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Semimetrics, semiécart in ordered semigroups


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Completing a metric space is a classic construction. Starting with the space of rationals $\mathbb{Q}$ (or $\mathbb{Q}^n$) with its metric taking values in the ordered semigroup of positive rationals, its completion $\mathbb{R}$ (or $\mathbb{R}^n$) has a metric in the completion, in a suitable sense, of $\mathbb{Q}^+$, which is the ordered semigroup of positive reals. That this result admits natural generalizations is the main contention of this paper.

First, we abstract the properties of the semigroups $\mathbb{Q}^+$ or $\mathbb{R}^+$ in the definition of the "Abelian perfectly ordered semigroup".

**Definition.** An Abelian perfectly ordered semigroup (or Apo-semigroup, for short) is a triple $(S, +, \preceq)$ consisting of a set $S$, a binary operation $+$ defined on $S$ under which $(S, +)$ is a commutative semigroup with zero element $0$, and a relation of partial order $\preceq$ defined on $S$ such that the following conditions are satisfied:

(a) For arbitrary $x, y, z$ from $S$, $x \preceq y$ if, and only if, $(x + z) \preceq (y + z)$

(b) The set $H = \{ x \in S ; 0 < x \}$ is down-directed and weakly divisible: that is, if $x, y$ are in $H$, there is a $z$ in $H$ which is $\preceq x, \preceq y$; and if $x$ is in $H$, there is a $x'$ in $H$ such that $x' + x' \preceq x$.

It is not hard to show that the last condition (b) is true for $H$ if, and only if, it is true of some coinitial subset of $(H, \preceq)$.

We next define a semimetric or semiécart for a set $X$ into such a Apo-semigroup.

**Definition.** Given a set $X$, and an Apo-semigroup $(S, +, \preceq)$, a mapping $d$ of $X \times X$ in $S$ is called a semimetric for $X$ in $(S, +, \preceq)$ (is called a semiécart for $X$ in $(S, +, \preceq)$), if it satisfies the following condition: for any $x, y, z$ from $X$, $d(x, z) \preceq d(x, y) + d(y, z)$ (if it satisfies the following condition: given $h$ in $H$ there is $h'$ in $H$ such that for arbitrary $x, y, z$ from $X$, $d(x, y) \preceq h'$ and $d(y, z) \preceq h'$ imply $d(x, z) \preceq h$).

In view of our assumption (b) for $H$, it follows that a semimetric $d$ for $X$ in $(S, +, H)$ is ipso facto a semiécart for $X$ in $(S, +, \preceq)$.

Since the condition (a) for the ordered semigroup implies that it is cancellative (being also abelian), the semigroup \((S, +)\) can be isomorphically imbedded as a subsemigroup of a group \((G, +)\) of differences; and we can also now extend the partial order \(\leq\) from the subsemigroup (of elements of the form \(x - 0\)) to the whole group, by setting: 
\[(x - x') \leq (y - y') \text{ if, and only if, } (x + y) \leq (x' + y) \text{ in } (S, \leq).\]
Then, it is seen that \((G, +, \leq)\) is also an Apo-(semi-)group. We call it the "group-completion" of the Apo-semigroup.

Given the Apo-semigroup \((S, +, \leq)\), the set \(S\) has a "intrinsic" semimetric in the Apo-group \((G, +, \leq)\) which is the group completion of \((S, +, \leq)\); namely \(d\), given by 
\[d(x, x') = x' - x.\]

Note also that when \(d\) is a semimetric (or semiécart) for \(X\) in \((S, +, \leq)\), there is a conjugate semimetric \(d'\) given by 
\[d'(x, y) = d(y, x)\]
for any \(x, y\) from \(X\).

We pass on to define the "semiuniform spaces" and their "completions".

**Definition.** - A family \(U = (U_j; j \in J)\) of binary relations on a set \(X\) (indexed by a set \(J\)) is called a **semiuniformity** (or semiuniform structure) for \(X\) if the following conditions are true:

1. (U1) For each \(x\) of \(X\) and each \(j\) of \(J\), \((x, x) \in U_j\), that is all the relations \(U_j\) are reflexive.

2. (U2) Given \(j \in J\), there is a \(j' \in J\) such that the relational product \(U_j \circ U_{j'}\) is contained in \(U_j\).

We may call the family \(U\) a transitive family of relations when (U2) holds, and

3. (U3) For \(j, j' \in J\), there is a \(j'' \in J\), such that \(U_{j''}\) is contained in both \(U_j\) and \(U_{j'}\).

The semiuniformity is called a quasuniformity, if it satisfies also the following "symmetry" condition:

4. (U4) Given \(j \in J\), there is \(j'' \in J\), such that the reverse relation \(U_{j''} \circ U_j\) is contained in \(U_j\).

And finally, the quasuniformity is a uniformity (in the sense of A. WEIL), if the intersection of the \(U_j\) is the identity relation on \(X\).

A semiuniformity \(U\) for \(X\) determines a "conjugate" semiuniformity
\[U^{-1} = (U_j^{-1}; j \in J)\]
on obtained by taking the reverse relations for all the \(U_j\). \(U\) (and its conjugate) also determine a "symmetric" associate semiuniformity (or quasuniformity)
A semiuniformity \( U \) for \( X \) determines a topology \( T(U) \) for \( X \) when we take as a base of neighbourhoods at a point \( x \) of \( X \) the sets \( (U_j(x); j \in J) \) where, as usual, \( U_j(x) \) consists of the points \( y \) of \( X \) for which \( (x, y) \in U_j \). The topology \( T(S(U)) \) determined by the symmetric associate \( S(U) \) of \( U \), we shall call the "star topology" determined by \( U \), and denote it by \( T^*(U) \).

If now \( (D, \preceq) \) is any down-directed (indexing) set, a function \( s \) of \( D \) in \( X \) is called a \( (D, \preceq) \)-sequence in \( X \). Such a sequence is said to converge to a point \( x \) of \( X \) under a topology \( T \) for \( X \) if, for each neighbourhood \( N(x) \) of \( x \) in \( T \), we can find a \( d \) in \( D \) such that \( s(e) \) belongs to \( N(x) \) for each \( e \) (of \( D \)) which is \( \preceq d \). And such a \( (D, \preceq) \)-sequence \( s \) in \( X \) is called a Cauchy sequence of the semiuniform space \( (X, U) \). If, for each \( U_j \) in \( U \), we can find a \( d \) in \( D \) such that \( (s(e), s(e')) \in U_j \) whenever \( e, e' \) (of \( D \)) are \( \preceq d \). Clearly, a Cauchy sequence of \( (X, U) \) is also a Cauchy sequence of \( (X, S(U)) \), and vice-versa. It can be shown that any \( (D, \preceq) \)-sequence of \( X \), which converges to some point of \( X \) under \( T^*(U) \), is a Cauchy sequence of \( (X, U) \). The semiuniform space \( (X, U) \) is called a complete semi-uniform space if every Cauchy sequence of \( (X, U) \) converges to some point of \( X \) under \( T^*(U) \). It follows that \( (X, U) \) is complete if, and only if, \( (X, S(U)) \) is complete.

We state then the main theorem regarding completing a semiuniform space (which I have proved elsewhere).

**THEOREM 1.** - Given a semiuniform space \( (X, U) \) there is an associated complete semiuniform space \( (X^*, U^*) \), which we call the canonical completion of \( (X, U) \), such that: there is a bi-uniform bijection between \( (X, U) \) and a semiuniform subspace of \( (X^*, U^*) \); and every point of \( X^* \) is a limit of a Cauchy sequence of \( (X^*, U^*) \) consisting of points of this subspace only, the convergence being under the star topology of \( X^* \) determined by \( U^* \).

When we consider a set \( X \) with a semiécart (or semimetric) \( d \) in Apo-semigroup \( (S, +, \preceq) \), we get an associated semiuniformity \( U = (U_h; h \in H) \) for \( X \), when we set \( ((x, y) \in U_h) \iff (d(x, y) \leq h) \). This semiuniformity is symmetric if the semimetric is symmetric. In particular, for a Apo-semigroup the intrinsic semimetric for \( S \) in the "group completion" \( (G, +, \preceq) \) gives rise to an intrinsic semiuniformity for \( (S, +, \preceq) \). Then we have the following main results.

**THEOREM 2.** - The completion of an Apo-semigroup is also an Apo-semigroup; the completion of an Apo-group is an Apo-group. Upto an order- and semigroup-iso- morphism, the semiuniform completion of the group completion of an Apo-semigroup...
is the same as the group completion of the semiuniform completion of the Apo-semi-
group.

If a set $X$ has a semiécart $d$ in an Apo-semigroup $(S, +, \leq)$, then its cano-
nical completion (as a semiuniform space) has its semiuniformity derivable from a
semiécart in the canonical completion of the Apo-semigroup. This can also be trea-
ted as a semiécart in the group completion of this last complete semigroup.

If $X$ has a semimetric in a totally ordered Apo-semigroup or group, its canoni-
cal completion, as a semiuniform space, has its semiuniformity derivable from a
semimetric in the canonical completion of the Apo-semigroup or group, which would
also be totally ordered.

Details of proofs would be appearing in a paper shortly in the Proceedings of the
Czechoslovak Academy of Sciences [under a report of a Topology Conference, held at
Kanpur (India)].

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