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QUASI-REGULARITY IN SEMIGROUPS

by Robert H. OEHMKE

Recently there have been a large number of attempts to develop a structure theory for semigroups through results covering radicals, semisimplicity, simplicity, etc., in manners analogous to that accomplished for noncommutative rings.

In ring theory, there are a great many equivalent characterizations of the Jacobson radical for noncommutative rings. Some of these are in terms of external objects, e. g. modules, and others internal, e. g. maximal modular right ideals, quasi-regularity. In all most of cases, suitable analogues of these concepts can be made to semigroups. In some cases, several nonequivalent analogues can be made. The only real test for such analogues is how successful they are in uncovering the structure of arbitrary semigroups or of a particular class of semigroup.

Here, I would like to comment on two such analogues, and list some results concerning them.

In 1965, Rebecca SLOVER defined a right quasi regular element of a semigroup S to be an element which is not a left identity element for any right congruence which has one of its equivalence classes a right ideal, except the universal congruence.

In 1969, Wendell JONES modified the definition slightly to define a right quasi regular element to be one which is not a left identity element for any right congruence, except the universal congruence.

The effective difference between these two definitions can easily be seen in the class of groups. In a group, every element is right quasi regular under the Slover definition, but only in a cyclic group does one have right quasi regular elements under Jones' definition. In this case, the generators are the right quasi regular elements.

It can easily be seen that

J-<u>right quasi regularity</u> <u>implies</u> S-<u>right quasi regularity</u> .

With a definition of right quasi regularity, one is lead to a definition of a radical, namely the maximal ideal all of whose elements are right quasi regular. Clearly, such an ideal exists if one admits $\emptyset$ as an ideal.

THEOREM. - Jones radical = Slover radical or $\emptyset$.

THEOREM. - If S has a zero element, then Jones radical = Slover radical .

THEOREM. - If S contains a modular transitive right congruence, then the Jones radical is $\emptyset$.

Either radical has the properties :

(a) S/R is radical free ;

(b) If T is a right ideal of S , then the radical of T is the intersection of the radical of S and T ;

(c) If S and T are semigroups, and θ is a homomorphism of S onto T , then the homomorphic image of the radical of S is contained in the radical of T .

Hence, in the most immediate sense, either definition seems to qualify for a definition of a radical.

A successful radical theory must accomplish three things :

1° Describe the semisimple semigroups ;

2° Describe the radical semigroup ;

3° Describe how the arbitrary semigroup can be constructed from the above information.

Some partial results in this direction are available.

THEOREM. - A Jones radical semigroup is a periodic semigroup in which every idempotent is a left zero element.

THEOREM. - A Slover semisimple commutative semigroup in which either the acc or dcc holds on ideals is a direct sum of a finite number of copies of the simple lattice (0 , 1) .

THEOREM. - A Jones semisimple commutative semigroup in which either the acc or dcc holds on congruences is a direct sum of a finite number of copies of the simple lattice (0 , 1) and a finite number of simple groups.

There are other types of problems that hold some interest, mainly arising from analogous problems in ring theory. In particular, it can be shown that a ring in which there is precisely one element that is not r. g. r. is a division ring.

A similar question can be asked for semigroups, and resolved as the following.

We assume there is precisely one element e in S such that e is not quasi regular under Jones' definition. We denote this class of semigroups by C .

Now let V be any set containing an element e , and V^* be free semigroup generated by V . Define ω to be a weighting function on V into the integers modulo p for some prime p such that $\omega(x) = 0$ if, and only if, $x = e$. Extend ω to all of V^* by the property

$$\omega(bc) = \omega(b) + \omega(c) \quad .$$

Let α be the congruence defined by the relations

$$xyz = xez \ , \qquad xy = xe \ , \qquad yz = ez$$

$$y = e \ ,$$

for $\omega(y) = 0$, and let D be the class of all semigroups that are homomorphic images of $V^*|\alpha$ for arbitrary V and ω .

Let E be the class of all semigroups S that can be decomposed as the union of an idempotent e and a subsemigroup U such that e acts associatively on U , and U is a semigroup in which every element is of finite order and in which every idempotent is a left zero.

THEOREM. - C is the union of the classes D and E .

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