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MINIMAL INJECTIVE RESOLUTIONS

by Robert M. FOSSUM

0. Introduction.

The many results about commutative noetherian rings which have a non -zero module of finite type with finite injective dimension seem to indicate that the minimal injective resolution of a module of finite type should contain a great amount of information about the module. See for example PESKINE and SZIRO's paper [5]. In this report, I will outline a proof of a result due principally to FOXBY and GRIFFITH (and proved independently by P. ROBERTS using differents methods) which states :

If A is a noetherian local ring with maximal ideal $\mathfrak{m}$ and if M is an A-module of finite type, then $\operatorname{Ext}_A^j(A/\mathfrak{m} , M) \neq 0$ for all j in the range

$$\operatorname{depth} M \leq j \leq \operatorname{id}_A M .$$

This can be interpreted as a rigidity result. It also gives information about the minimal injective resolution of M. For if

$$0 \longrightarrow M \longrightarrow I^0 \longrightarrow I' \longrightarrow \cdots$$

is a minimal injective resolution of M, then the result states that the injective envelope of the residue class field is a direct summand of I^j for those integers j in the range $\operatorname{depth} M \leq j \leq \operatorname{id} M$. An interesting aspect of the proof is that is uses HOCHSTER's result establishing the existence of a maximal Cohen-Macaulay module (not necessarily of finite type) for a local ring of characteristic p (see HOCHSTER [4]), while the result itself is independent of characteristic.

Complete details can be found in a paper by FOSSUM, FOXBY, GRIFFITH and REITEN [2].

1. Preliminary results.

Let A be a commutative noetherian local ring with maximal ideal $\mathfrak{m}$ and residue class field $k = A/\mathfrak{m}$. Let M be an A-module of finite type. It is standard that

$$\operatorname{depth}_A M = \inf\{i \ ; \ \operatorname{Ext}_A^i(k , M) \neq 0\}$$

and

$$\operatorname{id}_A M = \sup\{i \ ; \ \operatorname{Ext}_A^i(k , M) \neq 0\} .$$

So the question is : what happens to $\operatorname{Ext}_A^j(k , M)$ for j in the interval between $\operatorname{depth}_A M$ and $\operatorname{id}_A M$?

BASS reported two results [1].

PROPOSITION 1. – *If* $id_A M < \infty$, *then* $id_A M = depth_A A$.

PROPOSITION 2. – *If* $id_A M = \infty$, *then* $Ext_A^j(k , M) \neq 0$ *for all* j *with* $j \geq \dim A$.

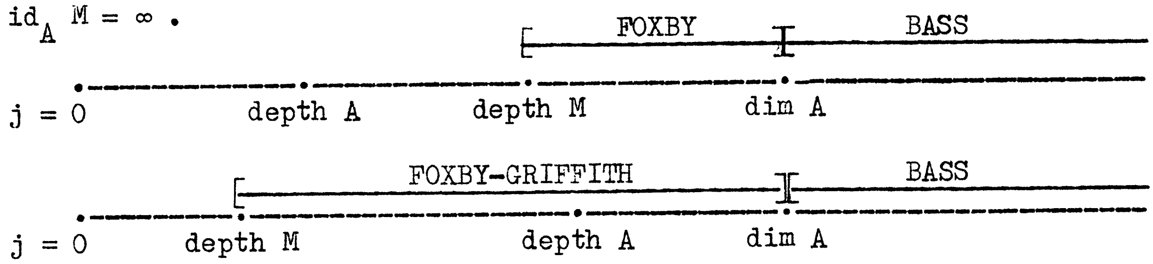
Much later FOXBY [3] extended the range in which $Ext_A^j(k , M) \neq 0$ for very special modules.

PROPOSITION 3. – *If* depth $A \leq$ depth M , *then* $Ext_A^j(k , M) \neq 0$ *for those* j *with*

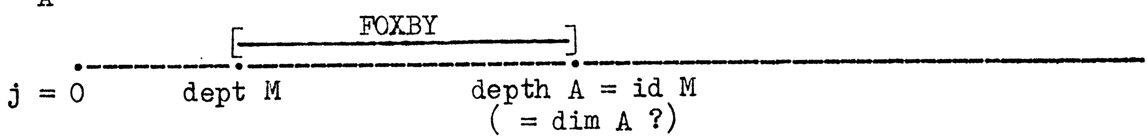
$$\text{depth } M \leq j \leq \text{id } M .$$

A diagram explains these results. The intervals with solid lines indicate the range of j where $Ext_A^j(k , M) \neq 0$.

1° $id_A M = \infty$.



2° $id_A M < \infty$



2. Main theorem.

The main theorem, which is stated in the local case in the introduction, follows :

THEOREM 1. – *Let* A *be a noetherian ring and* M *an* A-*module of finite type. Let*

$$0 \longrightarrow M \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \dots$$

be a minimal injective resolution of M . *If* j *is an integer, if* $p \in \operatorname{spec} A$, *and if* $\text{depth}_{A_p} M_p \leq j \leq id_{A_p} M_p$, *then the injective envelope of* A/p *is a direct summand of* I^j .

It is clear that we may assume that A is local and even complete, if necessary. Furthermore we can assume that depth $M <$ depth A by FOXBY's result.

Reduction. – Let $f_1 , \dots , f_r$ be a set of elements in A that forms a regular M-sequence and a regular A-sequence. Let $\mathfrak{f}$ denote the ideal generated by these elements. Since

$$Ext_A^j(k , M) \cong Ext_{A/\mathfrak{f}}^{j-r}(k , M/\mathfrak{f} M) ,$$

it may be assumed that depth $M = 0$.

3. The main lemma.

The principal lemma that connects the maximal Cohen-Macaulay modules with the problem under consideration follows. Let $E(k)$ denote the injective envelope of k as an A-module. The functor, that to M associates $\mathrm{Hom}_A(M , E(k))$ is denoted by $M^\vee$.

LEMMA 1. - Suppose N is an A-module (not necessarily of finite type), suppose $x_1 , \ldots , x_n$, is a regular N-sequence such that the annihilator

$$\mathrm{Ann}(N/(x_1 , \ldots , x_n)N)$$

is proper and $\mathfrak{m}$-primary. If M is an A-module of finite type with depth $M = 0$, then

$$\mathrm{Ext}_A^i(N/(x_1 , \ldots , x_j)N , M) \neq 0$$

for all i and j with $0 \leqslant i \leqslant j$.

Proof. - The proof goes by induction on j . Suppose $j = 0$. Since

$$\mathrm{Hom}_A(N , k) \simeq \mathrm{Hom}_A(N , \mathrm{Hom}_A(k , E(k)) \cong \mathrm{Hom}_A(N \otimes_A k , E(k)) ,$$

it is sufficient to show that $N \otimes_A k \neq 0$. But it is assumed that

$$\mathrm{Ann}(N/(x_1 , \ldots , x_n)N)$$

is $\mathfrak{m}$-primary and therefore $\mathfrak{m}(N/(x_1 , \ldots , x_n)N) \neq N/(x_1 , \ldots , x_n)N$. Hence $N/\mathfrak{m}N \neq 0$. The assumption depth $M = 0$ is equivalent to the assumption that k is isomorphic to a submodule of M . Therefore $\mathrm{Hom}_A(N , k) \neq 0$ implies $\mathrm{Hom}_A(N,M)\neq 0$.

The induction step uses the isomorphisms

$$\mathrm{Ext}_A^i(N/(x_1 , \ldots , x_j)N , M) \cong (\mathrm{Tor}_A^i(N/(x_1 , \ldots , x_j) , M^\vee)^\vee$$

and the exact sequences

$$0 \longrightarrow N/(x_1 ,\ldots, x_{j-1})N \xrightarrow{.x_j} N/(x_1 ,\ldots, x_{j-1})N \longrightarrow N/(x_1 ,\ldots, x_j)N \longrightarrow 0$$

to show that $\mathrm{Tor}_i^A(N/(x_1 , \ldots , x_j)N , M^\vee) \neq 0$ and therefore

$$\mathrm{Ext}_A^i(N/(x_1 , \ldots , x_j)N , M) \neq 0 .$$

LEMMA 2. - If $\mathrm{Ext}_A^j(k , M) = 0$, then $\mathrm{Ext}_A^j(T , M) = 0$ for all A-modules T with $\mathrm{Supp}\, T \leqslant \{\mathfrak{m}\}$.

Proof. - By induction on length, it is clear that $\mathrm{Ext}_A^j(T , M) = 0$ for all A-modules T of finite length. Otherwise write $T = \varinjlim T_\alpha$ where each T_α has finite length. Then

$$\mathrm{Ext}_A^j(T , M) = \varprojlim \mathrm{Ext}_A^j(T_\alpha , M) .$$

4. Proof of the theorem.

We assume, which we may, that A is a complete local ring and that depth $M = 0$. Suppose j is an integer in the range $0 < j < \dim A$. Let $d = \dim A$.

Suppose p is the characteristic of the residue class field. Let $R = A/pA$. We now quote a result due to HOCHSTER [4].

THEOREM 2. - *If R is an equi-characteristic local ring of dimension t, then there is an R-module T (not necessarily of finite type) such that if $x_1, \dots, x_t$ is a system of parameters of R, then $(x_1, \dots, x_t)T \neq T$ and these elements form a regular T-sequence. Such a T is called a maximal Cohen-Macaulay module.*

Apply this theorem to the ring R above. If $\dim R = d - 1$, pick elements $x_1, \dots, x_{d-1}$ in A that form a system of parameters in R and if $\dim R = \dim A$, pick $x_1, \dots, x_d$ in A forming a system of parameters in R. Let T be a maximal Cohen-Macaulay module for R. Set $N = T/x_d T$ (where $x_d = 0$ in case $\dim R = -1 + \dim A$). Then $x_1, \dots, x_{d-1}$ is a regular N-sequence and $\operatorname{Ann} N/(x_1, \dots, x_{d-1})N$ is $\mathfrak{m}$-primary. Apply lemma 1 to get

$$\operatorname{Ext}_A^r(N/(x_1, \dots, x_{d-1})N, M) \neq 0 \quad \text{for} \quad 0 \leqslant r \leqslant d-1 .$$

Apply lemma 2 to get $\operatorname{Ext}_A^r(k, M) \neq 0$ for $0 \leqslant r \leqslant d-1$. This proves the theorem.

COROLLARY 1. - *If j is an integer with $j > \operatorname{depth} M$, then $\operatorname{Ext}_A^j(k, M) = 0$ if, and only if, $\operatorname{id}_A M < j$.*

COROLLARY 2. - *If $0 \to M \to I^0 \to I^1 \to \dots$ is an injective resolution of M, then $E(k)$ is a direct summand of I^j if, and only if, $\operatorname{depth} M \leqslant j \leqslant \operatorname{id} M$.*

Remark 1. - It does not follow, nor as examples show is it even true, that the local cohomology modules $H_{\mathfrak{m}}^j(M) \neq 0$.

Remark 2. - If M is a nonzero module of finite type and finite injective dimension, then $\operatorname{id} M = \operatorname{depth} A$. If $\operatorname{depth} A \leqslant \operatorname{depth} M$, then A is Cohen-Macaulay. If $\operatorname{depth} M < \operatorname{depth} A$, then it is clear from the last paragraph of the proof that $\dim A - \operatorname{depth} A \leqslant 1$. In particular, if $\dim A = \dim(A/pA)$, then the proof shows that A is Cohen-Macaulay. But this also is easily obtained from [5]

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