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Inverse and partially ordered semigroups


<http://www.numdam.org/item?id=SD_1971-1972__25_2_A10_0>
We follow the notation and terminology of CLIFFORD and PRESTON [2].

Let \( S \) be an inverse semigroup with semilattice of idempotents \( E \), and let \( \rho \) denote the minimum group congruence [5] on \( S \). Then \( S \) is said to be reduced if \( E\rho = E \) (SAITO [7] used the term proper), and a congruence \( \tau \) on \( S \) is called reduced if \( S/\tau \) is reduced.

**THEOREM 1.** - Let \( S \) be an inverse semigroup. Then the congruence generated by \( \rho \cap \mathcal{R} \) is the minimum reduced congruence on \( S \).

**COROLLARY [7].** - If \( S \) is a reduced inverse semigroup, then \( \rho \cap \mathcal{R} \) is the identity congruence on \( S \).

The next result gives the structure of an arbitrary reduced inverse semigroup. The main idea behind the theorem is that each \( \rho \)-class of a reduced inverse semigroup is \( V \)-completed so as to build up an \( F \)-inverse semigroup; the structure of the latter is known [4], and the structure of the reduced inverse semigroup is then recovered. First, we introduce some notation.

Let \( E \) be a semilattice; then \( M(E) \) denotes the semilattice \( \{ a \leq H \leq E \mid EH = H \} \) under the operation of set multiplication. The mapping \( j : e \mapsto \text{End}(E) \) embeds \( E \) isomorphically in \( M(E) \). Further, given a group \( G \), a family \( \theta(G) = \{ \theta_g \mid g \in G \} \) of endomorphisms of \( M(E) \) is called compatible if it satisfies conditions (i), (ii) and (iii) of [4], theorem 4 for the semilattice \( M(E) \), together with the further condition:

(iv) \( \text{For each } F \in M(E) \text{ and } g \in G, F\theta_g = \bigcup \{(Ef)\theta_g \mid f \in F\} \).

Thus the family \( \theta(G) \) is specified by its action on \( E\jmath \).

**THEOREM 2.** - Let \( E \) be a semilattice, \( G \) a group, and \( \theta(G) \) a compatible family of endomorphisms of \( M(E) \). Denote by \( [E; G; \theta] \) the set \( \{ (Ee, g) \mid e \in E, g \in G, e \in E\theta_g \} \) under the operation

\[
(Ee, g)(Ef, h) = (Ee, (Ef)\theta_g, gh).
\]
Then \([E ; G ; \phi]\) is a reduced inverse semigroup, with semilattice of idempotents isomorphic to \(E\), and maximal group homomorphic image \(G\).

Conversely, given a reduced inverse semigroup \(S\) with semilattice of idempotents \(E\), \(S = [E ; S/\rho ; \phi]\) where for each \(H \in \mathcal{W}(E)\) and \(a \in S\), \(H \phi_a\) equals the set product \(a \rho. H.(a \rho)^{-1}\).

**COROLLARY.** - An inverse semigroup \(S\) with semilattice of idempotents \(E\) is isomorphic to a semidirect product of a semilattice and a group if and only if

\[E = \{xx^{-1} \mid x \in a \rho\}\]

for each \(a \in S\) and \(S\) is reduced; equivalently, if and only if \(E = a \rho. (a \rho)^{-1}\) for each \(a \in S\).

The theory has interesting specialisations to the semilattice of groups and bi-simple inverse cases.

The \(V\)-completion of the \(\rho\)-classes is accomplished by applying a theorem in the theory of partially ordered semigroups ([6], theorem 3 with \(S\) a reduced inverse semigroup under the natural order, \(\alpha = \rho^\mathcal{W}\) and \(D = S/\rho\) under the trivial order). For partially ordered semigroups, the following weaker result is obtained, generalising the main result of [3]:

**THEOREM 3.** - Let \(S\) be a partially ordered semigroup. Then \(S\) is a strict A-nomal quasi residuated semigroup whose maximal elements form the group of units of \(S\) if and only if \(S\) is a semidirect product of \(E\) by \(G\), where \(E\) is a quasi residuated semigroup with maximum element which is its identity element, and \(G\) is a trivially ordered group.

In theorem 3, \(\rho\) is taken to be the zig-zag congruence \([1]\) (see \([8]\)), and \(S\) being strict means that each \(\rho\)-class has a maximum element and that \(S\) has an identity \(1\) which is the maximum element in \(1 \rho\). In the semidirect product, the Cartesian ordering is employed, and the structural anti-homomorphism maps the \(G\) into the group of multiplicative, and order, automorphisms of \(E\).

**BIBLIOGRAPHY**


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