring of order n , where

$$n = \prod_{i=1}^k p_i^{t_i}, \quad i = 1, \dots, k, \quad t_i > 0.$$

Then R is isomorphic to the direct sum of p_i -rings R_i , $i = 1, \dots, k$ of order $p_i^{t_i}$

THEOREM 3. - Let S be a finite semigroup of order n ,

$$n = \prod p_i^{t_i}, \quad i = 1, \dots, k.$$

S is an R-semigroup if, and only if, S is the direct sum of R-semigroups S_i ,
of order p_i^{t_i} , i = 1 , ... , k .

THEOREM 4. - Let S be a finite cyclic semigroup ; then S is an R-semigroup
if, and only if, S is of order 1 or 2 .

If $n = p$, a prime number, then there exists two non-isomorphic rings of order p . The ring with trivial multiplication, all products equal to zero and the Galois field of order p .

THEOREM 5. - A finite semigroup S with zero of order p , a primer number, is
an R-semigroup if, and only if, S is a cyclic group with zero or a zero semigroup

COROLLARY. - A finite semigroup S with zero, of square-free order is an
R-semigroup if, and only if, S is a direct product of zero semigroups and/or
cyclic groups with zero.

THEOREM 6. - A finitely generated commutative semigroup S is an R-semigroup
if, and only if, S is finite.

THEOREM 7. - A finite R-semigroup of cube-free order is commutative.

For the integers modulo n , Z_n , we can obtain the following interesting result.

THEOREM 8. - Let $\bar{Z}_p^t = Z_{p^t} - \{0\}$. Then the multiplicative semigroup \bar{Z}_p^t
contains a subgroup $G(p^t)$ isomorphic to \bar{Z}_p , given by $G(p^t) = \langle X_p^t \rangle$, where,
for the prime $p \neq 2$ and $t > 1$,

$$X_p^t = X_p^{t-1} \quad (\text{modulo } p^{t-1})$$

and

$$(X_p)^{\frac{p-1}{2}} = p^t - 1 \quad (\text{modulo } p^t) ,$$

for $p = 2$, $G(2^t) = \langle 1 \rangle$,

for $t = 1$, $G(p) = \bar{Z}_p$.

REFERENCES

- [1] DUBREIL (P.). - Quelques problèmes d'algèbre liés à la théorie des demi-groupes, Colloque d'algèbre supérieure [1956. Bruxelles], p. 29-44. - Louvain, Centre-ric, 1957 (Centre belge de Recherches mathématiques).
- [2] KOGALOVSKIJ (S. R.). - On the multiplicative semigroups of rings [in Russian], Doklady Akad. Nauk. S. S. S. R., t. 140, 1961, p. 1005-1007.
- [3] PEINADO (R. E.). - On semigroups admitting ring structure, Semigroup Forum (to appear).

Rolando PEINADO
University of Puerto Rico
MAYAGUEZ (Porto-Rico)

(Texte reçu le 23 septembre 1970)
